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Multi-scaling analysis of a logistic model with slowly varying coefficients

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ABSTRACT

All single-species differential-equation population models incorporate parameters which define the model - for example, the *rate constant*, r, and *carrying capacity*, K, for the Logistic model. For constant parameter values, an exact solution may be found, giving the population as a function of time. However, for arbitrary time-varying parameters, exact solutions are rarely possible, and numerical solution techniques must be employed. In this work, we demonstrate that for a Logistic model in which the rate constant and carrying capacity both vary slowly with time, an analysis with multiple time scales leads to approximate closed form solutions that are explicit, are valid for a range of parameter values and compare favourably with numerically generated ones.

1. Introduction

Modelling the development of populations using differential equations has a long history. Although the number of individuals in a population changes in discrete steps, representing them as continuous variables has been shown to yield useful results in many applications. Neglecting spatial effects such as diffusion and dispersion reduces the mathematical model of a single-species population to an initial-value problem involving a single ordinary differential equation — see for example, [1] or [2]. While a gross simplification, such models are applicable to modelling of such unrelated phenomena as the evolution of fish school populations, and the spread of a single innovation (see [1,3] or [4]).

Such single-species models involve parameters, that define the evolutionary characteristics of the population. In general, these may themselves vary with time, t. Thus, the general Logistic model

$$\frac{\mathrm{d}P(t)}{\mathrm{d}t} = r(t)P(t)\left(1 - \frac{P(t)}{K(t)}\right), \qquad P(t=0) = P_0, \tag{1}$$

incorporates the *growth coefficients* $r(t)$ and $K(t)$- the *growth rate* and *carrying capacity* respectively - which are positive valued functions on $t \geq 0$. While it is possible to write down an explicit solution of (1)

$$P(t) = \frac{P_0 \exp \int_0^t r(\xi)\mathrm{d}\xi}{1 + P_0 \int_0^t (r(\xi)/K(\xi)) \exp(\int_0^\xi r(\eta)\mathrm{d}\eta)\mathrm{d}\xi}, \tag{2}$$

the integrals involved in (2) may only be evaluated for a very limited choice of $r(t)$ and $K(t)$ (including that of r and K being positive constants). For other cases, approximate methods must be used, to solve (1) or evaluate (2)- in particular, numerical techniques. These have the disadvantage of applying only to particular instances of the functions r and K.

In many cases, r and K vary slowly, relative to the changing population. This may arise from slow changes in the population species itself, or in the background environment, or a combination of these. In such cases, r and K may be

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represented as (positive valued) functions $r(\varepsilon t)$ and $K(\varepsilon t)$, where t is time, and ε is a small, positive parameter. Thus, "normal size" (i.e., $O(1)$ as $\varepsilon \to 0+$) changes in the argument of these functions, εt, correspond to "large" (i.e., $O(1/\varepsilon)$ as $\varepsilon \to 0+$) changes in t, i.e., $r(\varepsilon t)$ and $K(\varepsilon t)$ are slowly varying compared to $P(t, \varepsilon)$. In this case, the Logistic model (1) becomes

$$\frac{\mathrm{d}P(t,\varepsilon)}{\mathrm{d}t} = r(\varepsilon t)P(t,\varepsilon)\left(1 - \frac{P(t,\varepsilon)}{K(\varepsilon t)}\right), \qquad P(t=0,\varepsilon) = P_0. \tag{3}$$

In what follows, we will demonstrate a technique that constructs an approximation to the solution of (3), valid for all times $t \geq 0$. This method belongs to a class of related methods, termed *multi-timing* or *multi-scaling* methods, well-established in the physical and engineering science literature — see, for example, [5], Ch. 3, or [6], Ch. 6, among many. Such methods exploit the disparate time variation of components of a system, to produce an algorithm capable of generating an approximate solution. In the present case, the disparity is between the rate of variation of the population and that of r and K. It should be noted that such a method has been applied to (3) in the case where only K varies slowly, and r is constant (see [7,8]). In the present case, the method must be modified to deal with two slowly varying parameters.

2. The multi-scale Logistic equation

The model (3) depends on the two time scales t and εt. However, while the analysis of [7,8] successfully applied a multi-timing method based on these two scales, in the general case here, of both r and K slowly varying, it turns out to be appropriate to consider the generalized time scales

$$t_0 = \frac{1}{\varepsilon}h(t_1) \quad \text{and} \quad t_1 = \varepsilon t, \tag{4}$$

where $h(t_1)$ is a function to be found. Note that $t_1 \geq 0$, and we expect $h(t_1)$ to be a positive valued function on all $t_1 > 0$, with $h(0) = 0$. Note also that for t_0 and t_1 to be dimensionless and dimensionally consistent, we require that ε and h have dimensions of reciprocal time.

Here t_0 is the "normal" time variable and t_1 is the "slow" time variable. This makes $K(\varepsilon t)$ and $r(\varepsilon t)$ functions of t_1 only. The "normal" time scale, t_0, is determined by the form of the function $h(t_1)$. Noting that $t_1 = \varepsilon t$, we have, on taking differentials, $\mathrm{d}t_0 = h'(t_1)\mathrm{d}t$, so $O(1)$ as $\varepsilon \to 0+$ changes in t are reflected as $O(1)$ changes in t_0, provided $h'(t_1)$ is $O(1)$ as $\varepsilon \to 0+$. Further, since we require that there be a one-to-one correspondence between t_0 and t values, we need $h'(t_1)$ to be mono-signed; thus, we assume that $h'(t_1) > 0$.

Regarding $P(t, \varepsilon)$ as a function of these two time scales, i.e., $P(t, \varepsilon) \equiv p(t_0, t_1, \varepsilon)$, and applying the chain rule gives the Logistic differential Eq. (3) in multi-scaled form as

$$h'(t_1)D_0\, p + \varepsilon D_1 p = r(t_1)p\left(1 - \frac{p}{K(t_1)}\right), \tag{5}$$

where D_0 and D_1 are partial derivatives taken with respect to t_0 and t_1 respectively.

We note that the ordinary differential equation in (3) is now equivalent to the partial differential equation (5), for the unknown function $p(t_0, t_1, \varepsilon)$. This apparent increase in complexity is offset by the fact that now, the dependence on ε is displayed explicitly (rather that implicitly, as in (3)). Thus, Eq. (5) is now in a suitable form for solution by a perturbation method based on the limit $\varepsilon \to 0$. We carry this analysis out next.

3. Perturbation analysis

Expressing p as a Poincaré expansion in ε,

$$p(t_0, t_1, \varepsilon) = p_0(t_0, t_1) + \varepsilon\, p_1(t_0, t_1) + \varepsilon^2\, p_2(t_0, t_1) + \cdots, \tag{6}$$

and substituting into Eq. (5) gives

$$h'(t_1)D_0(p_0 + \varepsilon p_1 + \cdots) + \varepsilon D_1(p_0 + \varepsilon p_1 + \cdots) = r(t_1)(p_0 + \varepsilon p_1 + \cdots)\left(1 - \frac{(p_0 + \varepsilon p_1 + \cdots)}{K(t_1)}\right). \tag{7}$$

Equating like powers of ε in (7) gives a sequence of differential equations. For terms independent of ε,

$$h'(t_1)D_0 p_0 = r(t_1)p_0\left(1 - \frac{p_0}{K(t_1)}\right); \tag{8}$$

and for $O(\varepsilon)$ terms,

$$h'(t_1)D_0 p_1 + D_1 p_0 = r(t_1)p_1\left(1 - \frac{2p_0}{K(t_1)}\right), \tag{9}$$

with analogous equations for $p_2,\ p_3,\ p_4, \ldots$.

Now, solving the partial differential equation (8) gives

$$p_0(t_0, t_1) = \frac{K(t_1)}{1 + C(t_1)K(t_1)e^{-\theta(t_1)t_0}}, \tag{10}$$

where $C(t_1)$ is an arbitrary function of t_1, being introduced when integration with respect to t_0 is carried out, and $\theta(t_1) = r(t_1)/h'(t_1)$. Note that $\theta(t_1)$ is a positive function of t_1 on $t_1 \geq 0$. Note also that Eq. (9) is a linear differential equation for p_1 as a function of t_0, that is readily solved, given p_0.

In what follows, we seek a particular solution only of (9), since the original differential equation is first order, so the overall solution should involve only one arbitrary "constant" (a function of t_1 here).

Solving (9) for p_1 gives this particular solution as

$$p_1 = \frac{-K'(t_1)/\theta(t_1) + \frac{1}{2}K(t_1)^2\theta'(t_1)C(t_1)t_0^2e^{-\theta(t_1)t_0} - K(t_1)^2C'(t_1)^2t_0e^{-\theta(t_1)t_0}}{h'(t_1)\left(1 + C(t_1)K(t_1)e^{-\theta(t_1)t_0}\right)^2}, \tag{11}$$

where the primes denote derivatives taken with respect to the argument, t_1.

Substituting p_0 and p_1 into the Poincaré expansion (6) gives a two-term expansion for the function $p(t_0, t_1, \varepsilon)$, assumed valid on all $t_0, t_1 \geq 0$.

We note from (10) that as $t_0 \to \infty$, $p_0 \to K(t_1)$; and $p_1 \to -K'(t_1)/(\theta(t_1)h'(t_1))$. However, the presence of the $t_0e^{-\theta(t_1)t_0}$ and $t_0^2e^{-\theta(t_1)t_0}$ terms in p_1 means that convergence of p_1 to its limit is slower than that of p_0; and, as $t_0 \to \infty$ the difference between p_1 and its limit becomes larger than the corresponding difference for p_0. To ensure that this condition does not occur, we set the coefficients of $t_0e^{-\theta(t_1)t_0}$ and $t_0^2e^{-\theta(t_1)t_0}$ to zero. Thus, to do this, we choose $C'(t_1) = 0$ and $\theta'(t_1) = 0$, which condition may be met by choosing $C(t_1)$ and $\theta(t_1)$ to be constants. In particular, we may choose $\theta(t_1) = 1$, giving $h'(t_1) = r(t_1)$ on $t_1 \geq 0$. This leads, with (4), to

$$t_0 = \frac{1}{\varepsilon}\int_0^{t_1} r(s)ds, \tag{12}$$

defining the timescale t_0. This reinforces our earlier contention that here, the time scales are not the same as those adopted in [8].

With the choices above, expansion (6) then becomes

$$p(t_0, t_1, \varepsilon) = \frac{K(t_1)}{1 + cK(t_1)e^{-t_0}} - \varepsilon\frac{K'(t_1)}{r(t_1)\,(1 + cK(t_1)e^{-t_0})^2} + \cdots, \tag{13}$$

where c is an arbitrary constant. Since our expansion consists of both leading order terms and $O(\varepsilon)$ terms, we assume the constant c takes on the same form, i.e. $c = c_0 + \varepsilon c_1 + \cdots$.

Substituting the initial condition from (3) into the expansion (13) gives

$$P_0 = \frac{K(0)}{1 + (c_0 + \varepsilon c_1 + \cdots)K(0)} - \varepsilon\frac{K'(0)}{r(0)\,(1 + (c_0 + \varepsilon c_1 + \cdots)K(0))^2} + \cdots. \tag{14}$$

By expanding this out using series, collecting coefficients of like powers of ε, then solving for c_0 and c_1, we find our two-term expansion (13) for the solution of the slowly varying Logistic model to be

$$P(t, \varepsilon) = \frac{K(\varepsilon t)P_0K(0)}{P_0K(0) - [K(\varepsilon t)P_0 - K(\varepsilon t)K(0)]e^{-t_0}} - \frac{\varepsilon P_0^2[K'(\varepsilon t)K(0)^2r(0) - K(\varepsilon t)^2K'(0)r(\varepsilon t)e^{-t_0}]}{r(\varepsilon t)r(0)[P_0K(0) + (K(\varepsilon t)K(0) - K(\varepsilon t)P_0)e^{-t_0}]^2} + \cdots, \tag{15}$$

with t_0 as in (12).

4. Results and discussion

The expression (15) provides a straightforward explicit approximation to the evolving population $P(t, \varepsilon)$ for given functions r and K. From this, we see that as $t \to \infty$ the population will tend to

$$K(\varepsilon t) - \frac{\varepsilon}{r(\varepsilon t)}K'(\varepsilon t) + \cdots, \tag{16}$$

i.e., a varying small neighbourhood of the carrying capacity $K(\varepsilon t)$. For many carrying capacities of interest, the derivative $K'(\varepsilon t)$ will tend to zero as $t \to \infty$, so in these cases the population, $P(t, \varepsilon)$, will tend to the carrying capacity plus/minus an $O(\varepsilon^2)$ term.

As we have noted above, there are no examples of exact solutions of (3) in the literature, for which *both* r and K vary - slowly or otherwise. Thus, we have no exact solutions to compare with the results of applying the expansion (15) to slowly varying r and K, and the only option is to compare with numerical solutions of (3). However, Banks [9] gives a comprehensive list of examples of solutions of (3) where r and K vary with time *separately* (but not together). In what follows, we consider three examples from [9] and show that these may be reconciled with the results obtained by applying (15).

We begin with an exponentially varying K, $K = K_0 e^{\varepsilon t}$ and r = constant ([9], Sec. 5.1), where ε is an arbitrary constant. Here, the problem (3) has an exact solution, given by (2) as

$$P = \frac{K_0}{(\frac{r}{r-\varepsilon})e^{-\varepsilon t} + [K_0/P_0 - (\frac{r}{r-\varepsilon})]e^{-rt}}. \tag{17}$$

Substitution of $K = K_0 e^{\varepsilon t}$, r = constant (with ε small and positive, giving a slowly exponentially varying K) into (15) gives, after some manipulation,

$$P = \frac{K_0}{e^{-\varepsilon t} + (K_0/P_0 - 1)e^{-rt}} - \varepsilon \frac{K_0(e^{-\varepsilon t} - e^{-rt})}{r[e^{-\varepsilon t} + (K_0/P_0 - 1)e^{-rt}]^2} + \cdots. \tag{18}$$

It is a simple matter to show that when (17) is expanded for small ε (keeping the $e^{-\varepsilon t}$ intact), we obtain (18), thus confirming (15) in this case.

As a second example, we choose $K = K_0 + K_a \sin \varepsilon t$, r = constant. For small ε, this gives a slowly varying periodic K. For this K and r, the integrals in (2) cannot be evaluated, and there is no exact solution. However, when K_a/K_0 is small, the integrands may be expanded in powers of K_a/K_0, and an expression can be obtained for P as a power series in K_a/K_0 — see Banks ([9], Sec. 5.5). If we further expand this result for small positive ε, we obtain after some manipulation,

$$P = \frac{K_0}{1 + (K_0/P_0 - 1)e^{-rt}} + \frac{K_a(\sin \varepsilon t + \varepsilon(\cos \varepsilon t - e^{-rt})/r)}{(1 + (K_0/P_0 - 1)e^{-rt})^2} + \cdots. \tag{19}$$

Substituting this choice of K and r above into (15), and expanding in powers of K_a/K_0, gives this same result, after some manipulation, thus reconciling the two approaches.

A third example lets r vary, while keeping K constant. Thus, we consider the linearly varying rate $r = r_0(1 - \varepsilon t)$ and K = constant. Although no exact solution exists for finite K, it may be shown ([9], Sec. 4.1) that when $K = \infty$ (unconstrained growth), we get, from (2),

$$P = P_0 e^{r_0 t(1-\varepsilon t/2)}. \tag{20}$$

Applying this r and K to (15) gives

$$P = \frac{P_0}{P_0/K - (P_0/K - 1)e^{-r_0 t(1-\varepsilon t/2)}} + \cdots, \tag{21}$$

with the $O(\varepsilon)$ term vanishing identically. Letting $K \to \infty$ in (21) gives (20). Again, this confirms the validity of (15).

The distinct advantage of the result (15) is that it provides an approximation when *both* r and K are (slowly) varying. With no exact solutions to compare with in this case, we compare the result (15) with a numerical solution. As a typical example, we choose a periodically varying carrying capacity and growth rate, given by

$$K(\varepsilon t) = K_0 + \delta \sin \varepsilon t \qquad r(\varepsilon t) = r_0 + \Delta \sin \varepsilon t \tag{22}$$

where ε is small, while δ, Δ are the amplitudes of the oscillatory components. Here, the carrying capacity and rate oscillate around their initial values K_0 and r_0. Such behaviour is typical of environments which slowly fluctuate over time. For example, in a marine environment the changing tides resulting from the phases of the moon bring slow variation in the ability of the environment to support a given species.

Fig. 1 gives a comparison of the multi-timing approximation (15) with the numerical solution of (3) for a choice of parameter values in the periodic growth coefficients (22).

If we regard the numerical solution as an 'exact' solution, this shows that the expansion (15) provides an extremely good approximation to the solution (3) in this case. In fact, the two plots are almost coincident. Fig. 1 also shows that the population very quickly moves from its initial value to a small neighbourhood of the carrying capacity, and remains there for all subsequent time, alternating between values above and below the carrying capacity as the sign of $K'(\varepsilon t)$ changes. This reflects (16).

5. Conclusion

The multi-scaling technique has been successfully applied to a Logistic population model in which the defining parameters vary slowly with time, giving an explicit, closed form, easily used approximation for the evolving population, which may be shown to reduce to known results from the literature. It also compares very favourably with the results of numerical computations in particular cases, while being valid for a range of parameter values.

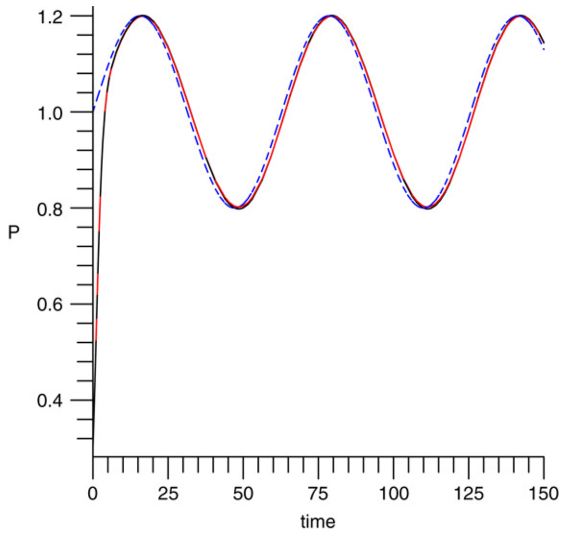

Fig. 1. Logistic multi-scale approximation (black) vs. numerical solution (red) using (22), with $K_0 = 1$, $r_0 = 0.9$, $\delta = 0.2$, $\Delta = 0.2$, $P_0 = 0.3$ and $\varepsilon = 0.1$. The carrying capacity is shown in blue (dashed line). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

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