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Locally strong endomorphisms of paths

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Abstract

We determine the number of locally strong endomorphisms of directed and undirected paths—direction here is in the sense of a bipartite graph from one partition set to the other. This is done by the investigation of congruence classes, leading to the concept of a complete folding, which is used to characterize locally strong endomorphisms of paths. A congruence belongs to a locally strong endomorphism if and only if the number l of congruence classes divides the length of the original path and the points of the path are folded completely into the l classes, starting from 0 to l and then back to 0, then again back to l and so on. It turns out that for paths locally strong endomorphisms form a monoid if and only if the length of the path is prime or equal to 4 in the undirected case and in the directed case also if the length is 8. Finally some algebraic properties of these monoids are described.

MSC: 00A22; 05A18; 05C25; 05C38; 05C20; 11A41; 20M19

Keywords: Graph endomorphism; Locally strong graph endomorphism; Directed path; Undirected path; Completely regular semigroup; Left group

A *graph endomorphism* (or *endomorphism*) of a graph (V, E) is a mapping $f : V \to V$ on the vertex set V of the graph which preserves edges, i.e. for all $x, y \in V$, $\{x, y\} \in E$ implies $\{f(x), f(y)\} \in E$. An endomorphism $f : V \to V$ is called *locally strong* if for all $x, y \in V$, $\{f(x), f(y)\} \in E$ implies that for all $x' \in f^{-1}(f(x))$ there exists $y' \in f^{-1}(f(y))$ such that $\{x', y'\} \in E$, and for all $y' \in f^{-1}(f(y))$ there exists $x' \in f^{-1}(f(x))$ such that $\{x', y'\} \in E$, compare [2] and [3]. We will write $(x, y) \in E$ if (V, E) is a directed graph.

In [1] we see an algorithm to determine the cardinalities of the endomorphism monoid of finite undirected paths. In this paper we provide an algorithm to determine the cardinality of the set of locally strong endomorphisms of finite undirected and of directed paths. In [3] it was proved that all endomorphisms of paths (as special trees) which are not automorphisms, but are stronger than half strong (for the definition see [2,3]) are locally strong. We show now that the set of locally strong endomorphisms on an undirected path will form a monoid if and only if the length of the path is prime or equal to 4. For directed paths the condition turns into “prime, length 4 or length 8”. A way of counting all endomorphisms of undirected paths by first counting the congruence classes has been introduced by Michels in [4]. Independently Michels obtained the results on the semigroup $\mathrm{LEnd}(P_n)$ for undirected paths.

The locally strong endomorphisms with only two congruence classes, that is with image paths of length 1, always form a monoid in the set of all locally strong endomorphisms. These monoids form left groups which are unions of copies $(\mathbb{Z}_2; +)$ or, in the directed case, of left zero semigroups. The other subsemigroups of locally strong endomorphisms become unions of groups if we take two elements out. For semigroup theoretical concepts see for example [5].

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1. Undirected paths

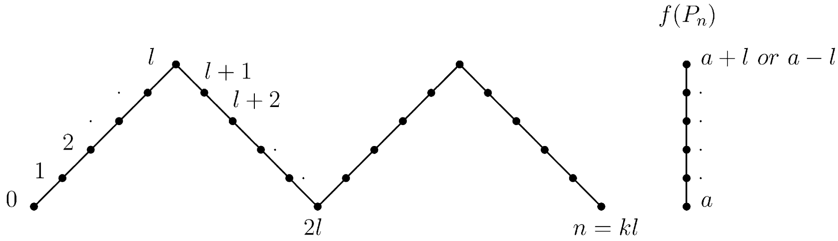
By P_n we denote an undirected path of length n with $n + 1$ vertices.

Let $f : P_n \to P_n$ be an endomorphism. The length of the image path of f is called the *length* of f.

An endomorphism $f : P_n \to P_n$ is called a *complete folding* if the congruence classes of the relation, $\ker f = \{(x, y) \in P_n \times P_n | f(x) = f(y)\}$, partition P_n into l classes where $l|n$ and the equivalence classes are of the form:

- $[0] = \{2ml \in P_n | m = 0, 1, \ldots\}$,
- $[l] = \{(2m + 1)l \in P_n | m = 0, 1, 2, \ldots\}$, and for r such that $0 < r < l$,
- $[r] = \{2ml + r \in P_n | m = 0, 1, \ldots\} \cup \{2ml - r \in P_n | m = 1, 2, \ldots\}$.

Clearly, such a complete folding has length l.



The following observation is clear.

Lemma 1.1. *Every complete folding of an undirected path is locally strong.*

Lemma 1.2. *If f is a locally strong endomorphism and $f(P_n) = \{a, a + 1, \ldots, a + l\} \subseteq P_n$, then $f(0) = a$ or $a + l$.*

Proof. Suppose that $f(0) = a + r$ for some r, $0 < r < l$, then $f(1) = a + r + 1$ or $a + r - 1$.

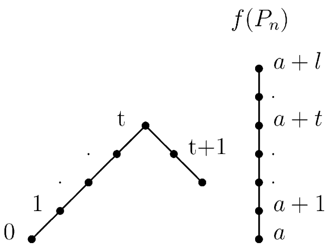
Case 1: If $f(1) = a + r + 1$, then we get that $\{a + r, a + r - 1\} \in E$ but there exists no $x \in f^{-1}(a + r - 1)$ such that $\{0, x\} \in E$. Thus f is not a locally strong endomorphism.

Case 2: If $f(1) = a + r - 1$, then as in Case 1 we get $\{a + r, a + r + 1\} \in E$ but there is no $x \in f^{-1}(a + r + 1)$ such that $\{0, x\} \in E$. Thus f is not a locally strong endomorphism.

Therefore $f(0) = a$ or $a + l$. $\square$

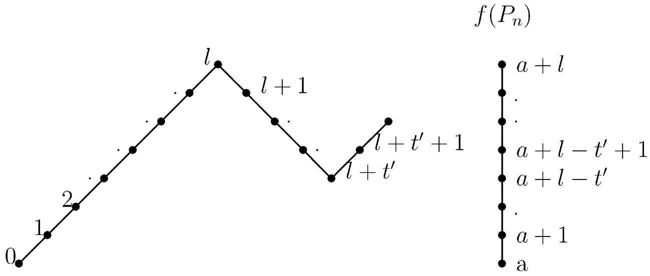
Lemma 1.3. *Every locally strong endomorphism on P_n is a complete folding.*

Proof. Let $f : P_n \to P_n$ be a locally strong endomorphism on P_n, and let $f(P_n) = \{a, a + 1, \ldots, a + l\}$. So, by Lemma 1.2 we get $f(0) = a$ or $a + l$, let us say $f(0) = a$. Then $f(1) = a + 1$. Next, we will show that $f(r) = a + r$ for all r, $0 \leqslant r \leqslant l$. Suppose there exists t, $0 < t < l$ such that $f(r) = a + r$ for all r, $0 \leqslant r \leqslant t$ but $f(t + 1) = a + t - 1$.



Since $\{a + t, a + t + 1\} \in E$, $t \in f^{-1}(a + t)$ and $t - 1, t + 1 \in f^{-1}(a + t - 1)$, there is no $x \in f^{-1}(a + t + 1)$ such that $\{t, x\} \in E$. So f is not a locally strong endomorphism. Thus $f(r) = a + r$ for all $r = 0, 1, \ldots, l$.

Suppose now that $f(l+r)=a+l-r$ for all $r=0,1,\ldots,t'$ but $f(l+t'+1)=a+l-t'+1$ for some t', $0<t'<l$.



Then $f(l+t'+1)=f(l+t'-1)=a+l-t'+1$. Therefore, there is no $x\in f^{-1}(a+l-t'-1)$ such that $\{x,l+t'\}\in E$. So f is not a locally strong endomorphism.

Altogether, the equivalence classes are of the form:

- $[0]=\{2ml\in P_n|m=0,1,2,\ldots\}$,
- $[l]=\{(2m+1)l\in P_n|m=0,1,2,\ldots\}$,
- $[r]=\{2ml+r\in P_n|m=0,1,\ldots\}\cup\{2ml-r\in P_n|m=1,2,\ldots\}$, for all r such that $0<r<l$.

If l does not divide n, then $n\in[r]$ for some r, $0<r<l$. Hence $f(n)=a+r$ and $f(n-1)=a+r-1$ (or $a+r+1$). Then $\{a+r,a+r+1\}\in E$ (or $\{a+r-1,a+r\}\in E$) but there is no $x\in f^{-1}(a+r+1)$ (or $x\in f^{-1}(a+r-1)$) such that $\{n,x\}\in E$. This is a contradiction to f being locally strong. Thus $l|n$. $\square$

Then from Lemmas 1.1 and 1.3, we get:

Corollary 1.4. *An endomorphism on undirected path is locally strong if and only if it is a complete folding.*

We will denote a locally strong endomorphism $f: P_n\to P_n$ of length l which maps 0 to a and 1 to $a+1$ (or $a-1$) by f_{l,a^+} (or f_{l,a^-}, respectively). For example:

$f_{3,2^+}: P_9\to P_9$ is

$$f_{3,2^+}=\begin{pmatrix}0&1&2&3&4&5&6&7&8&9\\2&3&4&5&4&3&2&3&4&5\end{pmatrix}.$$

$f_{3,6^-}: P_9\to P_9$ is

$$f_{3,6^-}=\begin{pmatrix}0&1&2&3&4&5&6&7&8&9\\6&5&4&3&4&5&6&5&4&3\end{pmatrix}.$$

Theorem 1.5. *Let* $|\mathrm{LEnd}_l(P_n)|$ *denote the cardinality of the set of locally strong endomorphisms of length* l *of the undirected path* P_n*. Then* $|\mathrm{LEnd}_l(P_n)|=2(n-l+1)$*. And the cardinality of the set of all locally strong endomorphisms on the undirected path* P_n *is* $|\mathrm{LEnd}(P_n)|=2\sum_{l|n}(n-l+1)$.

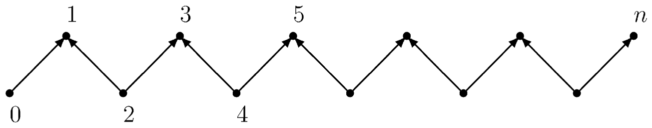
Proof. For $l|n$ we count the number of embeddings of P_l into P_n:

$$\begin{aligned}|\mathrm{LEnd}_l(P_n)|&=|\{f_{l,x^+}:P_n\to P_n|x=0,1,2,\ldots,n-l\}|\\&\quad+|\{f_{l,x^-}:P_n\to P_n|x=l,l+1,l+2,\ldots,n\}|\\&=|\{f_{l,x^+}:P_n\to P_n|x=0,1,2,\ldots,n-l\}|\\&\quad+|\{f_{l,x^-}:P_n\to P_n|x=l,l+1,l+2,\ldots,l+(n-l)\}|\\&=(n-l+1)+(n-l+1)\\&=2(n-l+1).\end{aligned}$$

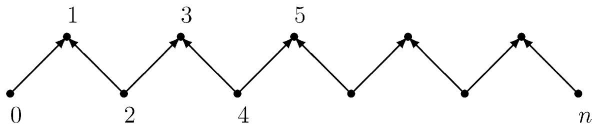
Therefore $|\mathrm{LEnd}(P_n)|=2\sum_{l|n}(n-l+1)$. $\square$

2. Directed paths

Consider directed paths $\overline{P_n}$ of length n. If n is odd, then the graph $\overline{P_n}$ is as follows:



If n is even then the graph $\overline{P_n}$ is as follows:



Lemma 2.1. *If $f : \overline{P_n} \to \overline{P_n}$ is an endomorphism on the directed path $\overline{P_n}$, then $f(x)$ is odd if and only if x is odd.*

Lemma 2.2. *Let* $\mathrm{Aut}(\overline{P_n})$ *denote the set of all automorphisms on the directed path $\overline{P_n}$. Then*

$$|\mathrm{Aut}(\overline{P_n})| = \begin{cases} 1 & \textit{if } n \textit{ is odd}, \\ 2 & \textit{if } n \textit{ is even}. \end{cases}$$

Proof. The statement is obvious when looking at the previous two pictures. □

In the same manner as for undirected paths case we can prove that

Theorem 2.3. *An endomorphism on the directed path is locally strong if and only if it is a complete folding.*

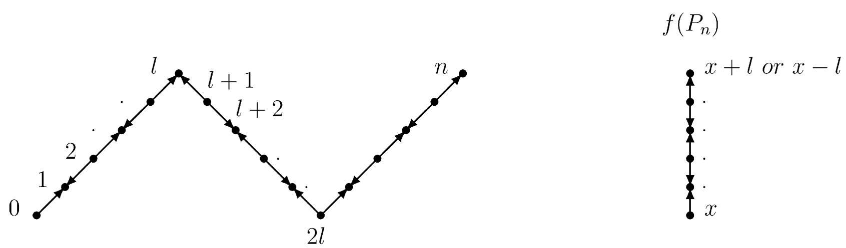
Theorem 2.4. *Let* $\mathrm{LEnd}_l(\overline{P_n})$ *denote the set of all locally strong endomorphisms of length l on the directed path $\overline{P_n}$, l divides n. Then*

$$|\mathrm{LEnd}_l(\overline{P_n})| = \begin{cases} n - l + 1 & \textit{if } l \textit{ is odd}, \\ n - l + 2 & \textit{if } l \textit{ is even}. \end{cases}$$

And

$$|\mathrm{LEnd}(\overline{P_n})| = \begin{cases} \sum_{l|n}(n - l + 1) & \textit{if } n \textit{ is odd}, \\ \sum_{\substack{l|n \\ l \text{ odd}}} (n - l + 1) + \sum_{\substack{l|n \\ l \text{ even}}} (n - l + 2) & \textit{if } n \textit{ is even}. \end{cases}$$

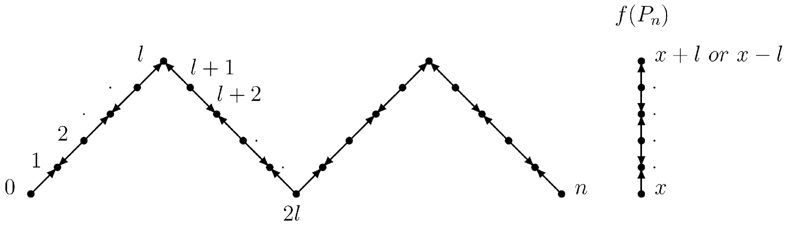
Proof. *Case* 1: Suppose l is odd, n is odd, and $l|n$.



Then

$$
\begin{aligned}
|\mathrm{LEnd}_l(P_n)| &= |\{f_{l,x^+} : P_n \to P_n | x = 0, 2, 4, \ldots, n-l\}| \\
&\quad + |\{f_{l,x^-} : P_n \to P_n | x = n-1, n-3, n-5, \ldots, l+1\}| \\
&= \left|\left\{ f_{l,x^+} : P_n \to P_n | x = 0, 2, 4, \ldots, 2\left(\frac{n-l}{2}\right)\right\}\right| \\
&\quad + \left|\left\{ f_{l,x^-} : P_n \to P_n | x = n-1, n-3, n-5, \ldots, n-\left(2\left(\frac{n-l}{2}\right)-1\right)\right\}\right| \\
&= \left(\frac{n-l}{2}+1\right) + \left(\frac{n-l}{2}\right) \\
&= n-l+1.
\end{aligned}
$$

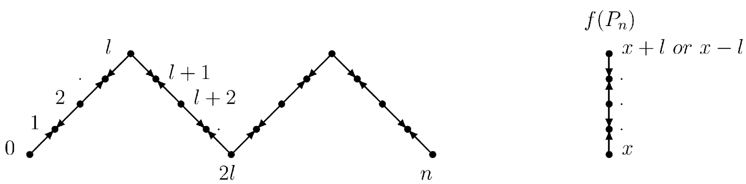
Case 2: Suppose now l is odd, n is even and $l|n$.



Then

$$
\begin{aligned}
|\mathrm{LEnd}_l(P_n)| &= |\{f_{l,x^+} : P_n \to P_n | x = 0, 2, 4, \ldots, n-l-1\}| \\
&\quad + |\{f_{l,x^-} : P_n \to P_n | x = n, n-2, n-4, \ldots, l+1\}| \\
&= \left|\left\{ f_{l,x^+} : P_n \to P_n | x = 0, 2, 4, \ldots, 2\left(\frac{n-l-1}{2}\right)\right\}\right| \\
&\quad + \left|\left\{ f_{l,x^-} : P_n \to P_n | x = n, n-2, n-4, \ldots, n-\left(2\left(\frac{n-l-1}{2}\right)\right)\right\}\right| \\
&= \left(\frac{n-l-1}{2}+1\right) + \left(\frac{n-l-1}{2}+1\right) \\
&= n-l-1+2 \\
&= n-l+1.
\end{aligned}
$$

Case 3: Let now l and n be even, and $l|n$. Note that the illustrating picture will end with n on the top level as in Case 1 if n/l is odd.



Then

$$
\begin{aligned}
|\mathrm{LEnd}_l(P_n)| &= |\{f_{l,x^+} : P_n \to P_n | x = 0, 2, 4, \ldots, n-l\}| \\
&\quad + |\{f_{l,x^-} : P_n \to P_n | x = n, n-2, n-4, \ldots, l\}| \\
&= \left|\left\{ f_{l,x^+} : P_n \to P_n | x = 0, 2, 4, \ldots, 2\left(\frac{n-l}{2}\right)\right\}\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\left\{f_{l,x^-} : P_n \to P_n \mid x = n, n-2, n-4, \ldots, n-2\left(\frac{n-l}{2}\right)\right\}\right| \\
&= \left(\frac{n-l}{2}+1\right) + \left(\frac{n-l}{2}+1\right) \\
&= n-l+2.
\end{aligned}
$$

Therefore, we get

$$
|\mathrm{LEnd}(P_n)| = \begin{cases} \sum_{l|n}(n-l+1) & \text{if } n \text{ is odd,} \\ \sum_{\substack{l|n \\ l \text{ odd}}}(n-l+1) + \sum_{\substack{l|n \\ l \text{ even}}}(n-l+2) & \text{if } n \text{ is even.} \end{cases}
$$

□

3. Algebraic properties of LEnd

The following two observations are clear.

Lemma 3.1. *Every endomorphism $f : P_n \to P_n$ of length 1 of a path P_n is a locally strong endomorphism.*

Lemma 3.2. *Let $n \geq 1$. If $f : P_n \to P_n$ is an endomorphism of length 1 and $g : P_n \to P_n$ is an endomorphism then $f \circ g$ and $g \circ f$ are endomorphisms of length 1.*

Theorem 3.3. *If n is prime, then the set $\mathrm{LEnd}(P_n)$, of all locally strong endomorphisms on the undirected path P_n forms a monoid, which is a left group consisting of copies of $\mathbb{Z}_2$ together with the automorphism group $\mathbb{Z}_2$. The set $\mathrm{LEnd}(P_4)$ forms a monoid which is a union of groups if we delete two elements.*

Proof. The first part follows from Lemmas 3.1 and 3.2, and Corollary 1.4. The algebraic structure of the monoid is obviously of the described form. On P_4 there are eight locally strong endomorphisms of length 1, $f_{1,0^+}$, $f_{1,1^-}$, $f_{1,1^+}$, $f_{1,2^-}$, $f_{1,2^+}$, $f_{1,3^-}$, $f_{1,3^+}$, $f_{1,4^-}$, there are six locally strong endomorphisms of length 2, $f_{2,0^+}$, $f_{2,1^+}$, $f_{2,2^-}$, $f_{2,2^+}$, $f_{2,3^-}$, $f_{2,4^-}$, and there are only two locally strong endomorphisms of length 4, $f_{4,0^+}$, $f_{4,4^-}$. We give the Multiplication table, omitting the two automorphisms. In the table, we write r_{x^+} for f_{r,x^+} and r_{x^-} for f_{r,x^-} for all $r, x \in P_4$.

$\circ$	1_{0+}	1_{1-}	1_{2-}	1_{1+}	1_{4-}	1_{3+}	1_{2+}	1_{3-}	2_{0+}	2_{2-}	2_{4-}	2_{2+}	2_{1+}	2_{3-}
1_{0+}	1_{0+}	1_{1-}	1_{0+}	1_{1-}	1_{0+}	1_{1-}	1_{0+}	1_{1-}	1_{0+}	1_{0+}	1_{0+}	1_{0+}	1_{1-}	1_{1-}
1_{1-}	1_{1-}	1_{0+}	1_{1-}	1_{0+}	1_{1-}	1_{0+}	1_{1-}	1_{0+}	1_{1-}	1_{1-}	1_{1-}	1_{1-}	1_{0+}	1_{0+}
1_{2-}	1_{2-}	1_{1+}	1_{2-}	1_{1+}	1_{2-}	1_{1+}	1_{2-}	1_{1+}	1_{2-}	1_{2-}	1_{2-}	1_{2-}	1_{1+}	1_{1+}
1_{1+}	1_{1+}	1_{2-}	1_{1+}	1_{2-}	1_{1+}	1_{2-}	1_{1+}	1_{2-}	1_{1+}	1_{1+}	1_{1+}	1_{1+}	1_{2-}	1_{2-}
1_{4-}	1_{4-}	1_{3+}	1_{4-}	1_{3+}	1_{4-}	1_{3+}	1_{4-}	1_{3+}	1_{4-}	1_{4-}	1_{4-}	1_{4-}	1_{3+}	1_{3+}
1_{3+}	1_{3+}	1_{4-}	1_{3+}	1_{4-}	1_{3+}	1_{4-}	1_{3+}	1_{4-}	1_{3+}	1_{3+}	1_{3+}	1_{3+}	1_{4-}	1_{4-}
1_{2+}	1_{2+}	1_{3-}	1_{2+}	1_{3-}	1_{2+}	1_{3-}	1_{2+}	1_{3-}	1_{2+}	1_{2+}	1_{2+}	1_{2+}	1_{3-}	1_{3-}
1_{3-}	1_{3-}	1_{2+}	1_{3-}	1_{2+}	1_{3-}	1_{2+}	1_{3-}	1_{2+}	1_{3-}	1_{3-}	1_{3-}	1_{3-}	1_{2+}	1_{2+}
2_{0+}	1_{0+}	1_{1-}	1_{2-}	1_{1+}	1_{0+}	1_{1-}	1_{2-}	1_{1-}	2_{0+}	2_{2-}	2_{0+}	2_{2-}	1_{1+}	1_{1+}
2_{2-}	1_{2-}	1_{1+}	1_{0+}	1_{1-}	1_{2-}	1_{1+}	1_{0+}	1_{1-}	2_{2-}	2_{0+}	2_{2-}	2_{0+}	1_{1-}	1_{1-}
2_{4-}	1_{4-}	1_{3+}	1_{2+}	1_{3-}	1_{4-}	1_{3+}	1_{2+}	1_{3-}	2_{4-}	2_{2+}	2_{4-}	2_{2+}	1_{3-}	1_{3-}
2_{2+}	1_{2+}	1_{3-}	1_{4-}	1_{3+}	1_{2+}	1_{3-}	1_{4-}	1_{3-}	2_{2+}	2_{4-}	2_{2+}	2_{4-}	1_{3-}	1_{3-}
2_{1+}	1_{1+}	1_{2-}	1_{3-}	1_{2+}	1_{1+}	1_{2+}	1_{3-}	1_{2-}	2_{1+}	2_{3-}	2_{1+}	2_{3-}	1_{2+}	1_{2+}
2_{3-}	1_{3-}	1_{2+}	1_{1+}	1_{2-}	1_{3-}	1_{2+}	1_{1+}	1_{2-}	2_{3-}	2_{1+}	2_{3-}	2_{1+}	1_{2-}	1_{2-}

When deleting the last two rows and columns we have a union of groups, namely $Z_2 \times L_4 \cup Z_2 \times L_2$. The automorphisms give another group $\mathbb{Z}_2$. □

Theorem 3.4. *If n is not a prime nor 4 then the set* $\mathrm{LEnd}(P_n)$ *is not closed under composition.*

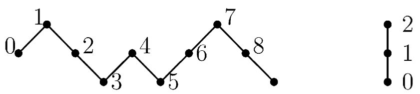
Proof. First, let $p > 2$ be a prime which divides n. Consider

$$f_{p,0^+} \circ f_{p,2^+} = \begin{pmatrix} 0 & 1 & 2 & 3 & \dots & p-1 & p & p+1 & \dots \\ 2 & 3 & 4 & 5 & \dots & p-1 & p-2 & p-1 & \dots \end{pmatrix}.$$

This is not a complete folding, thus $f_{p,0^+} \circ f_{p,2^+}$ is not a locally strong endomorphism.

If now $n = 2^k$, $k \geqslant 3$, consider

$$f_{2,0^+} \circ f_{4,1^+} = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \dots \\ 1 & 2 & 1 & 0 & 1 & 0 & 1 & 2 & 1 & \dots \end{pmatrix},$$



Therefore, again, $f_{2,0^+} \circ f_{4,1^+}$ is not a locally strong endomorphism. □

Corollary 3.5. *The set* $\mathrm{LEnd}(P_n)$ *forms a monoid if and only if n is a prime or 4.*

For directed paths we get:

Theorem 3.6. *The set* $\mathrm{LEnd}(\overline{P_n})$ *of a directed path* $\overline{P_n}$ *forms a monoid if and only if n is a prime or 4 or 8.*

Proof. In the case where the length n of the directed path has a prime divisor > 2 we use the same proof as for undirected paths. Locally strong endomorphisms of length 2 fulfill the conditions of Lemma 3.2 which is given there for locally strong endomorphisms of length 1 for undirected paths. To see this we interpret a two succeeding directed arcs, for example $(0, 1)$, $(1, 2)$ as one undirected arc.

With this argument we can use the second part of the proof of Theorem 3.4 to see that $\mathrm{LEnd}(\overline{P_{2^k}})$ is not closed starting with $\overline{P_{16}}$.

Consequently, for $\mathrm{LEnd}(\overline{P_8})$ we get the same multiplication table as for $\mathrm{LEnd}(P_4)$ as given in Theorem 3.3, we only have to add the eight endomorphisms of $\overline{P_8}$ of length 1, which again are locally strong.

For $\overline{P_4}$ we consider the multiplication table of $\mathrm{LEnd}(\overline{P_4})$ after deleting the two automorphisms which is a union of groups, four one-element groups and two copies of $\mathbb{Z}_2$, namely $L_4 \cup (\mathbb{Z}_2 \times L_2)$, where L_n denotes the left zero semigroup with n elements: □

$\circ$	1_{0^+}	1_{2^-}	1_{4^-}	1_{2^+}	2_{0^+}	2_{2^-}	2_{4^-}	2_{2^+}
1_{0^+}	1_{0^+}	1_{0^+}	1_{0^+}	1_{0^+}	1_{0^+}	1_{0^+}	1_{0^+}	1_{0^+}
1_{2^-}	1_{2^-}	1_{2^-}	1_{2^-}	1_{2^-}	1_{2^-}	1_{2^-}	1_{2^-}	1_{2^-}
1_{4^-}	1_{4^-}	1_{4^-}	1_{4^-}	1_{4^-}	1_{4^-}	1_{4^-}	1_{4^-}	1_{4^-}
1_{2^+}	1_{2^+}	1_{2^+}	1_{2^+}	1_{2^+}	1_{2^+}	1_{2^+}	1_{2^+}	1_{2^+}
2_{0^+}	1_{0^+}	1_{2^-}	1_{0^+}	1_{2^-}	2_{0^+}	2_{2^-}	2_{0^+}	2_{2^-}
2_{2^-}	1_{2^-}	1_{0^+}	1_{2^-}	1_{0^+}	2_{2^-}	2_{0^+}	2_{2^-}	2_{0^+}
2_{4^-}	1_{4^-}	1_{2^+}	1_{4^-}	1_{2^+}	2_{4^-}	2_{2^+}	2_{4^-}	2_{2^+}
2_{2^+}	1_{2^+}	1_{4^-}	1_{2^+}	1_{4^-}	2_{2^+}	2_{4^-}	2_{2^+}	2_{4^-}

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