Finite rank Toeplitz products with harmonic symbols

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Abstract

On the Bergman space of the unit ball of $\mathbb{C}^n$, we study the finite rank problem for Toeplitz products with harmonic symbols. We first solve the problem with two factors in case symbols have local continuous extension property up to the boundary. Also, in case symbols have additional Lipschitz continuity up to (some part of) the boundary, we solve the problem for multiple products with number of factors depending on the dimension $n$. Analogous theorems on the polydisk are also obtained.

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1. Introduction

Let $\Omega$ be the unit ball $B$ or unit polydisk $D^n$ (the Cartesian product of $n$ copies of the unit disk $D$) in the complex $n$-space $\mathbb{C}^n$. Let $L^p = L^p(\Omega)$ denote the usual Lebesgue space with respect to the volume measure $V$ on $\Omega$ normalized to have total mass 1. The Bergman space $A^2 = A^2(\Omega)$ is then the space of all $L^2$-holomorphic functions on $\Omega$. Due to the mean value property of holomorphic functions, the space $A^2$ is a closed subspace of $L^2$, and thus is a Hilbert space. The Bergman projection $P$ is defined to be the Hilbert space orthogonal projection from $L^2$ onto $A^2$. Since every point evaluation is a bounded linear functional on $A^2$, to each $a \in \Omega$ there corresponds a unique function $K_a \in A^2$ which has the following reproducing property:

$$f(a) = \langle f, K_a \rangle, \quad f \in A^2,$$

where the notation $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2$. By the reproducing property (1.1), the Bergman projection $P$ can be represented by

$$P\psi (a) = \frac{1}{V(\Omega)} \int_{\Omega} \psi K_a \, dV, \quad a \in \Omega,$$

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for functions $\psi \in L^2$. For a function $u \in L^\infty$, the Toeplitz operator $T_u$ with symbol $u$ is defined by

$$T_u f = P(uf)$$

for $f \in A^2$. It is clear that $T_u : A^2 \to A^2$ is a bounded linear operator.

**Note.** The domain, either $B$ or $D^n$ (or both), to which the notation introduced above refers will be clear from the context, if not specified.

In this paper we study the finite rank product problem of whether finite rank products of several Toeplitz operators have only the trivial solution. More explicitly, the problem we consider is:

*If $T_{u_1} \cdots T_{u_N}$ has finite rank, does it then follow that one of $u_j$ is identically zero?*

For general bounded symbols this problem is wide open. The case of single Toeplitz operator with rank at most one on $D$ is all that is known so far; see Ahern and Čučković [1] where their proof is attributed to R. Rochberg. More recently, Guo, Sun and Zheng [10] obtained some positive results on $D$ with certain additional assumptions on symbol functions. More explicitly, they [10, Theorem 2] showed that if $T_u$ has finite rank where $u$ is a finite sum of products of a holomorphic function and a co-holomorphic function, then $u = 0$. In the same paper, they [10, Theorem 7] also proved that if the product $T_u T_v$ with harmonic symbols $u, v$ has finite rank, then $u = 0$ or $v = 0$. The latter result was reproved by the authors [5, Corollary 3.11]. However, when one considers sums of Toeplitz products, the situation becomes completely different; the authors [5] have shown on $D$ that a sum of two Toeplitz products can have arbitrarily preassigned finite rank. Proofs of all these results depend on methods that are restricted to $D$ and do not seem to extend to higher dimensional cases.

The zero product problem, a special case of the finite rank product problem, is of independent interest and has been also settled only for a limited class of symbols. The zero product problem was first solved by Ahern and Čučković [1] for two factors with harmonic symbols on the disk. The polydisk version was proved in [7, Corollary 5.4] with pluriharmonic symbols. Their proofs do not seem to extend to the unit ball. The first two authors [4] devised a new approach to solve the zero product problem for multiple factors (the number of factors depending on dimension) with harmonic symbols (having certain smoothness up to the boundary) on the ball. Also, the polydisk version was proved by the authors [6]. In this paper we push the arguments of [4,6] one step further to settle the finite rank product problem with symbols considered in [4,6].

We first state our results on the ball. In what follows, we write $h^\infty(B)$ for the class of all bounded harmonic functions on $B$. The next theorem, which generalizes the corresponding zero product theorem [4, Theorem 1.1], is our first result.

**Theorem 1.1.** Suppose that $u_1, u_2 \in h^\infty(B)$ are continuous on $B \cup W$ for some relatively open set $W \subset \partial B$. If $T_{u_1} T_{u_2}$ has finite rank, then either $u_1 = 0$ or $u_2 = 0$.

For symbols having Lipschitz continuous extensions to the boundary, our method applies to multiple products. We first recall the notion of Lipschitz spaces. Given $0 < \alpha \leq 1$ and a complex function $f$ on a region $X \subset \mathbb{C}^n$, we let

$$\|f\|_{A_\alpha(X)} = \sup_{z \neq w} \frac{|f(z) - f(w)|}{|z - w|^\alpha}$$

where the supremum is taken over all $z, w \in X$ with $z \neq w$. We say $f \in A_\alpha(X)$ if and only if $\|f\|_{A_\alpha(X)} < \infty$. Equipped with the norm $|f(x_0)| + \|f\|_{A_\alpha(X)}$ where $x_0 \in X$ is any fixed point, the space $A_\alpha(X)$ becomes a Banach space. Note that Lipschitz functions on $X$ necessarily extend to Lipschitz functions on $\overline{X}$ of the same order. Given a point $\zeta$, we say $f \in A_\alpha(\zeta)$ if $f \in A_\alpha(U)$ for some neighborhood $U$ of $\zeta$.

Our next result solves the finite rank product problem for several Toeplitz operators with harmonic symbols that have Lipschitz continuous extensions to the boundary. Unfortunately, our proof under the weaker assumption “finite rank product” rather than “zero product” requires loss of one factor from the corresponding zero product theorem [4, Theorem 1.2].
Theorem 1.2. Let $u_1, \ldots, u_{n+2} \in h^\infty(B) \cap \Lambda_\alpha(B)$ for some $\alpha \in (0, 1]$. If $Tu_1 \cdots Tu_{n+2}$ has finite rank, then $u_j = 0$ for some $j$.

For harmonic symbols that have only local Lipschitz continuity up to the boundary, we can also apply the proof of Theorem 1.2, but with loss of another factor in the product.

Theorem 1.3. Let $u_1, \ldots, u_{n+1} \in h^\infty(B) \cap \Lambda_\alpha(\zeta)$ for some $\alpha \in (0, 1]$ and $\zeta \in \partial B$. If $Tu_1 \cdots Tu_{n+1}$ has finite rank, then $u_j = 0$ for some $j$.

We now turn to the polydisk case. Going from the ball to the polydisk, we need to adjust our setting suitable for the polydisk. A function $u \in C^2(D^n)$ is called $n$-harmonic as in [13] if $u$ is harmonic in each variable separately. It turns out that $n$-harmonic symbols on the polydisk are the right substitutes for harmonic ones on the ball and the distinguish boundary is the right substitute for the boundary of the ball. Recall that the distinguish boundary $T^n$ of $D^n$ is the Cartesian product of $n$ copies of the unit circle $T = \partial D$. We use the notation $h^\infty(D^n)$ for the class of all bounded $n$-harmonic functions on $D^n$.

Our results on the polydisk are as follows. The next theorem generalizes the corresponding zero product theorem [6, Theorem 1.1].

Theorem 1.4. Suppose $u_1, u_2 \in h^\infty(D^n)$ are continuous on $D^n \cup W$ for some relatively open set $W \subset T^n$. If $Tu_1 Tu_2$ has finite rank, then either $u_1 = 0$ or $u_2 = 0$.

For Lipschitz symbols, we lose one factor, as in the ball case, from the corresponding zero product theorem [6, Theorem 1.2].

Theorem 1.5. Let $u_1, u_2, u_3 \in h^\infty(D^n) \cap \Lambda_\alpha(U)$ for some $\alpha \in (0, 1]$ and for some open set $U$ containing $T^n$. If $Tu_1 Tu_2 Tu_3$ has finite rank, then $u_j = 0$ for some $j$.

Trying to prove our results on the polydisk, we are led to the problem of whether or not the Bergman projection maps Lipschitz spaces of order less than one into itself. Unlike the ball case (see the comments preceding Proposition 3.1), that problem on the polydisk appears to be still open. The original problem still being open, it turns out that the Bergman projection maps a given Lipschitz space of order less than one into any Lipschitz space of smaller order; see Theorem 4.2.

Remarks.

1. In case $n = 1$, Theorems 1.1, 1.3 and 1.4 are already proved in [10, Theorem 7] (also, [5, Corollary 3.11]) without any boundary continuity condition of symbol functions, as is mentioned earlier.
2. Note that the identity operator is also a Toeplitz operator (with constant symbol 1). Thus, if the finite rank product theorem holds for a certain number of factors, it also holds for any smaller number of factors.
3. In conjunction with Theorem 1.4, we note that, for a single Toeplitz operator $Tu$ with $u \in h^\infty(D^n)$, the compactness of $Tu$ on $A^2(D^n)$ implies $u = 0$. This follows from Theorem 2.1 below and the fact that the Berezin transform (see Section 2) fixes bounded $n$-harmonic functions. However, the analogue for higher dimensional balls seems more subtle than expected and remains open; see the Note after Proposition 2.2 below.
4. We do not know whether either boundary regularity or harmonicity of the symbols can be removed in the hypotheses of our theorems above when $n \geq 2$. Also, the number of factors comes from the methods we employ and may not be critical.

In Section 2, we recall and collect some known results which will be used in our proofs. In Section 3, we prove Theorems 1.1, 1.2 and 1.3. In Section 4, we prove Theorems 1.4 and 1.5.

Constants. In the rest of the paper we use the same letter $C$, often depending on the allowed parameters, to denote various positive constants which may change at each occurrence. For nonnegative quantities $X$ and $Y$, we often write $X \lesssim Y$ or $Y \gtrsim X$ if $X$ is dominated by $Y$ times some inessential positive constant. Also, we write $X \approx Y$ if $X \lesssim Y \lesssim X$. 
2. Preliminaries

In this section, we collect well-known results relevant to our proofs. Recall that, given a bounded linear operator \( S \) on \( A^2 \), its Berezin transform is the function \( \tilde{S} \) on \( \Omega \) defined by

\[
\tilde{S}(a) = \langle Sk_a, k_a \rangle, \quad a \in \Omega,
\]

where \( k_a \) denotes the normalized Bergman kernel, namely,

\[
k_a(z) = \frac{K(z, a)}{\sqrt{K(a, a)}}, \quad z \in \Omega.
\]

It is not hard to see that Berezin transforms are continuous on \( \Omega \). The boundary vanishing property of Berezin transform turns out to provide a compactness criterion for certain type of operators. Consider operators which are finite sums of finite products of Toeplitz operators with bounded symbols. Thus, such an operator \( S \) is of the form

\[
S = \sum_{i=1}^{M} T_{U_{i1}} \cdots T_{U_{iN_i}} \tag{2.1}
\]

where each \( U_{ij} \in L^\infty \). The compactness of operators of this form is characterized as in the next theorem due to Axler and Zheng [3] on the disk and Englis [9] on bounded symmetric domains. In fact, Englis [9] worked on irreducible bounded symmetric domains and one can check that irreducibility hypothesis can be removed. Here, \( \partial \Omega \) denotes the topological boundary of \( \Omega \) (even if \( \Omega = D^2 \)). Also, \( C_0 = C_0(\Omega) \) denotes the class of all functions on \( \Omega \) having continuous extensions on \( \Omega \) and vanishing on \( \partial \Omega \).

**Theorem 2.1.** Let \( S \) be as in (2.1). Then \( S \) is compact on \( A^2 \) if and only if \( \tilde{S} \in C_0 \).

Another aspect of the boundary behavior of Berezin transforms which is useful for our purpose is the fact that the Berezin transform preserves the boundary continuity of symbols as in the next proposition taken from [4, Proposition 2.1] and [6, Proposition 2.1]. Here, \( b \Omega \) denotes the distinguished boundary of \( \Omega \); thus \( b B = \partial B \) and \( b D^n = T^n \).

**Proposition 2.2.** Suppose that \( u_1, \ldots, u_N \in L^\infty \) are continuous on \( \Omega \cup \{ \zeta \} \) for some \( \zeta \in b \Omega \). Let \( S = T_{U_1} \cdots T_{U_N} \). Then \( \tilde{S} \) continuously extends to \( \Omega \cup \{ \zeta \} \) and \( \tilde{S}(\zeta) = (u_1 \cdots u_N)(\zeta) \).

**Note.** The admissible limit analogue of Proposition 2.2 (on the ball), which may be of some independent interest, also holds: If symbol functions have only admissible limits at \( \zeta \in \partial B \) (instead of being continuous at \( \zeta \)), then the corresponding Toeplitz product has an admissible limit at \( \zeta \) whose value is the product of admissible limits of symbol functions. We do not need this fact in this paper and the proof is thus not included. However, we do not know whether or not the nontangential limit analogue is also true. In conjunction with Remark (3) in the Introduction, note that a bounded harmonic function on \( B \) has nontangential limits almost everywhere on \( \partial B \), but admissible limits are not guaranteed in general.

Also, we need the following lemma which is taken from [4, Proposition 3.5]. In the following, the notation \( z \cdot \bar{a} = \sum_{j=1}^{n} z_j \bar{a}_j \) denotes the Hermitian inner product of points \( z = (z_1, \ldots, z_n) \) and \( a = (a_1, \ldots, a_n) \) in \( \mathbb{C}^n \).

**Lemma 2.3.** Given \( s \geq 0 \) and \( c \geq 0 \), define

\[
J_{c,s}(z) = \int_{B} \frac{|\log(1 - |w|^2)|^s}{|1 - z \cdot \bar{w}|^{n+1+c}} dV(w), \quad z \in B.
\]

Then the following estimates hold:

\[
J_{c,s}(z) \approx \begin{cases} 
\frac{1}{(1-|z|^2)^c} \left( \frac{\log \frac{1}{1-|z|^2}}{\log \frac{1}{1-|z|^2}} \right)^s & \text{if } c > 0, \\
\left( \frac{\log \frac{1}{1-|z|^2}}{\log \frac{1}{1-|z|^2}} \right)^{s+1} & \text{if } c = 0
\end{cases}
\]
as \( |z| \to 1 \).
Finally, we need the following lemma.

**Lemma 2.4.** Let \( \{g_j\}_{j=1}^N \) be a linearly independent collection of complex functions on a set \( X \) containing at least \( N \) distinct elements. Then there exist some \( x_1, \ldots, x_N \in X \) such that the matrix

\[
\begin{pmatrix}
g_1(x_1) & \cdots & g_1(x_N) \\
\vdots & & \vdots \\
g_N(x_1) & \cdots & g_N(x_N)
\end{pmatrix}
\]  

(2.2)
is invertible.

**Proof.** We prove by induction on \( N \). The case \( N = 1 \) is clear. Assume \( N \geq 2 \) and suppose our assertion holds for \( N - 1 \). In order to complete the induction step, we prove the contrapositive, so assume that our assertion does not hold for \( N \). We will derive a contradiction.

By induction hypothesis we can find some \( x_1, \ldots, x_{N-1} \in X \) such that the matrix

\[
M_1 := \begin{pmatrix}
g_1(x_1) & \cdots & g_1(x_{N-1}) \\
\vdots & & \vdots \\
g_{N-2}(x_1) & \cdots & g_{N-2}(x_{N-1}) \\
g_{N-1}(x_1) & \cdots & g_{N-1}(x_{N-1})
\end{pmatrix}
\]

is invertible. Note that the first \( N - 1 \) columns of the matrix (2.2) are linearly independent, because \( M_1 \) is invertible. On the other hand, assuming our assertion does not hold for \( N \), the matrix (2.2) with \( x_N = x \) is singular for arbitrary \( x \in X \).

It follows that the last column of the matrix (2.2) is a linear combination of the others. That is, given \( x \in X \), there are complex numbers \( c_1(x), \ldots, c_{N-1}(x) \) (also depending on \( x_1, \ldots, x_{N-1} \)) such that

\[
\begin{pmatrix}
g_1(x) \\
\vdots \\
g_N(x)
\end{pmatrix} = M_2 \begin{pmatrix}
c_1(x) \\
\vdots \\
c_{N-1}(x)
\end{pmatrix},
\]

(2.3)

Thus, setting

\[
M_2 := \begin{pmatrix}
g_1(x_1) & \cdots & g_1(x_{N-1}) \\
\vdots & & \vdots \\
g_{N-2}(x_1) & \cdots & g_{N-2}(x_{N-1}) \\
g_{N-1}(x_1) & \cdots & g_{N-1}(x_{N-1})
\end{pmatrix},
\]

we see from (2.3) that

\[
\begin{pmatrix}
g_1(x) \\
\vdots \\
g_{N-2}(x) \\
g_{N}(x)
\end{pmatrix} = M_2 \begin{pmatrix}
c_1(x) \\
\vdots \\
c_{N-1}(x)
\end{pmatrix} = M_2 M_1^{-1} \begin{pmatrix}
g_1(x) \\
\vdots \\
g_{N-2}(x) \\
g_{N}(x)
\end{pmatrix}.
\]

Since \( x \in X \) is arbitrary and components of the matrices \( M_1 \) and \( M_2 \) are independent of \( x \), we deduce from the above that \( g_N \) is a linear combination of \( g_1, \ldots, g_{N-1} \), which contradicts to the linearly independence. This completes the induction and the proof of the lemma.

---

3. The ball case

In this section we prove our main results on the ball. Before doing so, we need some preliminary results. Recall that the Bergman kernel \( K_a \) on the ball is given by

\[
K_a(z) = \frac{1}{(1 - z \cdot \bar{a})^{n+1}}, \quad z \in B.
\]
and thus the Bergman projection $P$ from $L^2(B)$ onto $A^2(B)$ is represented by

$$P \psi(a) = \int_B \frac{\psi(w)}{(1 - a \cdot \overline{w})^{n+1}} dV(w), \quad a \in B,$$

for functions $\psi \in L^2(B)$.

It is well known that the Bergman projection $P$ (for the ball) maps a given Lipschitz space of non-integral order into itself. This is a special case of a general theorem due to Ahern and Schneider [2]. Also, see [11, Theorem 7.7.10 and Remark 7.7.11]. Working on the ball, one may also easily modify the proof of [12, Theorem 6.4.9] for Lipschitz spaces of order less than one. Here, we need a local version of this fact as in the next proposition. Given $\zeta \in \partial B$, we let $H(\zeta)$ denote the class of all functions holomorphic on some open set containing $B \cup \{\zeta\}$.

**Proposition 3.1.** Let $0 < \alpha < 1$ and $\zeta \in \partial B$. Then

$$P \left[ L^2(B) \cap A_\alpha(\zeta) \right] \subset A_\alpha(B) + H(\zeta).$$

In particular,

$$T_u \left[ A^2(B) \cap A_\alpha(\zeta) \right] \subset A_\alpha(\zeta)$$

for $u \in L^\infty(B) \cap A_\alpha(\zeta)$.

**Proof.** Assume $f \in L^2(B) \cap A_\alpha(U)$ for some neighborhood $U$ of $\zeta$. Choose a neighborhood $U_1$ of $\zeta$ such that $\overline{U_1} \subset U$. Now, pick a smooth cut-off function $\psi$ on $C^n$ with $0 \leq \psi \leq 1$ such that $\psi = 1$ on $\overline{U_1}$ and $\psi = 0$ on $C^n \setminus U$. We certainly have $f \psi \in A_\alpha(\overline{B} \setminus U)$. Also, we have $f \psi \in A_\alpha(U \cap \overline{B})$, because $\psi$ is smooth. Let $z \in \overline{B} \cap U$ and $w \in \overline{B} \setminus U$. Note that $|f|$ continuously extends to $\overline{U} \cap \overline{B}$ and thus bounded on $U \cap \overline{B}$, say by $C_1 > 0$. Also, we have by smoothness of $\psi$

$$|\psi(z)| = |\psi(z) - \psi(w)| \leq C_2|z - w|$$

for some constant $C_2 > 0$ independent of $z$ and $w$. It follows that

$$|f(z)\psi(z) - f(w)\psi(w)| = |f(z)\psi(z)| \leq C_1C_2|z - w|.$$ 

This yields $f \psi \in A_\alpha(B)$ and thus $P(f \psi) \in A_\alpha(B)$ by the Ahern–Schneider theorem mentioned above. Meanwhile, since

$$P(f - f \psi)(z) = \int_{\overline{B} \setminus U_1} \frac{f(w)(1 - \psi(w))}{(1 - z \cdot \overline{w})^{n+1}} dV(w),$$

we see that $P(f - f \psi)$ extends holomorphically across $U_1 \cap \partial B$. This completes the proof of the first part of the proposition. Given $u \in L^\infty(B) \cap A_\alpha(\zeta)$, note that

$$T_u \left[ A^2(B) \cap A_\alpha(\zeta) \right] \subset P \left[ L^2(B) \cap A_\alpha(\zeta) \right].$$

Also, note $A_\alpha(B) + H(\zeta) \subset A_\alpha(\zeta)$, because functions in $A_\alpha(B)$ are easily seen to have extensions in $A_\alpha(C^n)$. Therefore the second part follows from the first part. □

We introduce some notation. In conjunction with Lemma 2.3 we let $Log^s(B)$, $s \geq 0$, denote the class of all measurable functions $f$ on $B$ such that

$$\|f\|_{Log^s(B)} := \text{ess sup}_{z \in B} |f(z)| \left(1 + \log \frac{1}{1 - |z|^2}\right)^{-s} < \infty.$$ 

Clearly, $Log^s(B) \subset L^p(B)$ for any $s \geq 0$ and $0 < p < \infty$. Given nontrivial functions $f, g \in A^2$, we let $f \otimes g$ denote the rank one operator on $A^2$ defined by

$$(f \otimes g)h = (h, g)f$$

for $h \in A^2$. The following lemma will play a crucial role in our proofs.
Lemma 3.2. Let \( u_1, \ldots, u_k \in L^\infty(B) \). Let \( \{ f_j \}_{j=1}^N \) and \( \{ g_j \}_{j=1}^N \) be linearly independent collections of functions in \( A^2(B) \). Assume

\[
T_{u_1} T_{u_2} \cdots T_{u_k} = \sum_{j=1}^N f_j \otimes g_j
\]
on \( A^2(B) \). Then \( f_j, g_j \in \text{Log}^k(B) \) for all \( j \). If, in addition, \( u_1, \ldots, u_k \in \Lambda_\alpha(\zeta) \) for some \( \alpha \in (0, 1) \) and \( \zeta \in \partial B \), then \( f_j, g_j \in \Lambda_\alpha(\zeta) \) for all \( j \).

Proof. Put \( S = T_{u_1} T_{u_2} \cdots T_{u_k} \) for brevity. We first show \( f_j \in \text{Log}^k(B) \) for all \( j \). Clearly, we have \( P \text{Log}^s(B) \subset \text{Log}^{s+1}(B) \) for general \( s \geq 0 \) by Lemma 2.3. Hence we have \( \text{SL}^\infty(B) = \text{SLog}^0(B) \subset \text{Log}^k(B) \). In particular, we see that \( SK_a \) belongs to \( \text{Log}^k(B) \) for each \( a \in B \). Note \( SK_a = \sum_{j=1}^N g_j(a) f_j \) by (1.1). Thus we have

\[
\begin{pmatrix}
SK_{a^1} \\
\vdots \\
SK_{a^N}
\end{pmatrix}
= \begin{pmatrix}
g_1(a^1) & \cdots & g_N(a^1) \\
\vdots & \ddots & \vdots \\
g_1(a^N) & \cdots & g_N(a^N)
\end{pmatrix}
\begin{pmatrix}
f_1 \\
\vdots \\
f_N
\end{pmatrix}
\]

for all points \( a^1, \ldots, a^N \) in \( B \). Now, since functions \( g_1, \ldots, g_N \) are linearly independent, we see from Lemma 2.4 that the \( N \times N \) matrix in the above displayed equation is invertible for some points \( a^1, \ldots, a^N \) in \( B \). Thus, each \( f_j \) is a linear combination of functions \( SK_{a^j} \) and thus belongs to \( \text{Log}^k(B) \), because functions \( SK_{a^j} \) all belong to \( \text{Log}^k(B) \). If, in addition, \( u_1, \ldots, u_k \in \Lambda_\alpha(\zeta) \) for some \( \alpha \in (0, 1) \) and \( \zeta \in \partial B \), then \( SK_{a^j} \) all belong to \( \Lambda_\alpha(\zeta) \) by Proposition 3.1 and thus \( f_j \in \Lambda_\alpha(\zeta) \) for all \( j \).

Since \( T^*_u = T_\sigma \) and \( (f \otimes g)^* = g \otimes f \) in general where the superscript * denotes the Hilbert space adjoint operator, we have

\[
S^* = T_{\bar{a}_k} \cdots T_{\bar{a}_1} = \sum_{j=1}^N g_j \otimes f_j.
\]

Now, what we have proved above implies the assertions on functions \( g_j \). The proof is complete. \( \square \)

The source of our test functions are sufficiently high powers of functions \( \lambda_t \) defined by

\[
\lambda_t(z) = \frac{1}{1 - tz_1}
\]

for \( 0 < t < 1 \) and \( z \in B \). Also, we let \( \mathbf{e} = (1, 0, \ldots, 0) \in \partial B \).

In conjunction with Proposition 3.1, we have the following estimate.

Lemma 3.3. Let \( k \geq n + 2 \) be an integer and assume \( g \in L^2(B) \cap \Lambda_\alpha(\mathbf{e}) \) for some \( \alpha \in (0, 1) \). Then there is a constant \( C = C(k, g) \) such that

\[
|\langle \lambda^k_t, g \rangle| \leq \frac{C}{(1 - t)^{k - n - 1 - \alpha/2}}
\]

for \( 0 < t < 1 \).

Proof. Assume \( g \in L^2(B) \cap \Lambda_\alpha(U) \) where \( U \) is some neighborhood of \( \mathbf{e} \). Let \( 0 < t < 1 \). Since \( \langle \lambda^k_t, \mathbf{e} \rangle = \lambda^k_t(0)g(\mathbf{e}) = g(\mathbf{e}) \), we have

\[
|\langle \lambda^k_t, g \rangle| \leq |\langle \lambda^k_t, g - \mathbf{e} \rangle| + |g(\mathbf{e})|.
\]
Meanwhile, we have

$$\left| \langle \lambda_k t, g - g(e) \rangle \right| \leq \int_{B \cap U} \frac{|g(w) - g(e)|}{|1 - tw_1|^{k-1/2}} dV(w) + \int_{B \setminus U} \frac{|g(w) - g(e)|}{|1 - tw_1|^{k-1/2}} dV(w).$$

Since $|w - e| \leq 2|1 - w_1|^{1/2}$, we have by Lemma 2.3

$$\int_{B \setminus U} \|g\|_{A_\omega(U)} \frac{dV(w)}{|1 - tw_1|^{k-\alpha/2}} \approx \|g\|_{A_\omega(U)} (1 - t)^{k-n-1-\alpha/2}$$

for all $t$. Also, since $|1 - tw_1|$ is bounded away from 0 on $B \setminus U$, we have

$$\int_{B \setminus U} \|g - g(e)\|_2$$

for all $t$. Combining these estimates, we conclude the lemma. $\Box$

Given a $C^1$-function $u$ on $B$, we let $\mathcal{R}u$ denote the radial derivative of $u$ defined by

$$\mathcal{R}u(z) = \sum_{j=1}^n \left[ z_j \frac{\partial u}{\partial z_j}(z) + \bar{z}_j \frac{\partial u}{\partial \bar{z}_j}(z) \right], \quad z \in B.$$

The following uniqueness result for harmonic functions is taken from [4, Proposition 4.1].

**Proposition 3.4.** Suppose that $u$ is a function harmonic on $B$ and continuous on $B \cup W$ for some relatively open set $W \subset \partial B$. If both $u$ and $\mathcal{R}u$ vanish on $W$, then $u = 0$ on $B$.

We are now ready to prove our theorems. Our proof of Theorem 1.1 depends on Theorem 1.3 whose proof in turn depends on that of Theorem 1.2. Thus we first prove Theorems 1.2 and 1.3.

**Proof of Theorem 1.2.** Put $S = T_{u_1} \cdots T_{u_{n+2}}$ and assume that $S$ has finite rank. Since $S$ has finite rank (and thus is compact), we have $\tilde{S} = 0$ on $\partial B$ by Theorem 2.1. Since each $u_j$ is continuous on $\bar{B}$, we have

$$0 = \tilde{S} = u_1 \cdots u_{n+2} = 0 \quad \text{on } \partial B$$

by Proposition 2.2. Since the product function $u_1 \cdots u_{n+2}$ is continuous and vanishes everywhere on the boundary, there exists a relatively open set $W \subset \partial B$ such that

- either $u_j(\zeta) \neq 0$, $\zeta \in W$, or
- $u_j = 0$ on $W$

holds for each $j$. If $u_1 = 0$, there is nothing to prove. So, we may assume that $u_1$ vanishes nowhere on $W$.

Suppose that, for some $j$, there is a relatively open set $W_j \subset W$ such that

$$u_j = \mathcal{R}u_j = 0 \quad \text{on } W_j. \quad (3.2)$$

Then, since $u_j \in C(\bar{B})$ is harmonic on $B$ by assumption, Proposition 3.4 and (3.2) will lead us to conclude $u_j = 0$ on $B$, which completes the proof.

Now, we assume that (3.2) does not hold for any $j$ and derive a contradiction. Since (3.2) does not hold for any $j$, we may shrink (if necessary) the set $W$ to get a smaller relatively open set, still denoted by $W$, such that

- either (i) $u_j(\zeta) \neq 0$, $\zeta \in W$, or
- $u_j(\zeta) = 0$, $\mathcal{R}u_j(\zeta) \neq 0$, $\zeta \in W$, \quad (3.3)

holds for each $j = 1, 2, \ldots, n + 2$. We may further assume that $e \in W$; this causes no loss of generality by rotation-invariance of radial differentiation.
We introduce more notation. In the rest of the proof we let $0 \leq t < 1$ and $z \in B$ represent arbitrary points. Let $\sigma$ be the function on $B$ defined by

$$\sigma(z) = 2(1 - \Re z_1) - \sum_{j=2}^{n} |z_j|^2. \quad (3.4)$$

Recall that $\mathcal{R}u_j(e) \neq 0$ by (3.3), in case $u_j(e) = 0$. Let $d_j = 1$ if $u_j(e) = 0$ and $d_j = 0$ otherwise. Note $d_1 = 0$. Now, define the major part of $u_j$ by

$$m_j := \begin{cases} u_j(e) & \text{if } d_j = 0, \\ -\mathcal{R}u_j(e)\sigma & \text{if } d_j = 1 \end{cases}$$

for each $j$. What we have done so far is almost the same as the corresponding part of the proof of [4, Theorem 1.2] and repeated here for reader’s convenience.

Put $M = T_{m_1} \cdots T_{m_{n+2}}$ and $R = S - M$. Note $M = S - R$. We will apply both $M$ and $S - R$ independently to the same test functions as in the proof of [4, Theorem 1.2], obtain pointwise estimates along the radius ending at $e$, and reach a contradiction.

Choose a sufficiently large positive integer $k$ ($k > 2n + 2$ is enough). Let $0 < t < 1$. We now estimate $M \lambda_k^t(e)$, $S \lambda_k^t(e)$ and $R \lambda_k^t(e)$ independently; recall that $\lambda_t$ is the function defined in (3.1). For the estimate of $M \lambda_k^t(e)$, following the proof of [4, Theorem 1.2], we have

$$M \lambda_k^t(e) \gtrsim \frac{1}{(1-t)^{k-d}} \quad (3.5)$$

as $t \to 1$ where $d = d_1 + \cdots + d_{n+2}$. Also, easily modifying the corresponding part of the proof of [4, Theorem 1.2], we see that there exists some $\beta \in (0, 1/2)$ such that

$$|R \lambda_t^k(e)| \lesssim \frac{|\log(1-t)|^{n+2}}{(1-t)^{k-d-\beta}} \quad (3.6)$$

as $t \to 1$.

We now estimate $S \lambda_t^k(e)$. Since $S$ has finite rank, say $N$, there exist linearly independent collections $\{f_j\}_{j=1}^{N}$ and $\{g_j\}_{j=1}^{N}$ of functions in $A^2(B)$ such that

$$S = \sum_{j=1}^{N} f_j \otimes g_j. \quad (3.7)$$

Thus we have

$$S \lambda_t^k(e) = \sum_{j=1}^{N} \langle \lambda_t^k, g_j \rangle f_j(e). \quad (3.8)$$

Note that functions $f_j, g_j$ all belong to $A_\alpha(e)$ by Lemma 3.2; we may assume $\alpha < 1$ without loss of generality. Thus we obtain by Lemma 3.3

$$|S \lambda_t^k(e)| \lesssim \left( \sup_{t,j} |f_j(e)| \right) \frac{1}{(1-t)^{k-n-1-\alpha/2}} \quad (3.9)$$

for all $t$.

We now have by (3.5), (3.6) and (3.8)

$$1 = \frac{|S \lambda_t^k(e) - R \lambda_t^k(e)|}{|M \lambda_t^k(e)|} \lesssim (1 + |\log(1-t)|^k) \left[ (1-t)^{n+1+\alpha/2-d} + (1-t)^{\beta} \right]$$

as $t \to 1$ and this estimate is independent of $t$. Note $d \leq n + 1$, because $d_1 = 0$; it is this step where we lose one factor from [4, Theorem 1.2]. Thus, taking the limit $t \to 1$, we reach a contradiction. The proof is complete. \qed
Next, we prove Theorem 1.3.

**Proof of Theorem 1.3.** In the proof of Theorem 1.2, the global Lipschitz hypothesis is used to choose \( W \) on which the first symbol vanishes nowhere. Thus one may repeat the proof of Theorem 1.2 with the Toeplitz products \( T_{u_0} T_{u_1} \cdots T_{u_{n+1}} \), where \( u_0 = 1 \). \( \square \)

Finally, we prove Theorem 1.1.

**Proof of Theorem 1.1.** As is mentioned in the Introduction, the theorem is a special case of [10, Theorem 7] for \( n = 1 \). Thus the proof here for \( n = 1 \) can be viewed as another proof under the additional hypothesis of local boundary continuity. We will also repeat some part of the proof of [4, Theorem 1.1] for reader’s convenience.

Put \( S = T_{u_1} T_{u_2} \) and assume that \( S \) has finite rank. As in the proof of Theorem 1.2, we have

\[
 u_1 u_2 = 0 \quad \text{on} \quad W \subset \partial B.
\]

There are two cases to consider: (i) Both \( u_1 \) and \( u_2 \) vanish everywhere on \( W \) and (ii) either \( u_1 \) or \( u_2 \) does not vanish on some relatively open subset of \( W \). In case of (i) we have \( u_1, u_2 \in A_1(\zeta) \) for some \( \zeta \in W \) by [4, Lemma 4.2]. Thus, the case (i) is contained in Theorem 1.3. So, we may assume (ii). Note that we may further assume that \( u_1 \) does not vanish on some relatively open set, still denoted by \( W \), because otherwise we can use the adjoint operator \( S^* = T_{u_2} T_{u_1} \). We now have \( u_2 = 0 \) on \( W \). Assume \( \epsilon \in W \) without loss of generality. By Proposition 3.4 we may further assume that \( R_{u_2} \) vanishes nowhere on \( W \). This will lead us to a contradiction.

Let \( c_1 = u_1(\epsilon) \neq 0 \) and \( c_2 = -\frac{R_{u_2}(\epsilon)}{2} \neq 0 \). Let \( e_1 = u_1 - c_1 \) and \( e_2 = u_2 - c_2 \sigma \) where \( \sigma \) is the function introduced in (3.4). Then we have

\[
 S = T_{e_1 + e_2} T_{e_2} = c_1 c_2 T_\sigma + c_2 T_{e_1} T_\sigma + T_{u_1} T_{e_2}
\]

and thus

\[
 c_1 c_2 T_\sigma = S - c_2 T_{e_1} T_\sigma - T_{u_1} T_{e_2}.
\]

Now we apply each side of the above to the same test functions \( \lambda^k_\sigma \) with a sufficiently large positive integer \( k \) (\( k > n + 2 \) will be enough) and derive a contradiction.

Given \( \epsilon > 0 \) sufficiently small, let

\[
 \omega(\epsilon) = \sup \{ |e_1(\xi)| : \xi \in B \cup W, |\epsilon - \xi| < \epsilon \}
\]

be the modulus of continuity of \( u_1 \) at \( \epsilon \). By continuity of \( u_1 \), we have \( \omega(\epsilon) \to 0 \) as \( \epsilon \to 0 \). Then we have the following estimates:

\[
 |T_\sigma \lambda^k_\tau (\epsilon)| \gtrsim \frac{1}{(1-t)^{k-1}}, \quad (3.11)
\]

\[
 |T_{u_1} T_{e_2} \lambda^k_\tau (\epsilon)| \lesssim \frac{\log(1-t)}{(1-t)^{k-3/2}} \quad (3.12)
\]

and

\[
 T_{e_1} T_\sigma \lambda^k_\tau (\epsilon) \lesssim \left[ \frac{\omega(\epsilon)}{(1-t)^{k-1}} + \frac{1}{\epsilon^2/4 - (1-t)k+2} + \frac{1}{(1-t)^{k-2}} \right] \quad (3.13)
\]

for all \( t \) sufficiently close to 1. The estimates (3.11)–(3.13) are independent of \( t \) and \( \epsilon \) and proved in the proof of [4, Theorem 1.1].

We now estimate \( S \lambda^k_\tau (\epsilon) \). Assume that \( S \) has finite rank \( N \) and represent \( S \) as in (3.7). Note that functions \( f_j, g_j \) all belong to \( \Log^2(B) \) by Lemma 3.2. Fix \( 1 < p < 2 \) and let \( q \) be the conjugate exponent of \( p \). Applying Hölder’s inequality, we obtain by Lemma 2.3

\[
 |S \lambda^k_\tau (\epsilon)| \lesssim \sum_{j=1}^{N} \| \lambda^k_\tau \|_{L^p(B)} \| g_j \|_{L^q(B)} |f_j(\epsilon)| \lesssim \frac{|\log(1-t)|^2}{(1-t)^{k-(n+1)/p}} \quad (3.14)
\]

as \( t \to 1 \).
Now, we have by (3.11)–(3.14)
\begin{equation}
1 = \frac{|S - c_2 T_{e_1} T_{e_2} - T_{u_1} T_{e_2}| \lambda_k(t \epsilon)}{|c_1 c_2 T_{e_2} \lambda_k(t \epsilon)|} \lesssim (1 + \omega(\epsilon) + \frac{(1 - t)^{k-1}}{[\epsilon^2/4 - (1 - t)]^{k+n}})
\end{equation}
as \ t \to 1 \text{ and this estimate is independent of } t \text{ and } \epsilon. \text{ So, first taking the limit } \ t \to 1 \text{ with } \epsilon > 0 \text{ fixed and then taking the limit } \epsilon \to 0, \text{ we have}
\begin{equation}
1 \lesssim \omega(\epsilon) \to 0,
\end{equation}
which is a contradiction. The proof is complete. 

4. The polydisk case

In this section we prove our main results on the polydisk. Recall that the Bergman kernel \( K_a \) on the polydisk is given by
\begin{equation}
K_a(z) = \prod_{j=1}^{n} \frac{1}{(1 - a_{j} z_j)^{2}}, \quad z, a \in D^n,
\end{equation}
and thus the Bergman projection \( P \) from \( L^2(D^n) \) onto \( A^2(D^n) \) is represented by
\begin{equation}
P \psi(a) = \int_{D^n} \frac{\psi(w)}{\prod_{j=1}^{n} (1 - a_{j} w_j)^2} dV(w), \quad a \in D^n,
\end{equation}
for functions \( \psi \in L^2(D^n) \).

Unlike the ball case it is not known whether the Bergman projection maps a given Lipschitz space (of non-integral order) into itself on the polydisk. Thus the polydisk version of Proposition 3.1 is not available for now. However, a weaker version, Theorem 4.3 below, is enough for our purpose.

The following characterization of \( \Lambda_\alpha(D) \), \( 0 < \alpha < 1 \), is well known and due to Hardy and Littlewood (see, for example, [8, Theorem 4.1]):
\begin{equation}
\|g\|_{\Lambda_\alpha(D)} \approx \sup_{a \in D} |g'(a)| (1 - |a|^2)^{1-\alpha}
\end{equation}
for functions \( g \) holomorphic on \( D \). One can easily extend this characterization to the polydisk as in the next proposition. This must be also well known to experts and we included a proof for completeness.

**Proposition 4.1.** Let \( \alpha \in (0, 1) \). Then the equivalence
\begin{equation}
\|f\|_{\Lambda_\alpha(D^n)} \approx \sup_{z \in D^n} \sum_{j=1}^{n} \left| \frac{\partial f}{\partial z_j}(z) \right| (1 - |z_j|^2)^{1-\alpha}
\end{equation}
holds for functions \( f \) holomorphic on \( D^n \).

**Proof.** Let \( M \) denote the quantity in the right side of (4.2). Let \( f \in \Lambda_\alpha(D^n) \). Then \( f \) is Lipschitz continuous on \( D \) of order \( \alpha \) in each variable separately and Lipschitz constants associated with the remaining variables are uniformly bounded by \( \|f\|_{\Lambda_\alpha(D^n)} \). Thus we have the estimate \( M \lesssim \|f\|_{\Lambda_\alpha(D^n)} \) by (4.1). Conversely, assume \( M < \infty \) and let \( z, w \in D^n \). Let \( \tilde{w}_j = (z_1, \ldots, z_{j-1}, w_{j+1}, \ldots, w_n) \) for \( j = 1, \ldots, n - 1 \). Also, let \( \tilde{w}_0 = w \) and \( \tilde{w}_n = z \). Then we have by (4.1)
\begin{equation}
|f(z) - f(w)| \lesssim \sum_{j=0}^{n-1} |f(\tilde{w}_j) - f(\tilde{w}_{j+1})| \lesssim M \sum_{j=1}^{n} |z_j - w_j|^\alpha \approx M |z - w|^\alpha
\end{equation}
so that \( \|f\|_{\Lambda_\alpha(D^n)} \lesssim M \). The proof is complete. 

We do not know whether \( \Lambda_\beta(D^n) \) can be extended to \( \Lambda_\alpha(D^n) \) in the conclusion of the next theorem.
Theorem 4.2. If $0 < \beta < \alpha < 1$, then $P \Lambda_\alpha(D^n) \subset \Lambda_\beta(D^n)$.

Proof. Let $0 < \beta < \alpha < 1$. We proceed by induction on $n$. For $n = 1$, our assertion is a consequence of the Ahern–Schneider theorem (which is mentioned before Proposition 3.1). Let $n \geq 2$ and assume that our assertion holds for $n - 1$.

Let $f \in \Lambda_\alpha(D^n)$ and put $F = Pf$. By Proposition 4.1 it is sufficient to show

$$\sup_{z \in D^n} \left| \frac{\partial F}{\partial z_j}(z) \right| (1 - |z_j|^2)^{1 - \beta} < \infty$$

for $j = 1, \ldots, n$. In order to prove this, we only need to consider the case $j = 1$ by symmetry. Given $a \in D$ and $\xi \in D^{n-1}$, let $G$ be the function on $D^n$ defined by

$$G(a, \xi) = P_{n-1} f_a(\xi)$$

where $f_a = f(a, \cdot)$ and $P_{n-1}$ denotes the Bergman projection on $D^{n-1}$. Given $\xi \in D^{n-1}$, let $F_\xi = F(\cdot, \xi)$ and define $G_\xi$ similarly. Then we have

$$F_\xi = P_1 G_\xi$$

where $P_1$ denotes the Bergman projection on $D$. Thus we have

$$\sup_{a \in D} |F_\xi'(a)| (1 - |a|^2)^{1 - \beta} \approx \|F_\xi\|_{\Lambda_\beta(D)} \lesssim \|G_\xi\|_{\Lambda_\beta(D)};$$

the first estimate holds by (4.1) and the second by the Ahern–Schneider theorem.

In order to complete the induction step, we prove $\sup_{\xi \in D^{n-1}} \|G_\xi\|_{\Lambda_\beta(D)} < \infty$, or equivalently,

$$|G(a, \xi) - G(b, \xi)| \lesssim |a - b|^\beta$$

for all $a, b \in D$ and $\xi \in D^{n-1}$. Fix $a, b \in D$ and define $h_{a,b} = f_a - f_b$ so that

$$G(a, \cdot) - G(b, \cdot) = P_{n-1} h_{a,b}.$$ 

Since $f \in \Lambda_\alpha(D^n)$, we have

$$\|h_{a,b}\|_{L^\infty(D^{n-1})} \leq \|f\|_{\Lambda_\alpha(D^n)} |a - b|^\alpha.$$ 

Also, one may easily verify via the triangle inequality

$$\|h_{a,b}\|_{\Lambda_\alpha(D^{n-1})} \leq 2 \|f\|_{\Lambda_\alpha(D^n)}.$$ 

It follows that

$$|h_{a,b}(\xi) - h_{a,b}(\eta)| \leq 2 \|f\|_{\Lambda_\alpha(D^n)} \min\{|a - b|\alpha, |\xi - \eta|\alpha\} \leq 2 \|f\|_{\Lambda_\alpha(D^n)} |a - b|^\beta |\xi - \eta|^{\alpha - \beta}$$

for all $\xi, \eta \in D^{n-1}$, or equivalently,

$$\|h_{a,b}\|_{\Lambda_{\alpha-\beta}(D^{n-1})} \leq 2 \|f\|_{\Lambda_\alpha(D^n)} |a - b|^\beta.$$ 

Thus, by induction hypothesis and the closed graph theorem, we have

$$\|P_{n-1} h_{a,b}\|_{\Lambda_{\alpha-\beta}(D^{n-1})} \lesssim \|h_{a,b}\|_{\Lambda_{\alpha-\beta}(D^{n-1})} \lesssim \|f\|_{\Lambda_\alpha(D^n)} |a - b|^\beta$$

and therefore

$$|P_{n-1} h_{a,b}(\xi)| \leq |P_{n-1} h_{a,b}(0)| + \|P_{n-1} h_{a,b}\|_{\Lambda_{\alpha-\beta}(D^{n-1})} |\xi|^{\alpha - \beta} \lesssim \|h_{a,b}\|_{L^\infty(D^{n-1})} + \|P_{n-1} h_{a,b}\|_{\Lambda_{\alpha-\beta}(D^{n-1})} \lesssim \|f\|_{\Lambda_\alpha(D^n)} |a - b|^\beta.$$
for all $\xi \in D^{n-1}$; it is the first two inequalities above where the hypothesis $\beta < \alpha$ is used. So, we have (4.4). This completes the induction and the proof of the theorem. □

We also have the following local version. Given $\zeta \in T^n$, we let $H(\zeta)$ denote the class of all functions holomorphic on some open set containing $D^n \cup \{\zeta\}$.

**Proposition 4.3.** Let $0 < \beta < \alpha < 1$ and $\zeta \in T^n$. Then

$$P\left[ L^2(D^n) \cap \Lambda_{\alpha}(\zeta) \right] \subset \Lambda_{\beta}(D^n) + H(\zeta).$$

In particular,

$$T_u\left[ A^2(D^n) \cap \Lambda_{\alpha}(\zeta) \right] \subset \Lambda_{\beta}(\zeta)$$

for $u \in L^\infty(D^n) \cap \Lambda_{\alpha}(\zeta)$.

**Proof.** Using Theorem 4.2 instead of the Ahern–Schneider theorem, one may easily modify the proof of Proposition 3.1 to verify the proposition. □

Given $s \geq 0$, we let $\log^s(D^n)$ denote the class of all measurable functions $f$ on $D^n$ such that

$$\|f\|_{\log^s(D^n)} := \text{ess sup}_{z \in D^n} |f(z)| \prod_{j=1}^{n} \left(1 + \log \frac{1}{1 - |z_j|^2}\right)^{-s} < \infty.$$  

Clearly, $\log^s(D^n) \subset L^p(D^n)$ for any $s \geq 0$ and $0 < p < \infty$. By an application of Lemma 2.3, we see $P\log^s(D^n) \subset \log^{s+1}(D^n)$ for general $s \geq 0$. Hence, for $S = T_{u_1}T_{u_2} \cdots T_{u_k}$ where $u_1, \ldots, u_k \in L^\infty(D^n)$, we have $S L^\infty(D^n) = S \log^0(D^n) \subset \log^k(D^n)$. So, a trivial modification of the proof of Lemma 3.2 yields the following lemma.

**Lemma 4.4.** Let $u_1, \ldots, u_k \in L^\infty(D^n)$. Let $\{f_j\}_{j=1}^N$ and $\{g_j\}_{j=1}^N$ be linearly independent collections of functions in $A^2(D^n)$. Assume

$$T_{u_1}T_{u_2} \cdots T_{u_k} = \sum_{j=1}^N f_j \otimes g_j$$

on $A^2(D^n)$. Then $f_j, g_j \in \log^k(D^n)$ for all $j$. If, in addition, $u_1, \ldots, u_k \in \Lambda_{\alpha}(\zeta)$ for some $\alpha \in (0, 1)$ and $\zeta \in T^n$, then $f_j, g_j \in \Lambda_{\beta}(\zeta)$ for all $j$ and $\beta \in (0, \alpha)$.

The source of our test functions are sufficiently high powers of functions $\lambda_t$ defined by

$$\lambda_t(z) = \frac{1}{1 - t_1 z_1} \cdots \frac{1}{1 - t_n z_n}, \quad z = (z_1, \ldots, z_n) \in D^n,$$

with suitably chosen $t = (t_1, \ldots, t_n)$ where $0 < t_j < 1$. Also, we let

$$p = (1, 1, \ldots, 1) \in T^n.$$

In conjunction with Proposition 4.3, we have the following estimate.

**Lemma 4.5.** Let $k \geq 3$ be an integer and assume $g \in L^2(D^n) \cap \Lambda_{\alpha}(p)$ for some $\alpha \in (0, 1)$. Then there is a constant $C = C(k, \alpha)$ such that

$$|\lambda_t^k, g| \leq C \left( \sum_{i=1}^n (1 - t_i^2)^{\alpha} \right) \left( \prod_{j=1}^n \frac{1}{(1 - t_j^2)^{k-2}} \right)$$

for all $t = (t_1, \ldots, t_n)$. 

Proof. Assume \( g \in L^2(D^n) \cap \Lambda_\alpha(U) \) where \( U \) is some neighborhood of \( p \). Following the proof of 3.3, we have
\[
\| \lambda_t^k g \| \leq \| \lambda_t^k g(p) \| + \| g(p) \|
\]
and
\[
\| \lambda_t^k g(p) \| \lesssim \int_{D^n \cap U} \frac{|g(w) - g(p)|}{\prod_{j=1}^n |1 - t_j w_j|^k} dV(w) + \| g - g(p) \|_2
\]
for all \( t \). Since \( g \in \Lambda_\alpha(U) \), we see from Lemma 2.3 that the integral in the right side is dominated by some constant (depending on \( g \)) times
\[
\sum_{i=1}^n \int_{D^n} \prod_{j=1}^n |1 - w_j|^\alpha |1 - t_j w_j|^k dV(w) \approx \sum_{i=1}^n \frac{1}{(1 - t_i^2)^{k-2-\alpha}} \prod_{j \neq i} \frac{1}{(1 - t_j^2)^{k-2}}.
\]
This completes the proof. \( \Box \)

We are now ready to prove our theorems. We first prove Theorem 1.5.

Proof of Theorem 1.5. As in the proofs of Theorems 1.2 and 1.1, our proof utilizes the estimates which are already established in an earlier work [6] of the authors. So, we will also repeat the same part of the proof of [6, Theorem 1.2] for reader’s convenience.

Before proceeding we first introduce some notation. Given an integer \( 0 \leq k \leq n \), let \( I(k) \) be the set of all multi-indices \( \nu = (\nu_1, \ldots, \nu_n) \) such that \( \nu_i \in \{0, 1\} \) for each \( i \) and \( |\nu| = k \). For a multi-index \( \mu \), we say \( \mu \in I^*(k) \) if there exists \( v \in I(k) \) such that \( \mu_i = v_i + 1 \) for some \( i \) and \( \mu_j = v_j \) for all \( j \neq i \). Also, we let
\[
\tau(z) = (1 - \Re z_1, \ldots, 1 - \Re z_n),
\]
\[
\eta(z) = (|1 - z_1|, \ldots, |1 - z_n|)
\]
for \( z \in \mathbb{C}^n \).

Put \( S = T_{u_1} T_{u_2} T_{u_3} \) and assume that \( S \) has finite rank. As in the proof of Theorem 1.2, we have
\[
u_1 u_2 u_3 = 0 \quad \text{on } T^n.
\]
Since \( u_1 u_2 u_3 \) is continuous and vanishes everywhere on the distinguished boundary, there exists a relatively open set \( W \subset T^n \) such that
\[
\begin{align*}
\text{either} & \quad u_j \text{ vanishes nowhere on } W, \\
\text{or} & \quad u_j = 0 \quad \text{on } W
\end{align*}
\]
holds for each \( j \). Since there is nothing to prove if \( u_1 = 0 \), we may assume that \( u_1 \) vanishes nowhere on \( W \). Now, assume that each \( u_j \) is not identically 0 on \( D^n \). We will get a contradiction.

Since each \( u_j \) is not identically 0 on \( D^n \), we may shrink (if necessary) the set \( W \) to get a smaller relatively open set, still denoted by \( W \), such that each \( u_j \) satisfies
\[
\begin{align*}
\text{either} & \quad (i) u_j|_W \text{ never vanishes}, \\
\text{or} & \quad (ii) u_j|_W = 0 \quad \text{but } u_j \neq 0.
\end{align*}
\]
By the proof of [6, Theorem 1.2], we may assume \( p \in W \). Given \( j \) for which the case (ii) holds, we have by [6, Lemma 3.1] a positive integer \( k_j \leq n \) and coefficients \( a_{j\nu} \), not all 0, such that
\[
u_j(z) = \sum_{v \in I(k_j)} a_{jv} \tau(z)^v + O\left( \sum_{\mu \in I^*(k)} \eta(z)^\mu \right)
\]
for \( z \in D^n \cup W \) near \( p \). Now, define the major part of \( u_j \) by
\[
m_j(z) := \begin{cases} 
    u_j(p) & \text{if } u_j(p) \neq 0, \\
    \sum_{v \in I(k_j)} a_{jv} \tau(z)^v & \text{if } u_j(p) = 0
\end{cases}
\]
for each \( j \). Put \( M = T_{m_1}T_{m_2}T_{m_3} \) and \( R = S - M \). We will estimate the same expression \( M\lambda^n_t(t) = (S - R)\lambda^n_t(t) \) in two different ways and reach a contradiction as in the proofs of the previous section. Here, \( m > 4 \) is any fixed integer and the parameter \( t = (t_1, \ldots, t_n) \) is chosen below.

Let \( \tilde{I} \) be the set of all multi-indices of the form \( v + \mu \) where \( v \in I(k_2) \) and \( \mu \in I(k_3) \). Given \( h \in \tilde{I} \), let

\[
a_h = \prod_{h_i \geq 1} \frac{1}{(m - 1) \cdots (m - h_i)}
\]

and consider a polynomial on \( \mathbb{R}^n \) given by

\[
F(x) := \sum_{h \in \tilde{I}} a_h \left( \sum_{v + \mu = h} a_{2v}a_{3\mu} \right) x^h
\]

for \( x \in \mathbb{R}^n \). The proof of [6, Theorem 1.2] shows that \( F \) is nontrivial. Thus there is some \( y \in (0,1)^n \) such that \( F(y) \neq 0 \). Given \( t \in (0,1) \), we now choose \( t_j = t_j(t) \) determined by the equation

\[
1 - t_j^2 = y_j(1 - t^2), \quad j = 1, \ldots, n, \tag{4.5}
\]

and let \( t = (t(t) = (t_1, \ldots, t_n) \) for the rest of the proof.

The proof of [6, Theorem 1.2] shows the following estimates independent of \( t \):

\[
|M\lambda^n_t(t)| \gtrsim \frac{1}{(1 - t^2)^{mn - k}},
\]

\[
|R\lambda^n_t(t)| \lesssim \frac{(1 - t)^{\alpha} |\log(1 - t)|^{3n}}{(1 - t)^{mn - k}} \tag{4.6}
\]

as \( t \to 1 \) where \( k = k_2 + k_3 \).

We now estimate \( S\lambda^n_t(t) \). Since \( S \) has finite rank, say \( N \), we may represent \( S \) by (3.7) (in the context of polydisk) with linearly independent collections \( \{f_j\}_{j=1}^N \) and \( \{g_j\}_{j=1}^N \) of functions in \( A^2(D^n) \). Fix \( \beta \in (0, \alpha) \). By Lemma 4.4 we have \( f_j, g_j \in A_\beta(p) \). Using Lemma 4.5 and proceeding as in the proof of Theorem 1.2, we have by (4.5)

\[
|S\lambda^n_t(t)| \lesssim \left( \sum_{i=1}^n (1 - t_i^2)^{\beta} \right) \left( \prod_{j=1}^n \frac{1}{(1 - t_j^{mn - 2n - \beta})} \right) \approx \frac{1}{(1 - t)^{mn - 2n - \beta}} \tag{4.7}
\]

as \( t \to 1 \).

Now, we have by (4.6) and (4.7)

\[
1 = \frac{|S\lambda^n_t(t) - R\lambda^n_t(t)|}{|M\lambda^n_t(t)|} \lesssim (1 - t)^{\beta} \left[ \log(1 - t) \right]^{3n} + (1 - t)^{2n - k}
\]

as \( t \to 1 \). The suppressed constant in this inequality is independent of \( t \). Note \( k \leq 2n \); it is this step where we lose one factor from [6, Theorem 1.2]. Thus, taking the limit \( t \to 1 \), we reach a contradiction. The proof is complete. \( \Box \)

As in the ball case, we have the following local version for two factor as a by-product of the proof of Theorem 1.5. This will be used in the proof of Theorem 1.4 below.

**Corollary 4.6.** Let \( u_1, u_2 \in h^\infty(D^n) \cap \Lambda_\alpha(\zeta) \) for some \( \alpha \in (0, 1) \) and \( \zeta \in T^n \). If \( T_{u_1}T_{u_2} \) has finite rank, then either \( u_1 = 0 \) or \( u_2 = 0 \).

We are now ready to prove Theorem 1.4.

**Proof of Theorem 1.4.** As in the proof of Theorem 1.5, we will also repeat the same part of the proof of [6, Theorem 1.1] for reader’s convenience.
Put $S = T_{u_1}T_{u_2}$ and assume that $S$ has finite rank. As in the proof of Theorem 1.2, we have $u_1u_2 = 0$ on $W$. We consider two cases separately:

(i) Both $u_1$ and $u_2$ vanish on $W$.

(ii) Either $u_1$ or $u_2$ vanishes nowhere on some relatively open subset of $W$.

First, consider the case (i). By [6, Lemma 3.1] we have $u_1, u_2 \in \Lambda^1(\zeta)$ for some $\zeta \in T^n$. The case (i) is thus contained in Corollary 4.6. Next, consider the case (ii). By considering adjoints as in the proof of Theorem 1.1, we may further assume that $u_1$ does not vanish on some relatively open set, still denoted by $W$. When $u_2$ vanishes nowhere on $W$, we have

$$
\sum_{\nu \in I(k)} a_\nu\tau(z)^\nu + O\left( \sum_{\mu \in I^*(k)} \eta(z)^\mu \right)
$$

for $z \in D^n \cup W$ near $p$. Put $c_1 = u_1(p) \neq 0$ and define major part of $u_2$ by

$$
m_2(z) := \sum_{\nu \in I(k)} a_\nu\tau(z)^\nu.
$$

Also, let $e_1 = u_1 - c_1$ and $e_2 = u_2 - m_2$. Then we have

$$
S = T_{e_1+e_2}T_{m_2+e_2} = c_1T_{m_2} + T_{e_1}T_{m_2} + T_{u_1}T_{e_2}
$$

and thus

$$
c_1T_{m_2} = S - T_{e_1}T_{m_2} - T_{u_1}T_{e_2}. \tag{4.8}
$$

As in the proof of Theorem 1.5, we will apply each side of the above to the same test functions $\lambda^m_t$ with $m > 4$ and derive a contradiction. This time the parameter $t$ is chosen as follows. Put $G(x) = \sum_{\nu \in I(k)} a_\nu x^\nu$ for $x \in \mathbb{R}^n$. Note that $G$ is a non-zero polynomial on $\mathbb{R}^n$, because some coefficient $a_\nu$ is not zero. Thus there is some $y \in (0, 1)^n$ such that $G(y) \neq 0$. Given $t \in (0, 1)$, we now choose $t_j = t_j(t) \in (0, 1)$ determined by Eq. (4.5) and let

$$
t = t(t) = (t_1, \ldots, t_n)
$$

for the rest of the proof.

Given $\epsilon > 0$ small and $0 < t < 1$ sufficiently close to 1, we put

$$
\varphi(\epsilon, t) = \frac{(1-t)^{m-1}}{[\epsilon - |y|(1-t)]^{m+2}}
$$

and

$$
\omega(\epsilon) = \sup\{|e_1(p)| : w \in D^n \cup W, \ |p - w| < \sqrt{n}\epsilon\}
$$

for simplicity. Then the proof of [6, Theorem 1.1] shows the following estimates independent of $t$ and $\epsilon$:

$$
|T_{m_2}\lambda^m(t)| \gtrsim \frac{1}{(1-t)^{mn-k}},
$$

$$
|T_{u_1}T_{e_2}\lambda^m(t)| \lesssim \frac{|\log(1-t)|^{2n}}{(1-t)^{mn-k-1}},
$$

$$
|T_{e_1}T_{m_2}\lambda^m(t)| \gtrsim \frac{\omega(\epsilon) + \varphi(\epsilon, t)}{(1-t)^{mn-k}} \tag{4.9}
$$

as $t \to 1$. 


Now, we estimate \( S_m^\lambda(t) \). Since \( S \) has finite rank, say \( N \), there exist linearly independent collections \( \{f_j\}_{j=1}^N \) and \( \{g_j\}_{j=1}^N \) of functions in \( A^2(D^n) \) such that (3.7) holds (in the context of polydisk). Thus we have

\[
S_m^\lambda(t) = \sum_{j=1}^N \langle \lambda^m, g_j \rangle f_j(t).
\]

Note that functions \( f_j, g_j \) all belong to \( \text{Log}^2(D^n) \) by Lemma 4.4. Pick \( p \) with \( 1 < p < \frac{2n}{k} \). By an application of Lemma 2.3, we have by (4.5)

\[
\|\lambda^m\|_{L^p(D^n)} \lesssim \prod_{j=1}^n \left( \frac{1}{1-t_j^2} \right)^{m-2/p} \approx \frac{1}{(1-t^2)^{(m-2/p)n}}
\]

for \( 0 < t < 1 \). Also, for each \( j \) and \( t \in (0, 1) \), we have by (4.5)

\[
\left| f_j(t) \right| \leq \|f_j\|_{\text{Log}^2(D^n)} \prod_{j=1}^n \left( 1 + \log \frac{1}{1-t_j^2} \right)^2
\]

\[
= \|f_j\|_{\text{Log}^2(D^n)} \prod_{j=1}^n \left( 1 + \log \frac{1}{1-t^2} + \log \frac{1}{y_j} \right)^2
\]

\[
\lesssim \left| \log(1-t) \right|^{2n}
\]

as \( t \to 1 \). Now, applying Hölder’s inequality, we obtain

\[
|S_m^\lambda(t)| \leq \sum_{j=1}^N \|\lambda^m\|_{L^p(D^n)} \|g_j\|_{L^q(D^n)} |f_j(t)| \lesssim \frac{\left| \log(1-t) \right|^{2n}}{(1-t)^{(m-2/p)n}}
\]

(4.10)

as \( t \to 1 \) where \( q \) is the conjugate exponent of \( p \).

It follows from (4.8)–(4.10) that

\[
1 = \frac{|(S - T_{e_1} T_{m_2} - T_{u_1} T_{e_2}) \lambda^m(t)|}{|c_1 T_{m_2} \lambda^m(t)|} \lesssim \omega(\epsilon) + \varphi(\epsilon, t) + (1-t)^{\delta} \left| \log(1-t) \right|^{2n}
\]

as \( t \to 1 \) where \( \delta = \min\{1, 2n/p - k\} > 0 \) and this estimate is independent of \( t \) and \( \epsilon \). Now, first taking the limit \( t \to 1 \) (with \( \epsilon \) fixed) and then taking the limit \( \epsilon \to 0 \), we obtain

\[
1 \lesssim \omega(\epsilon) \to 0
\]

by continuity of \( u_1 \) at \( p \), which is a contradiction. This completes the proof. \( \square \)

**Remark added after the acceptance of the paper.** After this paper was submitted, we were informed that Daniel Luecking proved that if a one-variable Toeplitz operator with a function symbol has finite rank, then its symbol must be zero. Also, the first author of this paper has recently generalized Luecking’s result to several variables.

**References**


