Representations of rank two affine Hecke algebras at roots of unity

Matt Davis

Department of Mathematics, Harvey Mudd College, 301 Platt Boulevard, Claremont, CA, United States

ABSTRACT

In this paper, we will fully describe the irreducible representations of the crystallographic rank two affine Hecke algebras using algebraic and combinatorial methods, for all possible values of $q$. The focus is on the case when $q$ is a root of unity of small order.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

The affine Hecke algebra $\mathcal{H}$ was introduced by Iwahori and Matsumoto [3]. Knowing the representations of $\mathcal{H}$ gives a substantial amount of information about the representations of a closely related $p$-adic group. The definition of $\mathcal{H}$ involves a parameter $q$ which can have a large effect on the structure of the algebra. In this paper, we will fully describe the irreducible representations of the affine Hecke algebras of type $C_2$ and $G_2$, for all possible values of $q$. The methods are essentially those introduced in [9], with the modifications required to deal with $q$ being a root of unity.

The representations in type $A$ were described in the non-root of unity case by Zelevinsky, in terms of combinatorial objects called multisegments (see [1] and [14]). In the root of unity case, these representations are indexed by the aperiodic multisegments (see the appendix of [7] for an argument relying on the results of [8]). The representations of $\mathcal{H}$ in all types have been classified geometrically by Kazhdan and Lusztig [6] in the non-root of unity case, and studied in the root of unity case by Grojnowski [2] and N. Xi [12,13], among others. In the root of unity case, Grojnowski gives a simple description of a geometric indexing set [2, Theorem 2] only in type $A$. However, Theorem 1 of [2] does not apply in all cases (see the remark on p. 524 of [12]). And, to this author, at least, it is not obvious how to turn the statement of Theorem 1 into Theorem 2. One hopes that a better understanding of the representations of $\mathcal{H}$ in some small cases will help clarify these issues.

E-mail address: davis@math.hmc.edu.

1 This material is based upon work supported by the National Science Foundation under Grant No. DMS-0839966.
We begin by defining the affine Hecke algebra $\mathcal{H}$, and recalling some basic facts about the representations of $\mathcal{H}$. We will make extensive use of $\mathbb{C}[X]$, a large commutative subalgebra of $\mathcal{H}$, and weights, elements of $\text{Hom}(\mathbb{C}[X], \mathbb{C})$, which describe the simple representations of $\mathbb{C}[X]$. An $\mathcal{H}$ module $M$ can be described in part by which weights appear, i.e. which simple $\mathbb{C}[X]$ modules are composition factors of it. The most important construction we will use is that of the principal series module $M(t)$ which can be constructed from any weight $t \in T$, since every simple $\mathcal{H}$-module is a quotient of $M(t)$ for an appropriate choice of a weight $t$ (Proposition 1(c)). We also recall from [9] several facts needed to analyze the modules $M(t)$, with some adaptations as necessary to deal with the root of unity case.

The main goal of the paper is to describe a way of visualizing and describing the composition factors of $M(t)$ directly from the combinatorial data of the weight $t$. This can be done with particular pictures based on the root system underlying $\mathcal{H}$. The following are examples in the type $A_2$ case.

![Picture](image_url)

The lines in this picture represent the hyperplanes perpendicular to the roots $\alpha$, and are drawn as solid, shaded, or dotted based on the values of the weight $t$ on the elements $X^\alpha \in \mathcal{H}$. Each dot in the picture represents one dimension of the module $M(t)$, and dots are connected if a single composition factor of $M(t)$ contains both of these basis elements. The general goal is to determine a few rules that determine which of these lines should be drawn. That is, we hope to find a few algebraic statements that describe how $M(t)$ breaks down into composition factors which can be translated into these pictures. Essentially, Theorem 3(b), Proposition 4, and Theorem 5 below are sufficient to complete the classification in the rank two cases, for all values of $q$. These pictures provide a very straightforward way of determining the composition factors of $M(t)$, without relying on heavy computations. One also hopes that a complete classification of the rank two crystallographic cases will facilitate a greater understanding of the representation theory of $\mathcal{H}$ in all types.

2. Definitions

In this section, we introduce the needed definitions and several preliminary results about the affine Hecke algebra. Proofs of most previously known results will not be given.

The affine Hecke algebra. Let $R$ be a root system in $\mathbb{R}^n$ with simple roots $\alpha_1, \ldots, \alpha_n$. Let $R^+$ be the set of positive roots and $R^-$ the set of negative roots. We define the rank of $R$ to be the number of simple roots $n$.

$$R = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm (2\alpha_1 + \alpha_2) \} \quad \text{and} \quad R^- = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2) \}$$

Two examples of root systems in $\mathbb{R}^2$

The reflection through $H_\alpha$ will be denoted by $s_\alpha$, or $s_i$ for the reflection through $H_\alpha$. If $\pi/mij$ is the angle between $H_\alpha$ and $H_\beta$, then $mij \in \{2, 3, 4, 6\}$ for $1 \leq i, j \leq n$, and the Weyl Group $W_0$ has presentation
\[
W_0 = \{s_1, \ldots, s_n \mid s_i^2 = 1, \frac{s_is_js_i}{m_{ij}} = \frac{s_j}{m_{ij}} \text{ for } 1 \leq i, j \leq n\}.
\]

Let \(P\) be the weight lattice, spanned by the elements \(\omega_i\) satisfying
\[
\langle \omega_i, \alpha_j \rangle = \delta_{ij} \cdot \frac{1}{2} \langle \alpha_j, \alpha_j \rangle,
\]
for \(\alpha_i\) and \(\alpha_j\) simple roots. Let \(Q\) be the lattice spanned by the simple roots \(\alpha_i\). Let
\[
X = \{X^\lambda \mid \lambda \in P\}, \quad \text{with } X^\lambda \cdot X^\mu = X^{\lambda + \mu}, \quad \forall \lambda, \mu \in P.
\] (1)

Then \(W_0\) acts on \(X\) by
\[
w \cdot X^\lambda = X^{w \cdot \lambda},
\]
and this action extends linearly to an action of \(W_0\) on the group algebra \(\mathbb{C}[X]\).

The affine Hecke algebra \(\mathcal{H}\) is the \(\mathbb{C}\)-algebra generated by \(\{T_i \mid i \in I\}\) and \(\{X^\lambda \mid \lambda \in P\}\), where \(\mathbb{C}[X]\) is a subalgebra of \(\mathcal{H}\), and subject to the relations
\[
T_i^2 = (q - q^{-1})T_i + 1, \quad \text{for } i = 1, 2, \ldots, n, \quad \text{(2)}
\]
\[
\underbrace{T_iT_jT_i \ldots} = \underbrace{T_jT_iT_j \ldots} \quad \text{for } i \neq j, \quad \text{and} \quad \text{(3)}
\]
\[
X^\lambda T_i = T_i X^{s_i^\lambda} + \frac{q - q^{-1}}{1 - X^{-\alpha_i}} X^\lambda, \quad \text{for } \lambda \in P, \ 1 \leq i \leq n. \quad \text{(4)}
\]

The rank of \(\tilde{\mathcal{H}}\) is defined to be the rank of the underlying root system \(R\). For \(w \in W_0\), let
\[
T_w = T_{s_1}T_{s_2} \ldots T_{s_k}
\]
for a reduced word \(w = s_{i_1}s_{i_2} \ldots s_{i_k} \in W_0\). Then \(\{T_wX^\lambda \mid w \in W_0, \lambda \in P\}\) is a \(\mathbb{C}\)-basis for \(\mathcal{H}\).

Weights. Let \(T = \text{Hom}(X, \mathbb{C}^\times)\) be the set of group homomorphisms from \(X\) to \(\mathbb{C}^\times\). Then \(T\) is an abelian group with \(W_0\)-action given by
\[
w \cdot t(X^\lambda) = t(X^{w^{-1} \cdot \lambda}) \quad \text{for } t \in T, \ w \in W_0, \ \lambda \in P.
\]
An element of \(T\) is called a weight. For a weight \(t\), the subgroup of \(W_0\) that fixes \(t\) under this action is generated by \(\{s_i \mid t(X^{Q_i}) = 1\}\). (This relies on the fact that we chose \(P\) rather than \(Q\) to build \(\mathcal{H}\). See [11], 3.15, 4.2, and 5.3).

For any finite-dimensional \(\mathcal{H}\)-module \(M\), define the \(t\)-weight space and the generalized \(t\)-weight space of \(M\) by
\[
M_t = \{m \in M \mid X^\lambda \cdot m = t(X^\lambda)m \text{ for all } X^\lambda \in X\}, \quad \text{and}
\]
\[
M_{t}^\text{gen} = \{m \in M \mid \text{for all } X^\lambda \in X, \ (X^\lambda - t(X^\lambda))^k m = 0 \text{ for some } k \in \mathbb{Z}_{>0}\},
\]
respectively. Then $M = \bigoplus_{t \in T} M_t^{\text{gen}}$ is a decomposition of $M$ into Jordan blocks for the action of $\mathbb{C}[X]$. An element $t \in T$ is a weight of $M$ if $M_t^{\text{gen}} \neq 0$.

**Induced modules and intertwining operators.** If $I \subseteq \{1, \ldots, n\}$, define $W_I = \langle s_i \mid i \in I \rangle$ and

$$\mathcal{H}_I = \{ T_w X^\lambda \mid \lambda \in \mathcal{P}, w \in W_I \}.$$  

For example, $\mathcal{H}_\emptyset = \mathbb{C}[X]$, while $\mathcal{H}_{\{i\}}$ is the subalgebra of $\mathcal{H}_{\{i\}}$ generated by $\mathbb{C}[X]$ and $T_i$. Then for $t \in T$ such that $t(X^{\alpha_i}) = q^2$ for $i \in I$, define $\mathbb{C}v_t$ to be the one-dimensional $\mathcal{H}_I$-module spanned by $v_t$, with $\mathcal{H}_I$ action given by

$$T_i \cdot v_t = q v_t \quad \text{and} \quad X^\lambda \cdot v_t = t(X^\lambda)v_t, \quad \text{for } X^\lambda \in X.$$  

**Proposition 1.** (See [9], Lemma 1.17.) Let $\mathbb{C}v_t$ be defined as above, and let $M = \text{Ind}_{\mathcal{H}_I}^{\mathcal{H}} \mathbb{C}v_t$. Let $W_I = \langle s_i \mid i \in I \rangle$, and let $W_0/W_I$ be a set of minimal length coset representatives of $W_I$-cosets in $W_0$.

(a) Then the weights of $M$ are $\{ wt \mid w \in W_0/W_I \}$, and

$$\dim(M_{\text{gen}}^{\text{w}^t}) = (\# \text{ of } v \in W_0/W_I \text{ with } vt = wt).$$  

(b) There is a basis of $M$ consisting of elements of the form

$$m_w = T_w v_t + \sum_{u < w, u \in W_0/W_I} p_{w,u} T_u v_t,$$

for $w \in W_0/W_I$, such that $m_w \in M_{\text{w}^t}$.

(c) If $t$ is a weight of an irreducible $\mathcal{H}$-module $N$ and $I = \emptyset$, then $N$ is a quotient of $M$. In fact, if $v \in N$ is a non-zero vector in $N_t$, then

$$\phi : M \rightarrow N,$$

$$v_t \mapsto v$$

extends to a surjective $\mathcal{H}$-module homomorphism.

In particular, if $I = \emptyset$, then we call

$$M(t) = \mathcal{H} \otimes_{\mathbb{C}[X]} \mathbb{C}v_t = \text{span}(T_w v_t \mid w \in W_0)$$

the principal series module for $t$.

Part (c) of this lemma implies that the weights of a single simple finite-dimensional module $M$ lie in a single orbit $\mathcal{O}_t$. We call this orbit (and, by abuse of terminology, any element of the orbit) the central character of $M$. In fact, $\mathcal{H}$ has finite dimension over its center, and thus all simple $\mathcal{H}$-modules are finite-dimensional (see [9], Section 2.3). Thus, this proposition tells us that understanding the composition factors of all the principal series modules $M(t)$ is sufficient for understanding all the simple $\mathcal{H}$-modules.

For a weight $t$ with $t(X^{\alpha_i}) \neq 1$ and an $\mathcal{H}$-module $M$, define a $\mathbb{C}$-linear operator $\tau_t : M_t^{\text{gen}} \rightarrow M$ by

$$\tau_t(m) = \left( T_i - \frac{q - q^{-1}}{1 - X^{-\alpha_i}} \right) \cdot m.$$  

(5)
Theorem 2. (See [9], Proposition 1.18.)

(a) $1 - X^{-\alpha_i}$ is invertible as an operator on $M^\text{gen}_t$, so that $\tau_i : M^\text{gen}_t \rightarrow M$ is well defined.
(b) As operators on $M^\text{gen}_t$, $X^\lambda \tau_i = \tau_i X^\lambda$ for all $X^\lambda \in X$, so that $\tau_i(M^\text{gen}_t) \subseteq M^\text{gen}_{st}$.
(c) As operators on $M^\text{gen}_t$,
$$\tau_i \tau_i = \frac{(q - q^{-1}X^{\alpha_i})(q - q^{-1}X^{-\alpha_i})}{(1 - X^{\alpha_i})(1 - X^{-\alpha_i})}.$$  
(d) The maps $\tau_i : M^\text{gen}_t \rightarrow M^\text{gen}_{st}$ and $\tau_i : M^\text{gen}_{st} \rightarrow M^\text{gen}_t$ are both invertible if and only if if $(X^{\alpha_i}) \neq q^\pm 2$.
(e) If $i \neq j$ and $m_{ij}$ is defined as in (3), then $\tau_i \tau_j \tau_i \ldots = \tau_j \tau_i \tau_j \ldots$, whenever both sides are well-defined.

For $t \in T$, the calibration graph of $t$ is the graph with vertices labeled by the elements of the orbit $W_0 t$ and edges $(wt, stwt)$ if $(wt)(X^{\alpha_i}) \neq q^\pm 2$. The $\tau$ operators are used to prove the following.

Theorem 3.

(a) (See [10], Proposition 2.3.) If $w \in W_0$ and $t \in T$ then $M(t)$ and $M(wt)$ have the same composition factors.
(b) (See [9], Proposition 1.6.) Let $M$ be a finite-dimensional $H$-module, and let $t$ and $wt$ be two elements of $W_0 t$ in the same connected component of the calibration graph for $t$. Then
$$\dim(M^\text{gen}_t) = \dim(M^\text{gen}_{wt}).$$
(c) (See [4].) $M(t)$ is irreducible if and only if $P(t) := \{ \alpha \in R^+ \mid t(X^\alpha) = q^\pm 2 \} = \emptyset$.

The structure of modules. Theorem 3(b) shows that the connected components of the calibration graph encode certain sets of weights whose corresponding weight spaces $M^\text{gen}_t$ must have the same dimension in any irreducible $H$-module $M$. These ideas lead us to the following propositions which will be fundamental in our later classification.

Proposition 4. Let $M$ be an irreducible 2-dimensional $H$-module and assume $q^2 \neq 1$.

(a) If $M$ has two different weight spaces $M_t$ and $M_{t'}$, then $t' = st$ for some $i$, and $t(X^{\alpha_i}) \neq q^\pm 2$ or 1, but $t(X^{\alpha_j}) = q^\pm 2$ and $s_i t(X^{\alpha_j}) = q^\mp 2$ for $j \neq i$. Moreover, there is a unique 2-dimensional module (up to isomorphism) containing these two weight spaces.
(b) If $M$ has only one weight space $M^\text{gen}_t$, then $t(X^{\alpha_i}) = 1$ for some $i$, and for $j \neq i$, $t(X^{\alpha_j}) = q^2$, and either $\langle \alpha_j, \alpha_i^\vee \rangle = 0$ or else $q^2 = -1$ and it is not the case that $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ and $\langle \alpha_j, \alpha_i^\vee \rangle = -2$.

Proof. (a) If $t(X^{\alpha_i}) = 1$ for some $i$, then consider $M$ as an $H_{(t)}$-module. By Kato’s criterion (Theorem 3(c)), the fact that $q^2 \neq 1$, and Proposition 1(c), there is only one irreducible $H_{(t)}$-module $N$ with central character $t$, where $t(X^{\alpha_i}) = 1$. This module is 2-dimensional with $\dim N^\text{gen}_t = 2$. Thus $M \cong N$ as $H_{(t)}$-modules and $t = t'$.

Assume $M$ has two different weight spaces $M_t$ and $M_{t'}$. Then since $M$ is irreducible, some $\tau_i$ must be non-zero on $M_t$, and $t' = st$. Then $\tau_i$ must also be non-zero on $M_{st}$, and $t(X^{\alpha_i}) \neq q^\pm 2$. Since $M_{st} = 0 = M_{st'}$ for $j \neq i$, $t(X^{\alpha_j}) = q^\mp 2$ and $s_i t(X^{\alpha_j}) = q^\pm 2$. The weight structure determines the action of $\mathbb{C}[X]$ on $M$, and since we know how the operators $\tau_i$ act, the actions of the $T_i$ are determined as well, so that the module structure of $M$ is determined by its weight structure.

(b) Assume $M$ consists of one generalized weight space $M^\text{gen}_t$, with $v_t \in M_t$. If all the operators $\tau_i$ were defined on $M_t$, then $\tau_i(v_t) = 0$ for all $i$. Hence $T_i v_t \in \mathbb{C}v_t$ for all $i$ and $v_t$ would span a submodule of $M$, a contradiction. Thus some $\tau_i$ is not well defined and $t(X^{\alpha_i}) = 1$.  


If \( t(X^{\alpha_j}) = 1 \) for any \( j \neq i \), then \( t(X^{\beta}) = 1 \) for any \( \beta \) in the span of \( \alpha_i \) and \( \alpha_j \). Then \( M \) as an \( \mathcal{H}_{i,j} \)-module contains a principal series module, and must have dimension at least as large as the number of roots in the root subsystem generated by \( \alpha_i \) and \( \alpha_j \), which is a contradiction since this number will be greater than 2. Then \( t(X^{\alpha_j}) \neq 1 \) for all \( j \neq i \).

Then consider \( M \) as an \( \mathcal{H}_{i,j} \) module, which is irreducible by Theorem 3(c). The action of \( \mathcal{H} \) on the basis \( \{v_t, T_i v_t\} \) is given by

\[
M(T_i) = \begin{bmatrix} 0 & 1 \\ 1 & q - q^{-1} \end{bmatrix}, \quad \text{and} \quad M(X^{\alpha_j}) = t(X^{\alpha_j}) \begin{bmatrix} 1 & (q - q^{-1})\langle \alpha_j, \alpha_i^\vee \rangle \\ 0 & 1 \end{bmatrix}.
\]

Then since \( \tau_j \) (which is well defined since \( t(X^{\alpha_j}) \neq 1 \)) must be the zero map on \( M_{i}^{\text{gen}} \), we have \( M(T_j) = M_{i}^{\frac{q - q^{-1}}{1 - X^{-\alpha_j}}} \), and

\[
M(T_j) = (q - q^{-1}) \left( \frac{1}{1 - t(X^{-\alpha_j})} \right) \begin{bmatrix} 1 & \frac{(q - q^{-1})t(X^{-\alpha_j})(-\alpha_j, \alpha_i^\vee)}{1 - t(X^{-\alpha_j})} \\ 0 & 1 \end{bmatrix}.
\]

However, since the relation (2) can be written \( (T_j - q)(T_j + q^{-1}) = 0 \), \( M(T_j) \) must have eigenvalues \( q \) or \( -q^{-1} \) and \( t(X^{-\alpha_j}) = q^{\pm 2} \).

If \( q^2 \neq -1 \), so that \( q \neq -q^{-1} \), then either \( M(T_j) - qI \) or \( M(T_j) + q^{-1}I \) is invertible, so that the other must actually be zero and so the off-diagonal term must be zero. The only way this can occur is if \( \langle \alpha_j, \alpha_i^\vee \rangle = 0 \).

If \( q^2 = -1 \) then

\[
M(T_j) = \begin{bmatrix} q & \langle \alpha_j, \alpha_i^\vee \rangle \\ 0 & q \end{bmatrix}.
\]

Then \( M(T_i) \) and \( M(T_j) \) must satisfy the same braid relation as \( T_i \) and \( T_j \), which is determined by the type of root system spanned by \( \alpha_i \) and \( \alpha_j \). A check of the possible root systems (\( A_1 \times A_1, A_2, C_2 \), and \( G_2 \)) shows that the braid relation is satisfied unless \( \langle \alpha_i, \alpha_j^\vee \rangle = -1 \) and \( \langle \alpha_j, \alpha_i^\vee \rangle = -2 \). \( \Box \)

**Theorem 5.** (See [9], Lemma 1.19.) Assume \( q^2 \neq 1 \). Let \( t \in T \) such that \( t(X^{\alpha_i}) = 1 \) and suppose that \( M \) is an \( \mathcal{H}(q) \)-module such that \( M_{i}^{\text{gen}} \neq 0 \). Let \( W_t \) be the stabilizer of \( t \) under the action of \( W_0 \) on \( T \). Assume that \( \bar{w} \in W_0/W_t \) is such that \( t \) and \( \bar{w}t \) are in the same connected component of the calibration graph for \( t \), and let \( w \) be a minimal length coset representative for \( \bar{W} \). Then

(a) \( \dim(M_{w}^{\text{gen}}) \geq 2 \) and \( \dim(M_{w}^{\text{gen}}) > \dim(M_{w}^{\text{gen}}) \).

(b) If \( M_{w_{ij}}^{\text{gen}} = 0 \) then \( (\bar{w}t)(X^{\alpha_i}) = q^{\pm 2} \) and if, in addition, \( q^2 \neq -1 \), then \( \langle w^{-1} \alpha_j, \alpha_i^\vee \rangle = 0 \).

**Visualizing modules.** For \( t \in T \), define

\[
Z(t) = \{ \alpha \in R^+ \mid t(X^{\alpha}) = 1 \} \quad \text{and} \quad P(t) = \{ \alpha \in R^+ \mid t(X^{\alpha}) = q^{\pm 2} \}.
\]

Notice that \( |Z(t)| \) and \( |P(t)| \) are constant on orbits \( W_0t \), since the action of \( W_0 \) permutes the multiset \( \{t(X^{\alpha}) \mid \alpha \in R \} \).

The \( \tau \) operators and the sets \( Z(t) \) and \( P(t) \) provide extensive information about the structure (and sometimes the composition factors) of \( M(t) \). Let \( H_\alpha \) be the hyperplane fixed by \( s_\alpha \) for \( \alpha \in R \). A chamber is a connected component of \( \mathbb{R}^n \setminus \cup_{\alpha \in R^+} H_\alpha \), and \( W_0 \) acts faithfully and simply transitively on the set of chambers. Choose a fundamental chamber \( C \) and define the positive side of a hyperplane \( H_\alpha \) to be the side on which \( C \) lies. The map
\{\text{chambers}\} \leftrightarrow W_0,
\quad w^{-1}C \mapsto w
\quad \quad \quad \quad \quad (6)

is a bijection.

By 3.15, 4.2, and 5.3 of Steinberg [11] the stabilizer of \( t \) is
\[ W_t = \langle s_\alpha \mid \alpha \in Z(t) \rangle. \]

Thus, if \( W_0/W_t \) is a set of minimal length coset representatives of \( W_t \)-cosets in \( W_0 \), then
\[ W_0/W_t \leftrightarrow W_0t \leftrightarrow \{ \text{chambers on the positive side of all } H_\alpha, \alpha \in Z(t) \}, \]
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
The structural theorems above tell us which dots to connect together in these drawings, and the
resulting picture describes the composition factors of \( M(t) \) – dots are connected via some path
exactly when the corresponding basis vectors lie in the same composition factor. In particular, the third
drawing gives a good picture of Theorem 3(b). If two chambers have a common boundary that is a
dotted hyperplane, then the \( \tau \) operator between the two corresponding weight spaces will be invert-
bile in both directions, and those weight spaces must have the same dimension in any irreducible
\( \mathcal{H} \)-module with the central character shown in the picture. Thus in this case, the basis vectors with
those weights lie in the same composition factor and are connected above. Similarly, the second draw-
ing demonstrates Theorem 5. For this central character, Theorem 5 implies that the \( t \)-weight space of
an irreducible \( \mathcal{H} \)-module must have dimension 0 or 2, and an irreducible containing a 2-dimensional
generalized \( t \) weight space must also have a non-zero \( s_2t \) weight space. Thus, in the picture, the dots in
the \( t \) chamber are connected, and are jointly connected to a dot in the \( s_2t \) chamber, since the
Corresponding basis vectors must lie in a 3-dimensional composition factor. In the first picture, \( M(t) \)
is irreducible by Theorem 3(c), so the dots are all connected.

Calibrated modules and weights. A weight \( t \) is defined to be regular if \( \mathcal{W}(t) \), the subgroup of the Weyl
group that fixes \( t \), is trivial. Then a weight \( t \) is regular if and only if \( \mathcal{Z}(t) = \emptyset \).

A representation \( M \) is calibrated if

\[
M^\text{gen}_t = M_t
\]

for all weights \( t \), i.e. the subalgebra \( C[X] \subseteq \mathcal{H} \) acts diagonally on \( M \).

Proposition 6. (See [9], Proposition 1.10.)

(a) If \( q^2 \neq 1 \), an irreducible \( \mathcal{H} \)-module is calibrated if and only if

\[
\dim(M^\text{gen}_t) = 1
\]

for all weights \( t \) of \( M \).

(b) If \( M \) is an \( \mathcal{H} \)-module with regular central character, then \( M \) is calibrated.

When \( q^2 = 1 \), all irreducible modules are calibrated, as will be shown by Theorem 9.

Calibrated modules with regular central character.

Theorem 7. (See [9], Proposition 1.11.) Assume \( q^2 \neq 1 \). Let \( t \) be a regular central character, and let \( G \) be a
component of the calibration graph. Define

\[
\mathcal{H}^{(t,G)} = \mathbb{C}\text{-span}\{v_w \mid \text{wt} \in G\}.
\]

Then the vector space \( \mathcal{H}^{(t,G)} \) is an irreducible calibrated \( \mathcal{H} \)-module with action

\[
X^\lambda \cdot v_w = (\text{wt}(X^\lambda))v_w \quad \text{for } X^\lambda \in X, \ w \in W_0, \quad \text{and}
\]

\[
T_i \cdot v_w = (T_i)_w v_w + (q^{-1} + (T_i)_w) v_{s_iw} \quad \text{for } 1 \leq i \leq n, \ w \in W_0.
\]

where \( (T_i)_w = \frac{q^{-1}}{1 - \text{wt}(X^{-\alpha_i})} \), and \( v_{s_iw} = 0 \) if \( s_i \text{wt} \notin G \).

Note that since \( t \) is a regular central character, \( \text{wt}(X^\alpha)_w \neq 1 \) for \( w \in W_0 \) and \( i = 1, \ldots, n \). Hence
\( (T_i)_w \) is always well defined. The most difficult part of this theorem is checking that the given
$H$-module structure satisfies the braid relation. Since $t$ is assumed to be regular, this essentially follows from the braid relation on the $\tau_i$. (See [9] for details.) In fact, more is true.

**Theorem 8.** (See [9], Proposition 1.11.) Assume $q^2 \neq 1$, and let $M$ be an irreducible $H$-module with regular central character $t$. (M is therefore calibrated.) Then if $wt$ is a weight of $M$, let $G$ be the component of the calibration graph containing $wt$. Then the weights of $M$ are exactly the vertices in $G$. In addition, $M$ is isomorphic to the module $H(t,G)$ given in Theorem 7.

**Clifford theory when $q^2 = 1$.** Let $q^2 = 1$. Then we can identify the subalgebra $H$ spanned by $\{T_w \mid w \in W_0\}$ with $\mathbb{C}[W_0]$, so that

$$H = \text{span}\{wX^\lambda \mid w \in W_0, \lambda \in P\}.$$  

Let $M$ be a finite-dimensional simple $H$-module and let $t \in T$ such that $M_t \neq 0$. Let $W_t$ be the stabilizer of $t$ in $W_0$. As vector spaces, $M_t \cong M_{wt}$ via the map $m \mapsto wm$, and

$$M = \bigoplus_{w \in W_0/W_t} M_{wt},$$

since $M$ is simple and the right-hand side is a submodule of $M$. (This implies that all $H$ modules are calibrated.)

**Theorem 9.** (See also [5].) Let $q^2 = 1$ and let $M$ be an irreducible $H$-module. Let $t \in T$ be such that $M_t \neq 0$. Define $H_{W_t} = \mathbb{C}\text{-Span}\{wX^\lambda \mid w \in W_t, \lambda \in P\}$, a subalgebra of $H$. Then

(a) $M_t$ is an irreducible $W_t$-module.
(b) $M_t$ is an $H_{W_t}$-module and

$$M \cong H \otimes_{H_{W_t}} M_t.$$  

Thus, when $q^2 = 1$, the standard conclusions of Clifford Theory completely describe the irreducible representations of $H$.

**3. Type $A_1$**

We begin with the type $A_1$ affine Hecke algebra. The results here are known, but this section serves as a model for the other types.

**The affine Hecke algebra.** The type $A_1$ affine Hecke algebra is built on the root data of $\text{SL}_2$. Let

$$R = \mathbb{Z}\alpha_1, \quad P = \mathbb{Z}\omega_1 \quad \text{and} \quad X = \{X^{k\omega_1} \mid k \in \mathbb{Z}\}$$

so that $X$ is the group generated by $X^{\omega_1}$ and is isomorphic to $P$. The Weyl group is $W_0 = \{1, s_1\}$ with $s_1^2 = 1$, and setting $s_1X^{\omega_1} = X^{-\omega_1}$ defines an action of $W_0$ on $X$. Let $q \in \mathbb{C}^\times$. The affine Hecke algebra of type $A_1$ is defined as in Section 2. We let $t_z$ denote the weight given by $t_z(X^{\omega_1}) = z$.

**Proposition 10.** Let $M(t_z)$ denote the principal series module for a weight $t_z$.

(a) If $z \neq \pm q^{\pm 1}$, then $M(t_z)$ is irreducible.
(b) If $z = \pm q^{\pm 1}$, then $M(t_z)$ has two 1-dimensional composition factors.
(c) If $q^2 = 1$ and $z = \pm q$, $M(t_z)$ is a direct sum of its composition factors.
Proof. Part (a) is given by Kato’s criterion (Theorem 3(c)), which also shows that $M(t)$ must be reducible if $t(X^{\omega_1}) = \pm q^{1/2}$. In parts (b) and (c), it is straightforward to explicitly calculate the action of $\mathcal{H}$ on the basis $\{v_t, T_1 v_t\}$. □

To visualize this classification, identify $\{t^{qx} \mid x \in \mathbb{R}\}$ with the real line. In this picture, the hyperplane $H_{\alpha_1}$ is marked with a solid line, while $H_{\alpha_1 \pm \delta}$ are denoted by dashed lines.

\[
\begin{array}{cccc}
H_{\alpha - \delta} & H_{\alpha} & H_{\alpha + \delta} \\
\uparrow & \uparrow & \uparrow \\
t_q^{-1} & t_q & t_q^3
\end{array}
\]

Characters $t_{qx}$, generic $q$

If $q$ is a primitive $2\ell$th root of unity then $\{t^{qx} \mid x \in \mathbb{R}\}$ is identified with $\mathbb{R}/2\ell\mathbb{Z}$ and $H_\alpha = \{k\ell \mid k \in \mathbb{Z}\}$. The following picture shows the specific case $\ell = 2$, so that $t_1 = t_q^2 = \cdots$, and $\ell = 1$, in which case $t_1 = t_q = t_q^3 = \cdots$. The periodicity is evident in the picture.

\[
\begin{array}{cccc}
H_{\alpha + \delta} & H_{\alpha} & H_{\alpha + \delta} \\
\uparrow & \uparrow & \uparrow \\
t_q & t_q & t_q^3
\end{array} \quad \begin{array}{cccc}
H_{\alpha} & H_{\alpha} & H_{\alpha} & H_{\alpha} \\
\uparrow & \uparrow & \uparrow & \uparrow \\
t_q & t_q & t_q & t_q
\end{array}
\]

Characters $t_{q^2}$, $q^4 = 1$ \quad \text{Characters} \quad \text{Characters} \quad t_{q^2}, \quad q^2 = 1

The following pictures show the chambers around $t$ as in Fig. 1, which give a picture of $M(t)$ and its composition factors.

\[
\begin{array}{c|c|c}
t_1, t_{-1}, q^2 \neq 1 & t_1, t_{-1}, q^2 = 1 & t_{q_1}, t_{-q_1}, q^2 \neq 1 \\
\end{array}
\quad \begin{array}{c|c|c}
t_2, z \neq \pm q_1, 1
\end{array}
\]

The visualization is not as clear in this case as in others, since it is the smallest example of the affine Hecke algebra, but the essential ingredients are present. The chamber pictures should be interpreted as those in Fig. 1. The weights of the $M(t)$ are all displayed, as are the actions of the $\tau$ operators that determine the composition factors of $M(t)$. Notice also the connection to the drawings of central characters above. The pictures of the $M(t_{q^2})$ are a picture of a small open neighborhood around the point $t_{q^2}$ in the picture of the characters. The complete classification of $\mathcal{H}$ modules is summarized in the following tables (see Table 1).

**Table 1**
Table of possible central characters in type $A_1$.

<table>
<thead>
<tr>
<th>Dims. of irreds. by weight</th>
<th>$t_1$</th>
<th>$t_{-1}$</th>
<th>$t_q$</th>
<th>$t_{-q}$</th>
<th>$t_{2z}, z \neq \pm 1$ or $\pm q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^4 \neq 1$</td>
<td>2</td>
<td>2</td>
<td>1.1</td>
<td>1.1</td>
<td>2</td>
</tr>
<tr>
<td>$q^2 = -1$</td>
<td>2</td>
<td>2</td>
<td>1.1</td>
<td>N/A</td>
<td>2</td>
</tr>
<tr>
<td>$q = -1$</td>
<td>1.1</td>
<td>1.1</td>
<td>N/A</td>
<td>N/A</td>
<td>2</td>
</tr>
</tbody>
</table>

The way that the representation theory of $\mathcal{H}$ varies with $q$ can be seen through a number of different lenses. The structure of $M(t)$ is controlled by the $\tau$ operators, which in turn are largely controlled by the sets $P(t)$ and $Z(t)$. In the picture of the characters, we see that the hyperplanes...
$H_\alpha$ and $H_{\alpha \pm \delta}$ are distinct unless $q^2 = \pm 1$. When these hyperplanes coincide, the sets $P(t)$ and $Z(t)$ change, changing the structure of the corresponding modules. Similar interpretations of course hold in all types.

4. Type $A_2$

The type $A_2$ root system is $R = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2) \}$, where $\langle \alpha_1, \alpha_2 \rangle = -1 = \langle \alpha_2, \alpha_1 \rangle$, with Weyl group $W_0 \cong S_3$. The simple roots are $\alpha_1$ and $\alpha_2$, and $\alpha_1 + \alpha_2$ is the only other positive root.

In this picture, $s_i$ is reflection through the hyperplane perpendicular to $\alpha_i$. The fundamental weights satisfy

$$
\omega_1 = \frac{1}{3}(2\alpha_1 + \alpha_2), \quad \alpha_1 = 2\omega_1 - \omega_2,
$$

$$
\omega_2 = \frac{1}{3}(2\alpha_2 + \alpha_1), \quad \alpha_2 = 2\omega_2 - \omega_1.
$$

Let

$$P = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\} \quad \text{and} \quad Q = \mathbb{Z}\text{-span}(R)$$

be the weight lattice and root lattice of $R$, respectively.

The affine Hecke algebra $\mathcal{H}$ is defined as in Section 2. Let

$$\mathbb{C}[Q] = \{X^\lambda \mid \lambda \in Q\} \quad \text{and} \quad T_Q = \text{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[Q], \mathbb{C}).$$

Define

$$t_{z,w} : \mathbb{C}[Q] \to \mathbb{C} \quad \text{by} \quad t_{z,w}(X^{\omega_1}) = z \quad \text{and} \quad t_{z,w}(X^{\omega_2}) = w.$$

For each $t_{z,w} \in T_Q$, there are 3 elements $t \in T$ with $t|_Q = t_{z,w}$, determined by

$$t(X^{\omega_1})^3 = z^2w \quad \text{and} \quad t(X^{\omega_2}) = t(X^{-\omega_1}) \cdot zw.$$

The dimension of the simple modules with central character $t$ and the submodule structure of $M(t)$ depends only on $t|_Q$. Thus we begin by examining the $W_0$-orbits in $T_Q$. For a generic weight $t$, $P(t)$ and $Z(t)$ are empty so that $M(t)$ is irreducible by Theorem 3(c). Thus we examine only the non-generic orbits.
Proposition 11. If \( t \in T_Q \), and \( P(t) \cup Z(t) \neq \emptyset \), then \( t \) is in the \( W_0 \)-orbit of one of the following weights:

\[
\{ t_{1,1}, t_{1,q^2}, t_{q^2,1}, t_{q^2, q^2}, \quad \{ t_{1,z} \mid z \in \mathbb{C}^* z \neq 1, \ q^{\pm 2} \}, \]

or \( \{ t_{q^2,z} \mid z \in \mathbb{C}^* z \neq 1, \ q^{\pm 2}, \ q^{-4} \} \).

Proof. The proof consists of exhausting all possibilities for \( Z(t) \) and \( P(t) \), up to the action of \( W_0 \).

Case 1: If \( Z(t) \) contains two positive roots, then it must contain the third. This implies \( t = t_{1,1} \).

Case 2: If \( Z(t) \) contains only one root, by applying an element of \( W_0 \), assume that it is \( \alpha_1 \). Then \( t(X^{\alpha_2}) = t(X^{\alpha_1 + \alpha_2}) \), so either \( P(t) = \emptyset \) or \( P(t) = \{ \alpha_2, \alpha_1 + \alpha_2 \} \). The first central character is \( t_{1,z} \) for some \( z \neq 1 \) or \( q^{\pm 2} \). (If \( z = 1 \) or \( z = q^{\pm 2} \), either \( P(t) \) or \( Z(t) \) would be larger.) For the second case, there are two potential choices for the orbit, arising from choosing \( t(X^{\alpha_2}) = q^2 \) or \( q^{-2} \). However, \( t_{1,q^{-2}} \) is in the same orbit as \( t_{q^2,1} \).

Case 3: Now assume that \( Z(t) = \emptyset \). If \( P(t) \) is not empty, assume that \( \alpha_1 \in P(t) \) and \( t(X^{\alpha_1}) = q^2 \). Then \( t(X^{\alpha_2}) \neq q^{-2} \) by assumption on \( Z(t) \). Then it is possible that \( \alpha_2 \in P(t) \), in which case \( t = t_{q^2, q^2} \).

Remark. Note that if \( q^2 = -1 \), then \( t_{1,q^2}, t_{q^2,1}, \) and \( t_{q^2,q^2} \) are all in the same \( W_0 \)-orbit. If \( q^2 = 1 \), then \( t_{1,1} = t_{q^2,1} = t_{1,q_2} = t_{q^2,q_2}, \) and \( t_{1,z} = t_{q^2,z} \). Also, for every generic weight \( t_{z,w} \), there are six weights in its \( W_0 \)-orbit, all of which are generic.

It is helpful to draw a picture of the weights \( \{ t_{q^2,x} \mid x \in \mathbb{R} \} \) for various values of \( q \). Solid lines in these pictures show the hyperplanes \( H_\alpha \), while dashed lines denote hyperplanes \( H_{\alpha \pm \delta} \), for \( \alpha \in \mathbb{R}^+ \). The weight \( t_{q^2,x} \) is the point \( x \) units away from \( H_{\alpha_1} \) and \( y \) units away from \( H_{\alpha_2} \).
Analysis of the characters.

Proposition 12. There are six 1-dimensional representations of $\mathcal{H}$. Three of these representations are given by the three weights $t$ with $t|Q = t_{q^2,q^2}$, with each $T_i$ acting with eigenvalue $q$. The other three are given by the three weights $t$ with $t|Q = t_{q^{-2},q^{-2}}$, with each $T_i$ acting with eigenvalue $-q^{-1}$.

Proof. The relation (2) determines the two possible eigenvalues for the action of $T_1$ on a 1-dimensional module. The relation in (4) relates the eigenvalues for $X_{\omega}^1$ and $T_1$. □

Principal series modules. We now examine the pictures of the chambers around a weight $t|Q$, as a way of visualizing $M(t)$. The solid, dashed and dotted hyperplanes hold the same interpretation as in Fig. 1. These hyperplanes encode the action of the $\tau$ operators, which largely determine the composition factors of $M(t)$. Assume for now that $q^2 \neq 1$.

Case 1: $P(t)$ empty. By Theorem 3(c), if $P(t) = \{ \alpha \in R^+ | t(X^\alpha) = 1 \}$ is empty, then $M(t)$ is irreducible and is the only irreducible module with central character $t$. This case includes the central characters $t_{1,1}$, $t_{1,z}$, and $t_{z,w}$ for generic $z, w$ – that is, any $z$ and $w$ for which $P(t_{z,w}) = Z(t_{z,w}) = \emptyset$.

Case 2: $Z(t) = \emptyset$, $P(t) \neq \emptyset$. This case includes the central characters $t_{q^2,q^2}$, and $t_{q^2,z}$. If $Z(t)$ is empty, then $M(t)$ is calibrated and the irreducible modules with central character $t$ are in one-to-one correspondence with the components of the calibration graph.

Case 3: $P(t) \neq \emptyset$, $Z(t) \neq \emptyset$. The only central characters with both $Z(t)$ and $P(t)$ non-empty are $t_{1,1}$ and $t_{1,z}$ when $q^2 = 1$, and $t_{q^2,1}$ and $t_{1,q^2}$ in all cases. If $q^4 = 1$, then $t_{q^2,1}$ and $t_{1,q^2}$ are in the same orbit, and are in the same orbit as $t_{q^2,1}$. If $q^2 = 1$, then $t_{q^2,1} = t_{1,q^2} = t_{1,1}$.

If $q^2 \neq \pm 1$ and $t|Q = t_{1,q^2}$ or $t_{q^2,1}$, then Theorem 5 shows that $M(t)$ has two 3-dimensional composition factors. When $q^2 = -1$ and $t|Q = t_{1,q^2}$, Proposition 4 shows the two-dimensional weight space $M(t)^{gen}_{t_{1,q^2}}$ makes up an entire composition factor, as does $M(t)^{gen}_{s_{1,s_2}}$. The remaining composition factors are two copies of the 1-dimensional module with weight $s_2 t$. 
Explicitly, let $C_{q^2,1}$ be the 1-dimensional $H_{(1)}$-module spanned by $v_t$ and let $C_{1,q^{-2}}$ be the 1-dimensional $H_{(2)}$-module spanned by $v_{s_2 s_1 t}$, given by

\[ X^{s_2} v_t = t(X^{s_2}) v_t \quad \text{and} \quad T_1 v_t = q v_t \quad \text{and} \quad X^{s_2} v_{s_2 s_1 t} = (s_2 s_1 t)(X^{s_2}) v_{s_2 s_1 t} \quad \text{and} \quad T_1 v_{s_2 s_1 t} = -q^{-1} v_{s_2 s_1 t}. \]

Then

\[ M = H \otimes H_{(1)} C_{q^2,1} \quad \text{and} \quad N = H \otimes H_{(2)} C_{1,q^{-2}} \]

are 3-dimensional $H$-modules with central character $t_{q^2,1}$.

**Proposition 13.** Let $M = H \otimes H_{(1)} C_{q^2,1}$ and $N = H \otimes H_{(2)} C_{1,q^{-2}}$.

(a) If $q^4 \neq 1$ then $M$ and $N$ are irreducible.

(b) If $q^2 = -1$ then $M_{s_1 t}$ is an irreducible submodule of $M$ and $N_{s_1 t}$ is an irreducible submodule of $N$. The quotients $N/N_{s_1 t}$ and $M/M_{s_1 t}$ are irreducible.

**Proof.** (a) Assume $q^4 \neq 1$. If either $M$ or $N$ were reducible, it would have a 1-dimensional submodule or quotient, which cannot happen since the 1-dimensional modules have central character $t_{q^2,q^2}$. Thus both $M$ and $N$ are reducible.

(b) If $t|_Q = t_{q^2,1}$, then the action of $\tau_1$ is non-zero on $M^\text{gen}_t$ by Proposition 1, and $M^\text{gen}_t$ is not a submodule of $M$. But $M$ is not irreducible, and the only possible remaining submodule is $M_{s_1 t}$. A similar argument shows the result for $N$ as well. \(\square\)

The modules with central character $t$ such that $t|_Q = t_{1,q^2}$ can be constructed in an entirely analogous fashion, for $q^2 \neq 1$. Finally, if $q^2 = 1$, then Theorem 9 suffices to classify the representations of $H$ with central characters $t_{1,1}$ and $t_{1,z}$ for $z \neq q^{\pm 2}$.

**Summary.** Table 2 summarizes the classification.
Table 2
Table of possible central characters in type $A_2$.

<table>
<thead>
<tr>
<th>Dims. of irreds. by weight</th>
<th>$t_{1,1}$</th>
<th>$t_{1,2}$</th>
<th>$t_{1,q^2}$</th>
<th>$t_{q^2,1}$</th>
<th>$t_{q^2,q^2}$</th>
<th>$t_{q^2,z}$</th>
<th>$t_{z,w}$, $z, w \neq \pm 1$ or $q^{\pm 2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q^6 \neq 1$, $q^4 \neq 1$</td>
<td>6</td>
<td>6</td>
<td>3, 3</td>
<td>3, 3</td>
<td>1, 1, 2, 2</td>
<td>3, 3</td>
<td>6</td>
</tr>
<tr>
<td>$q^6 = 1$</td>
<td>6</td>
<td>6</td>
<td>3, 3</td>
<td>3, 3</td>
<td>1, 1, 1, 1, 1, 1</td>
<td>3, 3</td>
<td>6</td>
</tr>
<tr>
<td>$q^2 = -1$</td>
<td>1, 1, 2</td>
<td>3, 3</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>$q = -1$</td>
<td>1, 1, 2</td>
<td>3, 3</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
</tbody>
</table>

5. Type $C_2$

The type $C_2$ root system is $R = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm (2\alpha_1 + \alpha_2) \}$, where $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ and $\langle \alpha_2, \alpha_1^\vee \rangle = -2$. Then the Weyl group is $W_0 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1s_2s_1s_2 = s_2s_1s_2s_1 \rangle$, which is isomorphic to the dihedral group of order 8. The simple roots are $\alpha_1$ and $\alpha_2$, with additional positive roots $\alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$. Then the action of $W_0$ on $R$ can be seen in the following picture, where $s_i$ acts by reflection through $H_{\alpha_i}$, the hyperplane perpendicular to $\alpha_i$.

$$\alpha_2 \quad \alpha_1 \quad \alpha_1 + \alpha_2 \quad \alpha_2 + 2\alpha_1$$

The type $C_2$ root system

The fundamental weights satisfy

$$\omega_1 = \alpha_1 + \frac{1}{2}\alpha_2, \quad \alpha_1 = 2\omega_1 - \omega_2,$$
$$\omega_2 = \alpha_1 + \alpha_2, \quad \alpha_2 = 2\omega_2 - 2\omega_1.$$ 

Let

$$P = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\} \quad \text{and} \quad Q = \mathbb{Z}\text{-span}(R)$$

be the weight lattice and root lattice of $R$, respectively.

Then the affine Hecke algebra $\mathcal{H}$ is defined as in Section 2.

The torus $T_Q$
For all weights $t_{z,w} \in T_Q$, there are 2 elements $t \in T$ with $t|_Q = t_{z,w}$, determined by

$$t(X^{w_1})^2 = z^2 w \quad \text{and} \quad t(X^{w_2}) = zw.$$ 

We denote these two elements as $t_{z,w,1}$ and $t_{z,w,2}$. Which particular weight $t_{z,w,i}$ is which is unimportant since we will always be examining them together. And in fact, most of the time, we will only refer to the restricted weight $t_{z,w}$, since the dimension of the modules with central character $t$ depends only on $t|_Q$. One important remark, though, is that if $t(X^{w_1}) = -1$, then the two weights $t$ with $t|_Q = t_{z,w}$ are in the same $W_0$-orbit and represent the same central character.

We begin by examining the $W_0$-orbits in $T_Q$. The structure of the modules with weight $t$ depends virtually exclusively on $P(t) = \{ \alpha \in R^+ \mid t(X^{\alpha}) = q^{\pm 2} \}$ and $Z(t) = \{ \alpha \in R^+ \mid t(X^{\alpha}) = 1 \}$. For a generic weight $t$, $P(t)$ and $Z(t)$ are empty, so we examine only the non-generic orbits.

**Proposition 14.** Let $q$ be generic. If $t \in T_Q$, and $P(t) \cup Z(t) \neq \emptyset$, then $t$ is in the $W_0$-orbit of one of the following weights:

$$t_{1,1}, \ t_{-1,1}, \ t_{1,q^2}, \ t_{q^2,1}, \ t_{\pm 1, q^2}, \ t_{1,q^2}, \ t_{-1,q^2}, \ \{ t_{1,z} \mid z \neq 1, \ q^{\pm 2}, \ \pm q^{\pm 1} \},$$

$$\{ t_{q^2,z} \mid z \neq 1, \ q^{\pm 2}, \ q^{-4}, q^{-6} \}, \ \text{or} \ \{ t_{z,q^2} \mid z \neq 1, \ q^{\pm 2}, \ -q^{-2}, q^{-4}, \ \pm q^{-1} \}.$$ 

**Proof.** The proof consists of exhausting all possibilities for $Z(t)$ and $P(t)$, up to the action of $W_0$. In the following, we refer to $\alpha_1$ and $\alpha_1 + \alpha_2$ as “short” roots, and $\alpha_2$ and $2\alpha_1 + \alpha_2$ as “long” roots.

**Case 1:** $|Z(t)| \geq 2$.

If $Z(t)$ contains a short root and any other root, then $t = t_{1,1}$. If $Z(t)$ contains two long roots, then $t = t_{-1,1}$.

**Case 2:** $|Z(t)| = 1$.

If $Z(t)$ contains exactly one root, we may assume it is either $\alpha_1$ or $\alpha_2$. If $t(X^{\alpha_1}) = 1$, then $t(X^{\alpha_2}) = t(X^{\alpha_1 + \alpha_2}) = t(X^{2\alpha_1 + \alpha_2})$. Thus either $P(t) = \{ \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 \}$, or $P(t) = \emptyset$. That is, $t$ is in the orbit of $t_{1,q^2}$ or $t_{1,z}$ for some $z \neq q^{\pm 2}, 1$. If $t(X^{\alpha_2}) = 1$, then $t(X^{\alpha_1}) = t(X^{\alpha_1 + \alpha_2})$. Then either $P(t) = \{ \alpha_1, \alpha_1 + \alpha_2 \}$, $P(t) = \{ 2\alpha_1 + \alpha_2 \}$, or $P(t) = \emptyset$. These are the orbits of $t_{q^2,1}$, $t_{q^2,1}$, and $t_{1,z}$, respectively, where $z \neq q^{\pm 2}, \pm q^{\pm 1}$, or 1.

**Case 3:** $Z(t) = \emptyset$.

Now assume that $Z(t)$ is empty, and $P(t)$ is not empty. First assume that $P(t)$ contains at least one short root. We can apply an element of $w$ to assume that $\alpha_1 \in P(t)$ and $t(X^{w_1}) = q^2$. Then if $\alpha_2 \in P(t)$, we must have $t(X^{w_2}) = q^2$ or else $Z(t)$ would be non-empty. Thus $t = t_{q^2, q^2}$. If $\alpha_1 + \alpha_2 \in P(t)$, then $t(X^{\alpha_1 + \alpha_2}) = q^{\pm 2}$, so that either $t(X^{\alpha_2}) = 1$ or $t(X^{2\alpha_1 + \alpha_2}) = 1$. If $2\alpha_1 + \alpha_2 \in P(t)$, then $t(X^{2\alpha_1 + \alpha_2}) = q^{-2}$ or else $\alpha_1 + \alpha_2 \in Z(t)$. Hence $t(X^{\alpha_1}) = q^{-6}$. But then $s_1s_2t = t_{q^2, q^2}$. If $P(t) = \{ \alpha_1 \}$, then $t = t_{q^2, z}$ for some $z \neq 1, \ pmq^1, q^{-4}, q^{-6}$.

Now, assume that $P(t)$ contains a long root but no short roots. Then we may assume that $t(X^{\alpha_1}) = q^2$. If $t(X^{2\alpha_1 + \alpha_2}) = q^2$ then $t(X^{\alpha_1}) = 1$. If $t(X^{2\alpha_1 + \alpha_2}) = q^{-2}$ then $t(X^{\alpha_1}) = q^{-2}$. However, $s_1s_2s_1t_{q^{-2}, q^2} = t_{-1, q^2}$. Thus $t$ is in the same orbit as $t_{-1, q^2}$ or $t_{z, q^2}$ for $z \neq 1, \ pmq^1, q^{-4}, q^{-2}, \ pmq^{-1}$.

**Remark.** If $q^2$ is a root of unity of order less than or equal to 4, there is redundancy in the list of characters given above. Essentially, this is a result of the periodicity in $T_Q$ when $q^2$ is an $\ell$th root of unity. If $q^2$ is a primitive fourth root of unity, then $t_{q^2, q^2} = s_1s_2s_1t_{-1, q^2}$.

If $q^2$ is a primitive third root of unity, $t_{q^2, q^2} = s_2s_1s_2t_{q^2, q^2}$. Also, one of $t_{q^2, 1}$ and $t_{-q^2, 1}$ and is in the same orbit as $t_{q^2, 1}$. (Which one it is depends on whether $q^2 = 1$ or $-1$. In either case, $t_{q^2+1, 1}$ is in a different orbit than $t_{q^2, 1}$, so $t_{q^2+1, 1}$ is our preferred notation for this character.)
If $q^2 = -1$, then $t_{-1,1} = t_{q^2,1}$, and $t_1 q^2$ is in the same orbit as $t_{q^2,1} = t_{-1,q^2}$. Also in this case, $t_{z,q^2} = t_{z,-1} = 52 t_{-z,-1}$. Finally, if $q = -1$, we have $t_{1.1} = t_{1,q^2} = t_{q^2,q^2} = t_{q^2,1}$. Also, $t_{-1,1} = t_{-1,q^2}$, while $t_{1,q^2} = t_{1,z}$ and $t_{z,q^2} = t_{z,1}$.

Analysis of the characters.

**Proposition 15.** There are eight 1-dimensional representations of $H$, one for each weight $t$ with $t|_Q = t_{q^2,1}$. In each of these representations, $T_i$ acts with eigenvalue $q$ or $-q^{-1}$ when $t(X^i) = q^2$ or $q^{-2}$, respectively.

**Proof.** As in Proposition 12. □

**Remark.** We will use the notation $L_{z,w,i}$ to denote the 1-dimensional representation with weight $t_{z,w,i}$, where each of $z$ and $w$ is either $q$ or $-q^{-1}$. Note that if $q$ is a primitive fourth root of unity, then $L_{q,w,i} \cong L_{-q^{-1},-q^{-1},3-i}$ for $i = 1$ or 2.

**Principal series modules.** A weight $t|_Q$ corresponds to a point in the root lattice $Q$ as described above. The composition structure of the principal series module $M(t)$ is largely determined by the structure of the operators $T_i$, which can be encoded in the following pictures of small neighborhoods of the various points $t$ in $Q$. The solid lines are the hyperplanes $H_{\alpha_i}$, while the dashed lines represent the hyperplanes $H_{\alpha_i \pm \delta}$. Thus the hyperplanes in the picture of the neighborhood of $t$ show which $T_i$ operators are invertible and which are not (or are not well defined). In most cases, this is enough to determine the exact composition factors of $M(t)$.

**Case 1:** $P(t) = \emptyset$.

If $P(t) = \emptyset$, then by Kato’s criterion (Theorem 3(c)), $M(t)$ is irreducible and is the only irreducible module with central character $t$. The weights of $M(t)$ are in bijection with $W_0/W_t$, the cosets of
the centralizer of $t$ in $W$, and $\dim(M(t)_{wt}) = |W_t|$. If $w$ and $s_iw$ are distinct weights in $M(t)$, then $\tau_i : M(t)_t \rightarrow M(t)_{s_it}$ is a bijection.

Case 2: $Z(t) = \emptyset$, but $P(t) \neq \emptyset$.

If $Z(t) = \emptyset$ then $t$ is a regular central character. Then the irreducibles with central character $t$ are in bijection with the connected components of the calibration graph for $t$, and can be constructed using Theorem 7.

Case 1:

Case 2:

$t_{-1,1}, q \neq 1$

$t_{1,1}, q^2 \neq 1$

$t_{1,2}, q^2 \neq 1$

$t_{2,1}, q^2 \neq 1$

$t_{z, w}$

$t_{q^2, q^2}, q \text{ generic}$

$t_{q^2, q^2}$, $q$ a primitive eighth root of unity
Case 3: $Z(t) \neq \emptyset, P(t) \neq \emptyset$.

The only central characters not covered in Cases 1 and 2 are those in the orbits of $t_{1,q^2}, t_{q^2,1},$ and $t_{\pm q,1}$.

If $q^2 = 1$, then Theorem 9 shows that $\mathcal{H}$ has five irreducible representations - four of them 1-dimensional, and one 2-dimensional.

Assume $q^2 \neq 1$ and let

\[
\begin{aligned}
  w_1 &= \left\{ 
    \begin{array}{ll}
      s_1 & \text{if } q^2 = -1, \\
      s_1s_2s_1 & \text{if } q^2 \neq -1.
    \end{array}
  \right.
\end{aligned}
\]

Then let $C_{q^2,1}$ and $C_{q^{-2},1}$ be the 1-dimensional $\mathcal{H}_{\{1\}}$-modules spanned by $v_t$ and $v_{w_1t}$, respectively, given by

\[
X^\lambda v_t = t(X^\lambda)v_t \quad \text{and} \quad T_1 v_t = q v_t, \quad \text{and}
\]

\[
X^\lambda v_{w_1t} = (w_1t)(X^\lambda)v_{w_1t} \quad \text{and} \quad T_1 v_{w_1t} = -q^{-1} v_{w_1t}.
\]

Then

\[
M = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} C_{q^2,1} \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} C_{q^{-2},1}
\]

are 4-dimensional $\mathcal{H}$-modules.

**Proposition 16.** If $q^2 = -1$ and $M = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} C_{q^2,1}$ and $N = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} C_{q^{-2},1}$ then

(a) $M$ is irreducible, and

(b) The map

\[
\phi : N \rightarrow M,
\]

\[
h v_{w_1t} \mapsto h v, \quad \text{for } h \in \mathcal{H}
\]

is an $\mathcal{H}$-module isomorphism, where $v = T_1T_2v_t - qT_2v_t - v_t \in M$, and

(c) Any irreducible $\mathcal{H}$-module $L$ with central character $t$ is isomorphic to $M$.

**Proof.** (a) If $q$ is a primitive fourth root of unity, then $M$ has weight spaces $M_{t}^{\text{gen}}$ and $M_{s_1t}^{\text{gen}}$, each of which is 2-dimensional. By Theorem 5 and Proposition 4, $M$ is irreducible.
(b) Let \( v = T_1 T_2 v_t - qT_2 v_t - v_t \). Then a straightforward computation using equation (4) shows that \( v \) spans a 1-dimensional \( \mathcal{H}_{(1)} \)-submodule of \( M \), given by
\[
T_1 v = q v, \quad \text{and} \quad X^t v = s_1 t(X^t) v.
\]
Then the \( \mathcal{H}_{(1)} \)-module map given by \( v_{wt} \mapsto v \) corresponds to \( \phi \) under the adjunction
\[
\text{Hom}_{\mathcal{H}}(\mathcal{H} \otimes \mathcal{H}_{(1)} \mathbb{C}_{q^{-2},1}, M) = \text{Hom}_{\mathcal{H}_{(1)}}(\mathbb{C}_{q^{-2},1}, M|_{\mathcal{H}_{(1)}}).
\]
Thus \( \phi \) is an \( \mathcal{H} \)-module map and since \( M \) is irreducible, the map is surjective. Since \( M \) and \( N \) have the same dimension, \( \phi \) is an isomorphism.

(c) Let \( L \) be an irreducible \( \mathcal{H} \)-module with central character \( t \), which must have weights \( t \) and \( s_1 t \). Then, viewing \( L \) as an \( \mathcal{H}_{(1)} \)-module, it must have all 1-dimensional composition factors, and it must have a 1-dimensional \( \mathcal{H}_{(1)} \)-submodule, with weight \( t \) or \( s_1 t \). Then the same argument as in part (b) gives an isomorphism from \( M \) to \( L \) or from \( N \) to \( L \). \( \square \)

**Proposition 17.**

(a) If \( q^2 \) is a primitive third root of unity then \( M_{s_1 t} \) is a submodule of \( M \) isomorphic to \( L_{-q^{-1},q, \pm 1} \) and \( M/M_{s_1 t} \) is irreducible. In addition, \( N_{s_1 t} \) is a submodule of \( N \) isomorphic to \( L_{q, -q^{-1}} \), and \( N/N_{s_1 t} \) is irreducible.

(b) If \( q^2 \) is not \( \pm 1 \) or a primitive third root of unity then \( M \) and \( N \) are irreducible and nonisomorphic.

**Proof.** (a) Assume \( q^2 \) is a primitive third root of unity. Then Proposition 1 shows that \( t_2 : M_{s_1 t} \to M_{s_1 t} \) is non-zero. But \( s_2 t_1 (X^{q^2}) = q^2 \) so that \( t_2 : M_{s_1 t} \to M_{s_1 t} \) is the zero map by Theorem 2, and \( M_{s_1 t} \) is a submodule of \( M \). By Theorem 5, \( M/M_{s_1 t} \) is irreducible. A parallel argument shows that \( N_{s_1 t} \) is a submodule of \( N \), with \( N/N_{s_1 t} \) irreducible.

(b) If \( q^2 \neq 1 \) and \( q^2 \neq 1 \), then \( P(t) = \{ \alpha_1, \alpha_1 + \alpha_2 \} \). Then Theorem 5 shows that the composition factor \( M' \) of \( M \) with \( \langle M' \rangle_t = 0 \) has \( \dim(M')_{t_{\text{gen}}} \geq 2 \) and \( \langle M' \rangle_{s_2 t_{1}} \neq 0 \). Then by Theorem 3(b), \( \langle M' \rangle_{s_2 t_{1}} \neq 0 \), so that \( M' = M \). Similarly, Theorem 5 and Theorem 3(b) show that \( N \) is irreducible. Since they have different weight spaces, they are not isomorphic. \( \square \)

Note that if \( q^2 = 1 \), then \( t_{1,q^2} = t_{1,1} = t_{1,1} \), so this case has already been addressed.

Let \( C_{1,q^2} \) and \( C_{1,q^{-2}} \) be the 1-dimensional \( \mathcal{H}_{(2)} \)-modules spanned by \( v_t \) and \( v_{wt} \), respectively, and given by
\[
T_2 v_t = q v_t \quad \text{and} \quad X^t v_t = t(X^t) v_t, \quad \text{and}
\]
\[
T_2 v_{wt} = -q^{-1} v_{wt} \quad \text{and} \quad X^t v_{wt} = w_0 t(X^t) v_{wt}.
\]
Then

\[ M = \mathcal{H} \otimes_{\mathcal{H}_{(2)}} C_{1, q^2} \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{(2)}} C_{1, q^{-2}} \]

are 4-dimensional \( \mathcal{H} \)-modules.

**Proposition 18.** Assume \( q^2 = -1 \) and let \( M = \mathcal{H} \otimes_{\mathcal{H}_{(2)}} C_{1, q^2} \) and \( N = \mathcal{H} \otimes_{\mathcal{H}_{(2)}} C_{1, q^{-2}} \). Then

(a) \( M_{s_2t} \) is a submodule of \( M \), and the image of \( M_{s_2t} \) is a submodule of \( M/M_{s_1s_2t} \). The resulting 2-dimensional quotient of \( M \) is irreducible. Also, \( N_{s_2t} \) is a submodule of \( N \) and the image of \( N_{s_2t} \) in \( N/N_{s_2t} \) is a submodule of \( N/N_{s_2t} \). The resulting 2-dimensional quotient of \( N \) is irreducible, and

(b) Any composition factor of \( M(t) \) is a composition factor of either \( M \) or \( N \).

**Proof.** (a) If \( q^2 = -1 \), then \( M \) has weight spaces \( M_{t}^{\text{gen}} \), which is two-dimensional, and \( M_{s_2t} \) and \( M_{s_1s_2t} \), both of which are 1-dimensional. Proposition 1 and Theorem 2 show that \( \tau_1 \) is non-zero on \( M_{s_2t} \), but zero on \( M_{s_1s_2t} \), so that \( M_{s_1s_2t} \) is a submodule of \( M \). The resulting quotient must be reducible, but since \( \nu_t \) generates all of \( M \), the generalized \( t \) weight space cannot be a submodule. Thus the \( s_2t \) weight space is the submodule, and its quotient must be the 2-dimensional module constructed in Proposition 4, since it accounts for the entire \( t \) weight space of \( M(t) \). A similar argument shows the result for \( N \).

(b) By counting dimensions of weight spaces, the remaining composition factor(s) of \( M(t) \) must have weights \( s_2t \) and \( s_1s_2t \). If there were only one composition factor \( L \) left, it would contain both weight spaces which would each have dimension 1, which is impossible by Proposition 4. Thus the remaining composition factors are more copies of the 1-dimensional modules. \( \Box \)

**Proposition 19.** Assume \( q^2 \neq \pm 1 \). Then \( M_{s_1s_2t} \) is a submodule of \( M \) and \( M/M_{s_1s_2t} \) is irreducible. Similarly, \( N_{s_2t} \) is a submodule of \( N \) and \( N/N_{s_2t} \) is irreducible.
Proof. If $q^4 \neq 1$, then by the same reasoning as in Proposition 18, $M_{s_1 s_2 t}$ must be a submodule of $M$. Similarly, $N_{s_1 s_2 t}$ is a submodule of $N$. Then Theorem 5 shows that the resulting 3-dimensional quotients of $M$ and $N$ are irreducible. \qed

If $q^2 \neq 1$, the composition factors of $M$ and $N$ account for all 8 dimensions of $M(t)$. Let $C_{\pm q, 1}$ and $C_{\pm q^{-1}, 1}$ be the 1-dimensional $\mathcal{H}(2)$-modules spanned by $v_{s_1 t}$ and $v_{s_2 s_1 t}$, respectively, and given by

$$T_2 v_{s_1 t} = q v_{s_1 t} \quad \text{and} \quad X^i v_{s_1 t} = s_1 t (X^i) v_{s_1 t},$$

$$T_2 v_{s_2 s_1 t} = -q^{-1} v_{s_2 s_1 t} \quad \text{and} \quad X^i v_{s_2 s_1 t} = s_2 s_1 t (X^i) v_{s_2 s_1 t}.$$

Then

$$M = \mathcal{H} \otimes \mathcal{H}(2) C_{\pm q, 1} \quad \text{and} \quad N = \mathcal{H} \otimes \mathcal{H}(2) C_{\pm q^{-1}, 1}$$

are 4-dimensional $\mathcal{H}$-modules.

If $t|Q = t_{\pm q, 1}$ and $q$ is a primitive sixth root of unity or if $t|Q = t_{q, 1}$ and $q$ is a primitive third root of unity, then $t|Q = t_{q^2, 1}$, which is in the same orbit as $t_{q^{-2}, 1}$, and the irreducibles with central character $t$ have already been analyzed.

Proposition 20. Let $M = \mathcal{H} \otimes \mathcal{H}(2) C_{\pm q, 1}$ and $N = \mathcal{H} \otimes \mathcal{H}(2) C_{\pm q^{-1}, 1}$. Unless $t|Q = t_{\pm q, 1}$ and $q$ is a primitive sixth root of unity or $t|Q = t_{q, 1}$ and $q$ is a primitive third root of unity, $M$ and $N$ are irreducible.

Proof. By assumption, $P(t) = \{2\alpha_1 + \alpha_2\}$. Then the claim follows from Theorem 5. \qed

Since they have different weight spaces and are thus not isomorphic, $M$ and $N$ are the only two irreducibles with central character $t$.

Summary. Table 3 summarizes the classification. Note that in this table, the entries for $t_{\pm q, 1}$ assume that $\pm q \neq q^{-2}$, as described above.

6. Type $G_2$

The type $G_2$ root system is

$$R = \{ \pm \alpha_1, \pm \alpha_2, \pm (\alpha_1 + \alpha_2), \pm (2\alpha_1 + \alpha_2), \pm (3\alpha_1 + \alpha_2), \pm (3\alpha_1 + 2\alpha_2) \}.$$
Table 3
Table of possible central characters in type $C_2$, with varying values of $q$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$q^8 \neq 1, q^6 \neq 1$</th>
<th>$q^6 = 1, q^4 \neq 1$</th>
<th>$q^6 = 1$</th>
<th>$q^3 = -1$</th>
<th>$q = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{1,1}$</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>1, 1, 1, 1, 2</td>
</tr>
<tr>
<td>$t_{-1,1}$</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>4</td>
<td>2, 2, 2</td>
</tr>
<tr>
<td>$t_{1,z}$</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>4, 4</td>
</tr>
<tr>
<td>$t_{1,q^2}$</td>
<td>1, 1, 3, 3</td>
<td>1, 1, 3, 3</td>
<td>1, 1, 3, 3</td>
<td>1, 1, 2, 2</td>
<td>N/A</td>
</tr>
<tr>
<td>$t_{q^2,1}$</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>N/A</td>
</tr>
<tr>
<td>$t_{q,1}$</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>N/A</td>
</tr>
<tr>
<td>$t_{-q,1}$</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>N/A</td>
</tr>
<tr>
<td>$t_{1}$</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>4, 4</td>
</tr>
<tr>
<td>$t_{q^2,q^2}$</td>
<td>1, 1, 3, 3</td>
<td>1, 1, 1, 2</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>$t_{q^2,z}$</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>N/A</td>
</tr>
<tr>
<td>$t_{-q^2}$</td>
<td>2, 2, 2</td>
<td>2, 2, 2</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>$t_{z,q^2}$</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>4, 4</td>
<td>N/A</td>
</tr>
<tr>
<td>$t_{z,w}$</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

with $\langle \alpha_1, \alpha_2' \rangle = -1$ and $\langle \alpha_2, \alpha_2' \rangle = -3$. Then the Weyl group is

$$W_0 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1 \rangle,$$

isomorphic to the dihedral group of order 12. The simple roots are $\alpha_1$ and $\alpha_2$, and $\alpha_1, \alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2$ will be referred to as short roots, while $\alpha_2, 2\alpha_1 + \alpha_2$, and $3\alpha_1 + 2\alpha_2$ will be referred to as long roots.

The fundamental weights satisfy

$$\omega_1 = 2\alpha_1 + \alpha_2, \quad \alpha_1 = 2\omega_1 - \omega_2,$$
$$\omega_2 = 3\alpha_1 + 2\alpha_2, \quad \alpha_2 = 2\omega_2 - 3\omega_1.$$

Let

$$P = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\}.$$
This is the same lattice spanned by $\alpha_1$ and $\alpha_2$. Then $W_0$ acts on $X$ by

$$s_1 \cdot X^{\omega_1} = X^{\omega_2 - \omega_1},$$
$$s_1 \cdot X^{\omega_2} = X^{\omega_2},$$
$$s_2 \cdot X^{\omega_1} = X^{\omega_1}, \quad \text{and}$$
$$s_2 \cdot X^{\omega_2} = X^{3\omega_1 - \omega_2}.$$

The affine Hecke algebra $H$ of type $G_2$ is defined as in Section 2.

Let $T = \text{Hom}_{C\text{-alg}}(C[X], C)$ and define

$$t_{z, w} : T \to C \quad \text{by} \quad t_{z, w}(X^{\alpha_1}) = z \quad \text{and} \quad t_{z, w}(X^{\alpha_2}) = w.$$

The structure of the modules with weight $t$ depends virtually exclusively on $P(t)$ and $Z(t)$. For a generic weight $t$, $P(t)$ and $Z(t)$ are empty, so we examine only the non-generic orbits.

**Theorem 21.** If $q^2$ is not a primitive $\ell$th root of unity for $\ell \leq 6$ and $Z(t) \cup P(t) \neq \emptyset$, then $t$ is in the same $W_0$-orbits as one of the following weights.

$$t_1, 1, t_{1, -1}, t_{1/3, 1}, t_{1, q^2}, t_{1, \pm q}, t_{q^2, 1}, t_{\pm q, 1}, t_{q^2/3, 1}, t_{q^2, -q^{-2}}, t_{1/3, q^2}, t_{q^2, q^2},$$

$$\{t_{1, z} \mid z \in \mathbb{C}^\times, z \neq \pm 1, q^{\pm 2}, \pm q^{\pm 1}\}, \quad \{t_{z, 1} \mid z \in \mathbb{C}^\times, z \neq \pm 1, 1^{1/3}, q^{\pm 2}, \pm q^{\pm 1}, q^{\pm 2/3}\},$$

The structure of the modules with weight $t$ depends virtually exclusively on $P(t)$ and $Z(t)$. For a generic weight $t$, $P(t)$ and $Z(t)$ are empty, so we examine only the non-generic orbits.

**Theorem 21.** If $q^2$ is not a primitive $\ell$th root of unity for $\ell \leq 6$ and $Z(t) \cup P(t) \neq \emptyset$, then $t$ is in the same $W_0$-orbits as one of the following weights.
\[ \{ t_{q^2, z} \mid z \in \mathbb{C}^\times, \{ 1, q^2, q^{-2} \} \cap \{ z, q^2 z, q^4 z, q^6 z, q^8 z^2 \} = \emptyset \}, \text{ or} \]
\[ \{ t_{z, q^2} \mid z \in \mathbb{C}^\times, \{ 1, q^2, q^{-2} \} \cap \{ z, q^2 z, q^2 z^2, q^2 z^3, q^2 z^4 \} = \emptyset \}. \]

**Proof.** In general, the third roots of unity in this theorem are assumed to be primitive, so that there are two different weights that we call \( t_{1/3, 1} \) and \( t_{1/3, q^2} \). Similarly, \( t_{q^2/3, 1} \) typically refers to one of three different characters, corresponding to the third roots of \( q^2 \). We refer to \( \alpha_1, \alpha_1 + \alpha_2, \) and \( 2\alpha_1 + \alpha_2 \) as “short” roots. The other roots are referred to as “long” roots.

**Case 1:** \( |Z(t)| \geq 2 \).

If \( Z(t) \) contains at least two roots, and one of them is short, then \( Z(wt) \) contains \( \alpha_1 \) for some \( w \in W_0 \). If \( Z(wt) \) also contains any of \( \alpha_2, \alpha_1 + \alpha_2, \) or \( 3\alpha_1 + \alpha_2, \) then \( wt = t_{11} \). It is also possible that \( Z(wt) = \{ \alpha_1, 3\alpha_1 + 2\alpha_2 \} \), in which case \( wt(X^q(t) + \alpha_2) = -1 \), and \( wt(X^q) = -1 \), so that \( wt = t_{11} \). If \( Z(t) \) contains no short roots, then \( \{ \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \} \). Then it must also contain the third, and \( Z(t) = \{ \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \} \). In this case, \( wt(X^q(t)) = 1 \), but \( wt(X^q(t)) \) is a third root of unity, so that \( wt = t_{11/3} \).

**Case 2:** \( |Z(t)| = 1 \).

If \( Z(t) \) has exactly one root, then there is some \( w \in W_0 \) with \( Z(wt) = \{ \alpha_1 \} \) or \( Z(wt) = \{ \alpha_2 \} \).

If \( Z(wt) = \{ \alpha_1 \} \), then \( P(t) \) either contains all of \( \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \) or \( 3\alpha_1 + \alpha_2, \) or it contains none of them. If it contains all of them, \( wt(X^q(t)) = q^2 \), and \( t \) is in the same \( W_0 \)-orbit as \( t_{11} \). If \( \alpha_1 + 2\alpha_2 \in P(wt) \), then \( wt(X^q(t)) = \pm q^\pm 1 \), and \( t \) is in the same orbit as \( t_{11} \). Otherwise, \( wt = t_{11} \) for some \( z \) besides \( \pm q^\pm 1 \) and \( q^{\pm 2} \). Also then, \( z \neq \pm 1 \) by assumption on \( Z(t) \).

If \( Z(wt) = \{ \alpha_2 \} \), then any two roots that differ by a multiple of \( \alpha_2 \) are either both in \( P(wt) \) or both not in \( P(wt) \). By applying \( w_0 \) if necessary, we can assume \( wt(X^q(t)) = q^2 \) for the \( \alpha \) that are in \( P(wt) \). If \( \alpha_1 \in P(wt) \), then \( wt(X^q(t)) = q^2 \), and \( wt = t_{q^2/2} \). If \( \alpha_1 + \alpha_2 \in P(wt) \), then \( wt = t_{q^2/2} \). If \( \alpha_1 + \alpha_2 \in P(wt) \), then \( wt(X^q(t)) = q^2 \) and \( wt = t_{q^2/3} \). Otherwise, \( wt = t_{q^2/2} \) for some \( z \) so that none of \( z, z^2, z^3 \) are equal to \( q^\pm 1 \) or \( q^\pm 2 \). That is, \( z \neq \pm 1, \pm 1^{1/3}, q^\pm 2, \pm q^\pm 1, q^\pm 2/3 \).

**Case 3:** \( |Z(t)| = \emptyset \).

If \( Z(t) \) is empty but \( P(t) \) contains a short root, then \( \alpha_1 \in P(wt) \) for some \( w \in W_0 \). If \( P(wt) \) contains another short root, then we can apply \( s_1 \) if necessary so that \( P(t) \) contains \( \alpha_1 \) and \( \alpha_1 + \alpha_2 \). Then \( wt(X^q(t)) = wt(X^q(t) + \alpha_2) \), so that \( wt(X^q(t)) = 1 \), or \( wt(X^q(t)) \) are \( q^2 \) and \( q^{-2} \) in some order, so that \( wt(X^q(t)+\alpha_2) = 1 \). Thus \( P(wt) \) contains at most one short root. If \( P(wt) \) also contains a long root, then applying \( s_1 \) if necessary, we can assume \( P(wt) \) contains either \( \alpha_2 \) or \( 3\alpha_1 + 2\alpha_2 \). If \( P(wt) \) contains \( \alpha_1 \) and \( \alpha_2 \), then we can apply \( w_0 \) to assume \( wt(X^q(t)) = q^2 \). If \( wt(X^q(t)) = q^2 \), then \( \alpha_1 + \alpha_2 \in Z(wt) \). Then \( wt(X^q(t)) = q^2 \) and \( wt = t_{q^2/2} \). If \( P(wt) \) contains \( \alpha_1 \) and \( 3\alpha_1 + 2\alpha_2 \), then \( \alpha_1 \) is perpendicular to \( 3\alpha_1 + 2\alpha_2 \), we can apply \( s_1 \) and/or \( s_{3\alpha_1+2\alpha_2} \) to assume \( wt(X^q(t)) = q^2 \) and \( wt(X^q(t)+\alpha_2) \). Hence \( wt(X^q(t)) = q^2 \) and by assumption, \( wt = t_{q^2/2} \). If \( P(wt) = \{ \alpha_1 \} \), then \( wt = t_{q^2/2} \) does not take the value 1 or \( q^\pm 2 \) on any other positive root. Then \( \{ 1, q^2, q^{-2} \} \cap \{ z, q^2 z, q^4 z, q^6 z, q^8 z^2 \} = \emptyset. \)

If \( P(wt) \) contains no short roots, but at least two long roots, then \( wt(X^q(t)) = q^2 \) and \( wt(X^q(t)+\alpha_2) \) for some \( w \in W_0 \). If \( wt(X^q(t)) = q^2 \), then \( wt(X^q(t)+2\alpha_2) = 1 \), a contradiction. Hence \( wt(X^q(t)) \) is a primitive third root of unity and \( wt = t_{1/3} \). If \( P(t) \) contains exactly one long root, then \( wt = t_{q^2} \) for some \( z \in \mathbb{C}^\times \) so that \( wt \) does not take the value 1 or \( q^\pm 2 \) on any other positive root. Thus \( \{ 1, q^2, q^{-2} \} \cap \{ z, q^2 z, q^4 z, q^6 z, q^8 z^2 \} = \emptyset. \)

**Remark.** There are some redundancies in this list for specific values of \( q \). If \( q^2 \) is a primitive fifth root of unity, then \( q \) and \(-q\) are equal to \( q^{-4} \) and \(-q^{-4} \) in some order depending on whether \( q^5 = 1 \) or \(-1 \). Then \( t_{q^2, q^2} \) is in the same orbit as \( t_{q^{-4}, 1} \), which is equal to either \( t_{q, 1} \) or \( t_{-q, 1} \).

If \( q^2 \) is a primitive fourth root of unity, then one note is necessary on the weight \( t_{q^2/3, 1} \). Since \( q^{-2} \) is a third root of \( q^2 \), we take \( q^{2/3} \) to mean a different third root of \( q^2 \) so that \( t_{q^{2/3}, 1} \) and \( t_{q^{-2}, 1} \) are in different orbits. In addition, \( t_{q^2, q^2} = t_{q^{-2}, q^2} \), which is in the same orbit as \( t_{q^2, 1} \).
If \( q^2 \) is a primitive third root of unity, then \( 1^{1/3} = q^2, q^{-2}, \) or \( 1. \) Then \( t_1^{1/3} \) is in the same orbit as \( t_{q^2,1} \) or \( t_{1,1}. \) Also, \( t_1^{1/3} \) is in the same orbit as \( t_{q^2,q^2}, \) which is in turn in the same orbit as \( t_{q^2,q}. \) In addition \( q \) and \( -q \) are equal to \( q^{-2} \) and \( -q^{-2} \) in some order depending on whether \( q^3 \) is 1 or \(-1. \) Then \( t_{q^2,q^2} \) is in the same orbit as either \( t_{1,q} \) or \( t_{1,-q} \) and \( t_{q^2,1} \) is in the same orbit as either \( t_{q,1} \) or \( t_{-q,1}. \)

If \( q^2 = -1, \) then \( t_{q^2,-q^2} = t_{q^2,1} = t_{-1,1}, \) which is in the same orbit as \( t_{q^2,q^2}, \) while \( t_{1,q^2} = t_{1,-1}. \) In fact, \( t_{-1,1} = s_1s_2t_{1,-1}. \) Also, since \( q = -q^{-1}, \) the weights \( t_{1,z,q} \) are in the same orbit as each other, as are the weights \( t_{z,q}. \) Finally, \( t_{1/3,q^2} \) is in the same orbit as \( t_{q^2,1/3}. \)

Finally, if \( q = 1, \) then \( t_{1,1} = t_{q^2,1} = t_{1,q} = t_{q^2,q^2} = t_{1,-q} = t_{-q,1}. \) Also, \( t_{q,1} = t_{-1,1}, \) which is in the same orbit as \( t_{1,-1} = t_{1,q} = t_{q^2,-q^2}. \) Finally, \( t_{1/3,1} = t_{q^2/3,1} = t_{1/3,q^2}, \) while \( t_{1,z} = t_{q^2,z} \) and \( t_{z,1} = t_{z,q^2}. \)

**Analysis of the characters.**

**Proposition 22.** There are four 1-dimensional representations of \( H, \) one for each weight \( t_{q^2,z,q^2}. \) In these modules, \( T_i \) acts with eigenvalue \( q \) or \( -q^{-1} \) when \( t(X^i) = q^2 \) or \( q^{-2}, \) respectively.

**Proof.** As in Proposition 12. \( \square \)

**Remark.** We will use the notation \( L_{z,w} \) to denote the 1-dimensional representation with weight \( t_{q^2,w^2}, \) where each of \( z \) and \( w \) is either \( q \) or \(-q^{-1}. \) Note that if \( q \) is a primitive fourth root of unity, then all four 1-dimensional representations are isomorphic.

**Principal series modules.** We now examine the principal series modules \( M(t) \) for all the possible central characters above.

**Case 1:** \( P(t) = \emptyset. \)

If \( P(t) = \emptyset \) then by Kato’s criterion, Theorem 3(c), \( M(t) \) is irreducible and thus is the only irreducible module with central character \( t. \)

**Case 2:** \( Z(t) = \emptyset. \)

If \( Z(t) = \emptyset \) then \( t \) is a regular central character. Then the irreducibles with central character \( t \) are in bijection with the connected components of the calibration graph for \( t, \) and can be constructed using Theorem 7.

The following graphs show the pictures of the central characters found in Cases 1 and 2 of Theorem 21, for the particular values of \( q \) for which either \( Z(t) \) or \( P(t) \) is empty. The remark after Theorem 21 details these values of \( q. \)

**Case 1:**

\[ C \]

\[ t_{1,1}, q^2 \neq 1 \]

\[ t_{q/3,1}, q^2 \neq 1, q^6 \neq 1 \]

\[ t_{1,-1}, q^2 \neq \pm 1 \]
Case 2:

For these central characters, rather than analyzing $M(t)$ directly, it is easier to construct several irreducible $H$-modules and show that they include all the composition factors of $M(t)$.

Case 3a: $t_1, q^2$.

Assume $\alpha_1 \in Z(t)$ and $\alpha_2 \in P(t)$. Then $t = t_{1,q^2}$, but $s_2 s_1 s_2 s_1 s_2 t_{1,q^{-2}} = t_{1,q^2}$, so that analyzing $M(t_{1,q^2})$ is sufficient. Then let $t = t_{1,q^2}$. We have $Z(t) = \{\alpha_1\}$ and $P(t) = \{\alpha_2\}$ unless $q^2 = \pm 1$. Hence the cases $q^2 = 1$ and $q^2 = -1$ will be treated separately.

If $q^2 = 1$, then $Z(t) = P(t) = R^+$, and the irreducibles with this central character can be constructed using Theorem 9.

If $q^2 \neq 1$, let $C_{\alpha_1}$ and $C_{\nu_{w_0}}$ be the 1-dimensional $H_{\{2\}}$-modules spanned by $\nu_1$ and $\nu_{w_0}$, respectively, and given by
Then define

\[ M = \mathcal{H} \otimes \mathcal{H}(t_{2}) \mathbb{C}v_{t} \quad \text{and} \quad N = \mathcal{H} \otimes \mathcal{H}(2) \mathbb{C}v_{w_{0}t}. \]

**Proposition 23.** Assume \( q^{2} \neq \pm 1 \). Let \( M = \mathcal{H} \otimes \mathcal{H}(2) \mathbb{C}v_{t} \) and \( N = \mathcal{H} \otimes \mathcal{H}(2) \mathbb{C}v_{w_{0}t} \), where \( t = t_{1}q^{2} \).

- (a) \( M_{s_{2}s_{3}s_{2}t} \) is a 1-dimensional submodule of \( M, M' \), the image of the weight spaces \( M_{s_{2}s_{3}s_{2}t} \) in \( M/M_{s_{2}s_{3}s_{2}t} \), is a submodule of \( M/M_{s_{2}s_{3}s_{2}t} \). The resulting quotient of \( M \) is irreducible.
- (b) If \( q^{2} \) is not a primitive third root of unity, then \( M' \) is irreducible.
- (c) If \( q^{2} \) is a primitive third root of unity, then \( (M')_{s_{2}s_{3}s_{2}t} \) is a submodule of \( M' \).
- (d) \( N_{s_{2}s_{3}s_{2}t} \) is a 1-dimensional submodule of \( N, N' \), the image of the weight spaces \( N_{s_{2}s_{3}s_{2}t} \) and \( N_{s_{2}s_{3}s_{2}t} \) in \( N/N_{s_{2}s_{3}s_{2}t} \), is a submodule of \( N/N_{s_{2}s_{3}s_{2}t} \). The resulting quotient of \( N \) is irreducible.
- (e) If \( q^{2} \) is not a primitive third root of unity, then \( N' \) is irreducible.
- (f) If \( q^{2} \) is a primitive third root of unity, then \( (N')_{s_{2}s_{3}s_{2}t} \) is a submodule of \( N' \).

**Proof.** Assume \( q^{2} \neq -1 \). Then \( Z(t) = \{\alpha_{1}\} \) and \( P(t) \) contains \( \alpha_{2}, \alpha_{1} + \alpha_{2}, 2\alpha_{1} + \alpha_{2} \), and \( 3\alpha_{1} + \alpha_{2} \). If \( q^{2} \) is a primitive third root of unity, then \( P(t) \) also contains \( 3\alpha_{1} + 2\alpha_{2} \). Then \( M \) has one 2-dimensional weight space \( M_{t}^{\text{gen}} \) and four 1-dimensional weight spaces \( M_{s_{2}t}, M_{s_{1}s_{2}t}, M_{s_{2}s_{1}t}, \) and \( M_{s_{2}s_{1}s_{2}t} \). For \( w \in \{s_{2}, s_{1}s_{2}, s_{2}s_{1}s_{2}, s_{1}s_{2}s_{1}s_{2}\} \), let \( m_{wt} \) be a non-zero vector in \( M_{wt} \). By a calculation as in Proposition 1(b),

\[ m_{wt} = T_{w}T_{1}v_{t} + \sum_{w'<w} a_{w,w'}T_{w'}T_{1}v_{t}, \]

for \( w \in \{s_{2}, s_{1}s_{2}, s_{2}s_{1}s_{2}, s_{1}s_{2}s_{1}s_{2}\} \), where \( a_{w,w'} \in \mathbb{C} \). Then if \( s_{1}w > w \), \( \tau_{m_{wt}} \neq 0 \) for \( w \in \{s_{2}, s_{1}s_{2}, s_{2}s_{1}s_{2}\} \), since the term \( T_{1}T_{w}T_{1} \) cannot be canceled by any other term in \( \tau_{m_{wt}} \).

Thus \( \tau_{1} : M_{s_{2}s_{1}s_{2}t} \to M_{s_{2}s_{1}s_{2}t} \) is the zero map since, by Theorem 2, \( \tau_{2}^{1} : M_{s_{2}s_{1}s_{2}t} \to M_{s_{2}s_{1}s_{2}t} \) is the zero map. Hence \( M_{s_{2}s_{1}s_{2}t} \) is a submodule of \( M \). Similarly, \( \tau_{1} : M_{s_{2}s_{1}t} \to M_{s_{2}s_{1}t} \) must be the zero map since, by Theorem 2, \( \tau_{2}^{1} : M_{s_{2}s_{1}t} \to M_{s_{2}s_{1}t} \) is the zero map. Let \( M_{1} = M/M_{s_{2}s_{1}s_{2}t} \). Then \( M' \), the subspace spanned by \( m_{s_{2}s_{1}s_{2}t} \) and \( m_{s_{2}s_{1}s_{2}t} \) in \( M_{1} \), is a submodule of \( M_{1} \), Theorem 5 shows that \( M_{2} = M_{1}/M' \) is irreducible.

(b) If \( q^{2} \) is not a primitive third root of unity, \( \tau_{2}^{1} : (M')_{s_{2}s_{1}s_{2}t} \to (M')_{s_{2}s_{1}s_{2}t} \) is invertible, so that \( M' \) is irreducible.

(c) If \( q^{2} \) is a primitive third root of unity, then \( \tau_{2}^{1} : (M')_{s_{2}s_{1}s_{2}t} \to (M')_{s_{2}s_{1}s_{2}t} \) is a 1-dimensional submodule of \( M' \), and \( M'/(M')_{s_{2}s_{1}s_{2}t} \) is 1-dimensional as well.

(d)-(f) The same argument used in (a)-(c) applies, with each weight space \( M_{wt} \) replaced by \( N_{w_{0}t} \).
However, the composition factors of $M$ and $N$ are not distinct. If $q^2$ is not a primitive third root of unity, then $M'$ and $N'$ are irreducible 2-dimensional modules with the same weight space structure. Then Proposition 4 shows that $M' \cong N'$. If $q^2$ is a primitive third root of unity, then note that two 1-dimensional modules $\mathbb{C}v_1$ and $\mathbb{C}v_{t'}$ are isomorphic if and only if they have the same weight. Then $M'_1$ and $N'_1$ have the same composition factors. In any case, the 3-dimensional modules are different since their weight space structures are different.

**Proposition 24.** If $q^2 \neq \pm 1$ then the composition factors of $M$ and $N$ are the only irreducibles with central character $t_{1,q^2}$.

**Proof.** Counting multiplicities of weight spaces in $M(t)$ and the distinct composition factors of $M$ and $N$ shows that the remaining composition factor(s) of $M(t)$ must contain an $s_1s_2t$ weight space and an $s_2s_1s_2t$ weight space, each of dimension 1.

If $q^2$ is not a primitive third root of unity then Theorem 3(b) shows that there must be one remaining composition factor with an $s_1s_2t$ weight space and an $s_2s_1s_2t$ weight space, and Proposition 4 shows that it is isomorphic to $M_1$.

If $q^2$ is a primitive third root of unity then $\tau_2^2M_{s_1s_2t} \to M_{s_1s_2t}$ is not invertible. Hence there cannot be an irreducible module consisting of an $s_1s_2t$ weight space and an $s_2s_1s_2t$ weight space, and the remaining composition factors of $M(t)$ are 1-dimensional. $\square$

If $q^2 = -1$, then $\dim M_{t}^{\text{gen}} = \dim M_{s_2t}^{\text{gen}} = \dim M_{s_1s_2t}^{\text{gen}} = 2$ and $\dim N_{t}^{\text{gen}} = \dim N_{s_2t}^{\text{gen}} = \dim N_{s_1s_2t}^{\text{gen}} = 2$. 

$t_{1,q^2}, q^2$ a primitive third root of unity

$t_{1,q^2}, q^2$ not a primitive third root of unity, $q^2 \neq -1$
Proposition 25. Assume \( q^2 = -1 \) and \( t = t_{1,q^2} \). Let \( M = H \otimes \mathcal{H}(2) \mathbb{C}v_t \) and \( N = H \otimes \mathcal{H}(1) \mathbb{C}v_{s_1 s_2 t} \).

(a) \( M \) and \( N \) each have two 1-dimensional modules and two 2-dimensional modules as composition factors.
(b) The composition factors of \( M \) and \( N \) are the only irreducible modules with central character \( t \).

Proof. By Proposition 4, there is a 2-dimensional module \( P \) with \( P = P^\text{gen}_{t} \). Let \( v \in P_t \) be non-zero. The map

\[
\mathbb{C}v_t \rightarrow P, \quad v_t \mapsto v
\]

is an \( \mathcal{H}(2) \)-module homomorphism. Since

\[
\Hom_{\mathcal{H}}(M, P) = \Hom_{\mathcal{H}(2)}(\mathbb{C}v_t, P),
\]

there is a non-zero map from \( M \) to \( P \). Since \( P \) is irreducible, this map is surjective and \( P \) is a quotient of \( M \). The kernel of any map from \( M \) to \( P \) must be

\[
M_1 = M^\text{gen}_{s_2 t} \oplus M^\text{gen}_{s_1 s_2 t},
\]

which is then a submodule of \( M \).

Then we note that \( m = T_1 T_2 T_1 T_2 T_1 v - q T_2 T_1 T_2 T_1 v - T_1 T_2 T_1 v + q T_2 T_1 v + T_1 v - q v \) spans a 1-dimensional submodule of \( M_1 \). Then let \( M_2 = M_1/m \), so that \( T_1 T_2 T_1 T_2 T_1 v = q T_2 T_1 T_2 T_1 v + T_1 T_2 T_1 v = q T_2 T_1 v - T_1 v + q v \) in \( M_2 \).

Then by a calculation as in Proposition 1(b), \( M_2 \) contains an element \( m' = T_2 T_1 v - q T_1 v - 3v \in M_{s_2 t} \). Then \( m' \), \( \tau_1(m') \) and \( T_2 \cdot \tau_1(m') \) are linearly independent (since their leading terms cannot be canceled) and span \( M_2 \). However, \( M_3 = (\tau_1(m'), T_2 \cdot \tau_1(m')) \) is clearly closed under the action of \( T_2 \). Also, \( \tau_1(m') \in M_{s_1 s_2 t} \), so that

\[
X^k \cdot T_2 \tau_1(m') = T_2 X^{s_2 \lambda} \tau_1(m') + (q - q^{-1}) \frac{X^k - X^{s_2 \lambda}}{1 - X^{-\alpha_2}} \tau_1(m'),
\]

which again lies in \( M_3 \). Finally, one can compute that

\[
T_1 \cdot \tau_1(m') = -q^{-1} \tau_1(m'), \quad \text{and} \quad T_1 \cdot T_2 \tau_1(m') = q(\tau_1(m')) + T_2 \tau_1(m').
\]

Thus \( M_3 \) is a submodule of \( M_2 \). By Theorem 5, \( M_3 \) is irreducible, and \( M_2/M_3 \) is a 1-dimensional module which is isomorphic to the 1-dimensional module spanned by \( m \).

An analogous argument proves the same result for \( N \). Let \( Q \) be the 2-dimensional module with \( Q = Q_{s_1 s_2 t}^\text{gen} \). Then there is a surjection from \( N \) to \( Q \), and the kernel of this map, \( N_1 \), consists of the \( t \) and \( s_2 t \) weight spaces of \( N \). Then \( n = T_2 T_1 T_2 T_1 T_2 T_1 - q T_1 T_2 T_1 T_2 T_1 - T_2 T_1 T_2 T_1 - q T_1 T_2 T_1 + T_2 + q v \) spans a 1-dimensional submodule of \( N_1 \). Let \( N_2 = N_1/Cn \).

Then \( N_{s_2 t} \) contains a non-zero element \( n' \), and \( n', T_2(n') \), and \( T_1 T_2(n') \) are linearly independent and span \( N_2 \). But \( T_2(n') \) and \( T_1 T_2(n') \) span a submodule of \( N_2 \), which is irreducible by Theorem 5.

(b) Let \( C\mathcal{V}_{s_2 t} \) be the one-dimensional \( \mathcal{H}(1) \)-module with weight \( s_2 t \), and define \( L = \mathcal{H} \otimes \mathcal{H}(1) C\mathcal{V}_{s_2 t} \). We claim that the composition factors of \( L \) are the same as those of \( M \). First, note that the one-dimensional \( \mathcal{H} \)-module \( L_{q,t} \) restricted to \( \mathcal{H}(1) \) is \( C\mathcal{V}_{s_2 t} \). Then there is an \( \mathcal{H}(1) \)-module map from \( C\mathcal{V}_{s_2 t} \) to \( L_{q,t} \), and thus there is a map from \( L \) to \( L_{q,t} \). Let \( L_1 \) be the kernel of this map. Then \( L_1 \) has a 1-dimensional \( s_2 t \) weight space, and 2-dimensional generalized \( t \) and \( s_1 s_2 t \) weight spaces. Also, \( L_1 \) contains \( l = T_2(v_{s_2 t}) = T_2 v_{s_2 t} - q v_{s_2 t} \), an element of the \( t \) weight space of \( L_1 \). Then we note that \( T_2 \cdot l \)}
\((T_2v_{s_2t} - qv_{s_2t}) = (q - q^{-1})T_2v_{s_2t} + v_{s_2t} - qT_2v_{s_2t} = q(T_2v_{s_2t} - qv_{s_2t})\), so that \(L\) spans a 1-dimensional \(\mathcal{H}_2\)-submodule of \(L_1\), with weight \(t\).

Thus, there is a \(\mathcal{H}_2\)-module map from \(Cv_t\) to \(L_1\), and thus an \(\mathcal{H}\)-module map from \(M\) to \(L\). This map is surjective since \(l, T_2l, T_1T_2l, T_2T_1T_2l\), and \(T_1T_2T_1T_2l\) are linearly independent and span \(L_1\). Then \(L_1\) is a quotient of \(M\) and its composition factors are composition factors of \(M\).

Now, let \(P\) be any irreducible \(\mathcal{H}\)-module with central character \(t_1, q^2\). If \(P\) is not a composition factor of \(M\) or \(N\), then \(P\) must be in the kernel of the (surjective) map from \(M(t)\) to \(M\). Hence \(P\) is at most 6-dimensional, and each of its generalized weight spaces is at most 2-dimensional. If \(P = \mathcal{P}^{\text{gen}}\), then \(P\) is 2-dimensional and must be the module described in Proposition 4. Otherwise, we note that \(\mathcal{P}_2^{\text{gen}} \oplus \mathcal{P}_s^{\text{gen}}\) is an \(\mathcal{H}_{(1)}\)-submodule of \(P\), since the action of \(t_1\) fixes this subspace of \(P\). Thus \(\mathcal{P}_2^{\text{gen}} \oplus \mathcal{P}_s^{\text{gen}}\) contains an irreducible \(\mathcal{H}_{(1)}\)-submodule. This subspace must be either \(\mathcal{P}_s^{\text{gen}}\) or a 1-dimensional module with weight \(s_2t\). Hence \(P\) is a quotient of either \(L\) or \(M\) and is isomorphic to a composition factor of \(M\). \(\square\)

**Case 3b: \(t_1, \pm q\).**

Let \(t' \in T\) and assume \(\alpha_1 \in Z(t')\) but \(\alpha_2 \notin P(t')\), so that none of \(\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2\) are in \(P(t')\). Since \(P(t') \neq \emptyset\), \(2\alpha_1 + 3\alpha_2 \notin P(t')\), so that \(t'(X^{2\alpha_2}) = q^{\pm 2}\) and \(t'(X^{\alpha_2}) = q^{\pm 1}\). By applying \(v_0\) if necessary, we may assume \(t'(X^{\alpha_2}) = \pm q\). Thus we will analyze the weights \(t_1, \pm q\) together, except in one case. If \(q\) is a primitive third root of unity then \(q = q^{-2}\) and \(q^{-1} = q^2\), so that \(t_1, -q = t_1, q^{-2}\) was analyzed in Case 3a. If \(q\) is a primitive sixth root of unity then \(-q = q^{-2}\) and \(-q^{-1} = q^2\) so that \(t_1, -q = t_1, q^{-2}\) was analyzed in Case 3a. Thus these cases are excluded from the following analysis by simply assuming that \(t'(X^{\alpha_2}) \neq q^{-2}\).

If \(q^2 = 1\), then \(Z(t') = P(t') = \{\alpha_1, 3\alpha_1 + \alpha_2\}\), and the irreducibles with central character \(t\) can be constructed using Theorem 9. Specifically, there are four 3-dimensional modules with central character \(t'\).

If \(q^2 \neq 1\), then \(s_1s_2t'(X^{\alpha_1}) = t'(X^{-\alpha_1 - \alpha_2}) = \pm q^{\mp 1}\) and \(s_1s_2t'(X^{\alpha_2}) = t'(X^{3\alpha_1 + 2\alpha_2}) = q^{\pm 2}\). Then by Theorem 3, \(M(t')\) and \(M(t)\) have the same composition factors, where \(t = s_1s_2t'\). Also by assuming that \(t'(X^{\alpha_2}) \neq q^{-2}\), we have \(Z(t) = \{2\alpha_1 + \alpha_2\}\) and \(P(t) = \{\alpha_2\}\). Let \(Cv_t\) and \(Cv_{w_0t}\) be the 1-dimensional \(\mathcal{H}_{(2)}\)-modules spanned by \(v_t\) and \(v_{w_0t}\), respectively, and given by

\[
T_2v_t = qv_t, \quad X^{\lambda}v_t = t(X^{\lambda})v_t, \quad T_2v_{w_0t} = -q^{-1}v_{w_0t}, \quad \text{and} \quad X^{\lambda}v_{w_0t} = w_0t(X^{\lambda})v_{w_0t}.
\]

Then define

\[
M = \mathcal{H} \otimes \mathcal{H}_{(2)} Cv_t \quad \text{and} \quad N = \mathcal{H} \otimes \mathcal{H}_{(2)} Cv_{w_0t}.
\]

![Diagram](https://via.placeholder.com/150)
Proposition 26. Assume \( q^2 \neq 1 \). Let \( t' = t_1 \pm q \), and define \( M \) and \( N \) as above. Assume that it is not true that 
\[
t'(X^{\alpha_2}) = q^{-2}.
\]
Then \( M \) and \( N \) are irreducible.

Let \( t = s_1 s_2 t \). Under the assumptions, \( Z(t) = \{ 2\alpha_1 + \alpha_2 \} \) and \( P(t) = \{ \alpha_2 \} \). Then \( \dim M_{\text{gen}}^t = \dim M_{\text{gen}}^{s_1 t} = \dim M_{\text{gen}}^{s_2 s_1 t} = 2 \). By Theorem 5, \( M \) has some composition factor \( M' \) with \( \dim M_{\text{gen}}^{t_1 s_2 s_1 t} = 2 \), and by Theorem 3(b), \( M' = M \) and \( M \) is irreducible. Similarly, Theorem 5 and Theorem 3(b) show that \( N \) is irreducible. \( \square \)

Under the assumptions of this theorem, since \( M \) and \( N \) are each 6-dimensional, they must be the only composition factors of \( M(t) \). If \( t'(X^{\alpha_2}) = q^{-2} \), then \( w_0 t' = t_1 q^2 \), which was discussed in the case above.

These two cases are the only weights \( t \) with \( P(t) \) non-empty and \( Z(t) \) containing a short root. Specifically, if \( t \) is any weight such that \( Z(t) \) contains \( \alpha_1, \alpha_1 + \alpha_2, \) or \( 2\alpha_1 + \alpha_2 \), there exists \( w \in W_0 \) so that \( \alpha_1 \in Z(w t) \). Then \( t \) is in the orbit of one of the weights in the previous cases. Then for the following cases, assume \( \alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 \notin Z(t) \).

Case 3c: \( t_{q^2,1} \).

If \( \alpha_2 \in Z(t) \) and \( \alpha_1 \in P(t) \), then \( \alpha_1 + \alpha_2 \in P(t) \) as well, and \( t = t_{q^2,1} \). These weights are in the same orbit, so we examine \( M(t_{q^2,1}) \). If \( q^2 = -1 \) then \( t(X^{2\alpha_1 + \alpha_2}) = 1 \), so that \( t \) is in the orbit of one of the weights considered in Cases 3a and 3b. If \( q^2 = 1 \), then \( t = t_{1,1} \) which has also already been considered. Then we assume \( q^2 \neq \pm 1 \).

Let \( \mathbb{C} v_t \) and \( \mathbb{C} v_{w_0 t} \) be the 1-dimensional \( \mathcal{H}_{\{1\}} \)-modules spanned by \( v_t \) and \( v_{w_0 t} \), respectively, and given by

\[
T_1 v_t = q v_t, \quad X^2 v_t = t(X^2) v_t, \\
T_1 v_{w_0 t} = -q^{-1} v_{w_0 t}, \quad \text{and} \quad X^2 v_{w_0 t} = w_0 t(X^2) v_{w_0 t}.
\]

Then define

\[
M = \mathcal{H} \otimes \mathcal{H}_{\{1\}} \mathbb{C} v_t \quad \text{and} \quad N = \mathcal{H} \otimes \mathcal{H}_{\{1\}} \mathbb{C} v_{w_0 t}.
\]

Proposition 27. If \( q^2 \) is a primitive third root of unity, then \( M \) and \( N \) are irreducible.

Proof. If \( q^2 \) is a primitive third root of unity, then \( Z(t) = \{ \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \} \) and \( P(t) = \{ \alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 \} \).
Then \( \dim M_{\text{gen}} = 4 \) and \( \dim M_{s1t}^\text{gen} = 2 \). By Theorem 5, if \( M' \subseteq M \) is a submodule of \( M \), then \( \dim(M'/t)_{s1t}^\text{gen} \geq 2 \) and \( \dim(M'/s1t)_{s1t}^\text{gen} \geq 2 \). Then \( \dim(M/M')_{s1t}^\text{gen} \leq 2 \), but \( \dim(M/M')_{s1t}^\text{gen} = 0 \), so that Theorem 5 implies that \( (M/M')_{s1t} \sim = 0 \). Thus \( M = M' = M \) and \( M \) is irreducible. Theorem 5 similarly implies that \( N \) is irreducible. \( \square \)

Then since \( M \) and \( N \) have different weight spaces, they are not isomorphic and are the only irreducibles with central character \( t \).

**Proposition 28.** Assume \( q^2 \neq \pm 1 \) and that \( q^2 \) is not a primitive third root of unity.

- (a) If \( q^2 \) is a primitive fourth root of unity then \( M_{s2s1s2s1t} \) is a 1-dimensional submodule of \( M \), and \( M' \), the image of the weight spaces \( M_{s1s2s1t} \) and \( M_{s2s1t} \) in \( M/M_{s1s2s1t} \), is an irreducible submodule of \( M/M_{s1s2s1t} \). The resulting quotient of \( M \) is irreducible.
- (b) If \( q^2 \) is a primitive fourth root of unity then \( N_{s1t} \) is a 1-dimensional submodule of \( N \), and \( N' \), the image of the weight spaces \( N_{s2s1t} \) and \( N_{s1s2s1t} \) in \( N/N_{s1t} \), is an irreducible submodule of \( N/N_{s1t} \). The resulting quotient of \( N \) is irreducible.
- (c) The composition factors of \( M \) and \( N \) are the only composition factors of \( M(t) \).
- (d) If \( q^2 \) is not a primitive third or fourth root of unity then \( M \) and \( N \) are irreducible, and are the only irreducible modules with central character \( t \).

**Proof.** If \( q^2 \) is not \( \pm 1 \) or a primitive third root of unity, \( Z(t) = \{ \alpha_2 \} \), so that \( M \) has one 2-dimensional weight space \( M_t^{\text{gen}} \) and four 1-dimensional weight spaces \( M_{s1t}, M_{s2s1t}, M_{s1s2s1t}, \) and \( M_{s2s1s2s1t} \).

- (a) If \( q^2 \) is a primitive fourth root of unity, then \( P(t) = \{ \alpha_1, \alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2 \} \). For \( w \in \{ s_1, s_2s_1, s_1s_2s_1, s_2s_1s_2s_1 \} \), let \( m_{wt} \) be a non-zero vector in \( M_{wt} \). By Proposition 1(b),

\[
m_{wt} = T_w T_2 v_t + \sum_{w' < w} a_{w,w'} T_w T_2 v_t,
\]

for \( w \in \{ s_1, s_2s_1, s_1s_2s_1, s_2s_1s_2s_1 \} \), where \( a_{w,w'} \in \mathbb{C} \). Then if \( s_iw > w \),

\[
\tau_i m_{wt} \neq 0
\]

for \( w \in \{ s_1, s_2s_1, s_1s_2s_1, s_2s_1s_2s_1 \} \), since the term \( T_j T_w T_2 \) cannot be canceled by any other term in \( \tau_i m_{wt} \).

Thus \( \tau_2 : M_{s2s1s2s1t} \rightarrow M_{s1s2s1t} \) is the zero map since, by Theorem 2, \( \tau_2^2 : M_{s1s2s1t} \rightarrow M_{s1s2s1t} \) is the zero map. Hence \( M_{s1s2s1t} \) is a submodule of \( M \). Let \( M_1 = M/M_{s1s2s1t} \). Similarly, \( \tau_2 : M_{s2s1t} \rightarrow M_{s1t} \) must be the zero map since, by Theorem 2, \( \tau_2^2 : M_{s1t} \rightarrow M_{s1t} \) is the zero map. Then \( M_1' \), the subspace spanned by \( \overline{M}_{s2s1t} \) and \( \overline{M}_{s1s2s1t} \) in \( M_1 \), is a submodule of \( M_1 \). Since \( \tau_2^2 : (M_1')_{s2s1t} \rightarrow (M_1')_{s2s1t} \) is invertible, \( M_1' \) is irreducible, and Theorem 5 shows that \( M_2 = M_1/M_1' \) is irreducible.

- (b) Replacing \( t \) by \( w_0t \) in this argument shows that \( N \) also has three composition factors. The weight space \( N_{s1t} \) is a submodule of \( N \), and \( N_1 = N/N_{s1t} \) has an irreducible 2-dimensional submodule \( N_1' \), consisting of the image of \( N_{s1s2s1t} \) and \( N_{s2s1s2s1t} \) in \( N_1 \). Theorem 5 shows that \( N_1/N_1' \) is irreducible.

- (c) The composition factors of \( M \) and \( N \) are not distinct, since \( M_1' \) and \( N_1' \) are irreducible 2-dimensional modules with the same weight spaces, and Proposition 4 shows that \( M_1' \equiv N_1' \). The 1-dimensional composition factors of \( M \) and \( N \) are not isomorphic since they have different weights, and the 3-dimensional modules are different since their weight space structures are different.

Counting multiplicities of weight spaces in \( M, N \), and \( M(t) \) shows that the remaining composition factor(s) of \( M(t) \) must contain an \( s_2s1t \) weight space and an \( s_1s2s1t \) weight space, each of dimension 1. But Theorem 3(b) shows that there must be one remaining composition factor, and Proposition 4 shows that it is isomorphic to \( M_1 \). Then the composition factors of \( M \) and \( N \) are all the composition factors of \( M(t) \).

- (d) Theorems 5 and 3(b) show that both \( M \) and \( N \) are irreducible if \( q^2 \) is not a primitive third or fourth root of unity. Since \( M \) and \( N \) are not isomorphic and are each 6-dimensional, they are the only composition factors of \( M(t) \). \( \square \)
Case 3d: $t_{\pm q, 1}$.

If $\alpha_2 \in \mathbb{Z}(t')$ and $2\alpha_1 + \alpha_2 \in P(t')$, then $t'(X^{2\alpha_1}) = q^{\pm 2}$ and $t'(X^{\alpha_1}) = \pm q^{\pm 1}$. By replacing $t'$ by $w_0t'$ if necessary, it suffices to assume that $t'(X^{\alpha_1}) = \pm q$. If $t'(X^{\alpha_1}) = q^{-2}$, then $t'$ was analyzed in Case 3c. This occurs when $q^3 = 1$ and $t' = tq_{2, 1}$, or when $q^3 = -1$ and $t' = t_{-q, 1}$. Thus the following analysis will apply to $tq_{2, 1}$ except if $q^3 = 1$, and $t_{-q, 1}$ except for when $q^3 = -1$. (This is tantamount to assuming that $P(t)$ and $Z(t)$ each contain exactly one element for this $t$.)

Also, if $t'(X^{3\alpha_1}) = q^{-2}$, then $P(t)$ also contains $3\alpha_1 + \alpha_2$ and $3\alpha_1 + 2\alpha_2$. This occurs when $q^5 = 1$ and $t'(X^{\alpha_1}) = q$ or when $q^5 = -1$ and $t'(X^{\alpha_1}) = -q$. When either of these hold, $t'$ is the same orbit as $t_{q^2, q^2}$. This case (which was specifically not addressed in Case 2 above) will be treated separately below.

Define $t = s_2 s_1 t'$ so that $t(X^{\alpha_1}) = q^2$ and $t(X^{\alpha_2}) = \pm q^{-3}$. Let $\mathbb{C}v_t$ and $\mathbb{C}v_{w_0t}$ be the 1-dimensional $\mathcal{H}_{(1)}$-modules spanned by $v_t$ and $v_{w_0t}$, respectively, and given by

$$T_1 v_t = q v_t, \quad X_1^2 v_t = t(X_1) v_t,$$
$$T_1 v_{w_0t} = -q^{-1} v_{w_0t}, \quad X_1^2 v_{w_0t} = w_0 t(X_1) v_{w_0t}.$$

Then define

$$M = \mathcal{H} \otimes \mathcal{H}_{(1)} \mathbb{C}v_t \quad \text{and} \quad N = \mathcal{H} \otimes \mathcal{H}_{(1)} \mathbb{C}v_{w_0t}.$$
Proof. Let \( t = s_2 s_1 t' \). Under the assumptions, \( Z(t) = \{3\alpha_1 + 2\alpha_2\} \) and \( P(t) = \{\alpha_1\} \). Then \( \dim M^\text{gen}_t = \dim M^\text{gen}_s = \dim M^\text{gen}_{s_1 s_2} = 2 \). Then Theorem 5 and Theorem 3(b) show that \( M \) is irreducible. \( N \) is also irreducible by the same reasoning. \( \square \)

Under the assumption of the theorem, since \( M \) and \( N \) are not isomorphic and are each 6-dimensional, they are the only composition factors of \( M(t) \). Note that \( t'(X^{\alpha_1}) = q^{-2} \) exactly if \( q^3 = 1 \) or \(-1\) and \( t'(X^{\alpha_2}) = q \) or \(-q\), respectively. In this case, the central character \( t' \) has been analyzed above (Case 3c) Also, \( t'(X^{\alpha_1}) = q^{-4} \) exactly if \( q^5 = 1 \) or \(-1\) and \( t'(X^{\alpha_2}) = q \) or \(-q\), respectively. In this case, \( t' \) is in the same orbit as \( t_{q^2,q^2} \).

Proposition 30. If \( t = t_{q^2,q^2} \) and \( q^2 \) is a fifth root of unity, then

(a) \( M \) has a 5-dimensional irreducible submodule \( M' \) and

(b) \( N \) has a 5-dimensional irreducible submodule \( N' \).

Proof. Given these assumptions, \( Z(t) = \{3\alpha_1 + 2\alpha_2\} \) and \( P(t) = \{\alpha_1, \alpha_2, 3\alpha_1 + \alpha_2\} \). Then \( \dim M^\text{gen}_t = \dim M^\text{gen}_s = \dim M^\text{gen}_{s_1 s_2} = 2 \). Let \( L_{q,q} = \mathbb{C} v \) be the 1-dimensional \( \mathcal{H} \) module given by

\[
T_1 v = q v, \quad X^{\alpha_i} = q^2 v, \quad \text{for } i = 1, 2.
\]

Since

\[
\hom_{\mathcal{H}}(\mathcal{H} \otimes \mathcal{H}_{(1)} \mathbb{C} v_t, L_{q,q}) = \hom_{\mathcal{H}_{(1)}}(\mathbb{C} v_t, L_{q,q} | \mathcal{H}_{(1)})
\]

and

\[
\phi: \mathbb{C} v_t \to L_{q,q},
\]

\[
v_t \mapsto v
\]

is a map of \( \mathcal{H}_{(1)} \)-modules, there is a non-zero map \( \theta: M \to L_{q,q} \). Then let \( M_1 \) be the kernel of \( \theta \), which is 5-dimensional. Similarly, there is a map \( \rho: N \to L_{q^{-1},q^{-1}} \), where \( L_{q^{-1},q^{-1}} = \mathbb{C} v \) is given by

\[
T_1 = -q^{-1} v, \quad X^{\alpha_i} = q^{-2} v, \quad \text{for } i = 1, 2.
\]

Then if \( N_1 \) is the 5-dimensional kernel of \( \rho \), Theorem 5 and Theorem 3(b) show that \( M_1 \) and \( N_1 \) are both irreducible. \( \square \)

These two 5-dimensional modules, plus the 1-dimensional modules \( L_{q,q} \) and \( L_{-q^{-1},-q^{-1}} \) account for all the composition factors of \( M(t) \).

Case 3e: \( t_{q^{6},3,1} \).

If \( \alpha_2 \in Z(t) \) and \( 3\alpha_1 + 2\alpha_2 \in P(t) \), then \( 3\alpha_1 + 2\alpha_2 \in P(t) \) as well. If \( t(X^{3\alpha_1+\alpha_2}) = q^{-2} \), then \( w_0 t(X^{\alpha_1}) = 1 \) and \( w_0 t(X^{3\alpha_1+\alpha_2}) = q^3 \), so by replacing \( t \) with \( w_0 t \) if necessary, assume that \( t(X^{\alpha_1}) = q^{-2} \). If \( \alpha_1 \in P(t) \), then this weight was analyzed in Case 3c, and if \( 2\alpha_1 + \alpha_2 \in P(t) \), then this weight was analyzed in Case 3d.

Then we assume \( Z(t) = \{\alpha_2\} \) and \( P(t) = \{3\alpha_1 + 2\alpha_2\} \). Let \( L_{t'} = s_1 t' \) so that \( Z(t') = \{3\alpha_1 + \alpha_2\} \) and \( P(t') = \{\alpha_2, 3\alpha_1 + 2\alpha_2\} \). Let \( \mathbb{C} v_t ' \) and \( \mathbb{C} v_{w_0 t'} \) be the 1-dimensional \( \mathcal{H}_{(1)} \)-modules spanned by \( v_t ' \) and \( v_{w_0 t'} \), respectively, and given by

\[
T_1 v_{t'} = q v_{t'}, \quad X^{\lambda} v_{t'} = t'(X^{\lambda}) v_{t'},
\]

\[
T_1 v_{w_0 t'} = -q^{-1} v_{w_0 t'}, \quad \text{and } X^{\lambda} v_{w_0 t'} = w_0 t'(X^{\lambda}) v_{w_0 t'}.
\]
Then define

\[ M = \mathcal{H} \otimes_{\mathcal{H}(1)} C v_t \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}(1)} C v_{w_0 t'} \]

Proposition 31. Assume \( t = t_{q^{2/3}, 1} \), where \( q^{2/3} \) is a third root of \( q^2 \) not equal to \( q^{\pm 2} \) or \( \pm q^{\pm 1} \), and that \( q^2 \neq \pm 1 \). Then \( M \) and \( N \) are irreducible.

Proof. Under the assumptions, \( Z(t')' = \{3\alpha_1 + \alpha_2\} \) and \( P(t) = \{\alpha_2, 3\alpha_1 + 2\alpha_2\} \), so that \( \dim M_{T_1}^{\text{gen}} = \dim M_{T_2}^{\text{gen}} = 2 \) while \( \dim M_{S_2 T_1} = \dim M_{S_2 S_1 T} = 1 \). Then Theorem 5 and Theorem 3(b) show that \( M \) is irreducible. Similarly, \( N \) is irreducible by the same reasoning, so that \( M \) and \( N \) are the only irreducible modules with central character \( t \). \( \square \)

Proposition 32. Assume \( q^2 = -1 \) and that \( t = t_{q^{2/3}, 1} \), where \( q^{2/3} \) is a third root of \( q^2 \) not equal to \( q^{\pm 2} \) or \( \pm q^{\pm 1} \). Then \( M \) and \( N \) each have an irreducible 2-dimensional submodule consisting of their \( s_2 s_1 t \) and \( s_1 s_2 s_1 t \) weight spaces. The resulting quotients are irreducible.

Proof. Let \( X = \{e, s_1, s_2 s_1, s_1 s_2 s_1\} \). By a calculation analogous to that in Proposition 1(b), the generalized \( t' \) weight space of \( M \) is generated by \( v_{t'} \) and a vector \( v \) of the form \( \sum_{x \in X} q_x T_x v_{t'} \), where the \( \alpha_x \) are in \( C \) and \( \alpha_{s_2 s_1} \neq 0 \). Then \( T_2(v) \neq 0 \), since it contains a non-zero \( T_2 T_1 T_2 T_1 v_{t'} \) term. But then \( \tau_2 T_2(v) = 0 \) by Theorem 2(c), so that the space \( M_1 = M_{S_2 T} \oplus M_{S_1 T_2} \) is actually a submodule of \( M \). The resulting quotient \( M/M_1 \) is irreducible by Theorem 3. A similar argument shows the same for \( N \). \( \square \)

Note that Proposition 4 shows that the 2-dimensional composition factors of \( M \) and \( N \) are isomorphic, and this proposition implies that when \( q^2 = -1 \), the composition factors of \( M \) and \( N \) are the only irreducibles with this central character. Counting dimensions of the weight spaces of these irreducibles shows that the final composition factor of \( M(t) \) must be 2-dimensional with weights \( s_2 t' \) and \( s_1 s_2 t' \), since there are no 1-dimensional modules with this central character. So the last composition factor of \( M(t) \) must also be isomorphic to the 2-dimensional submodule of \( M \).

Summary. We summarize the results of the previous theorems, including our choices of representatives for the various central characters, in Table 4. Some notes are necessary about Table 4. An entry of “N/A” means that the given central character is in the same orbit as a previous character for that particular value of \( q \), as described after Theorem 21.

If \( q^{10} = 1 \), we are assuming that \( q^5 = -1 \), so that \( t_{q^{2/3}, 1} = t_{\pm q^{-4}, 1} \).

If \( q^8 = 1 \), then only the central characters \( t_{q^{2/3}, 1}, t_{q^4, q^2}, \) and \( t_{q^2, q^2} \) change from the generic case. All three of these characters are now in the same orbit. Also, we assume that for the central character \( t_{q^{2/3}, 1} \), we choose a cube root of \( q^2 \) besides \( q^{-2} \).
Table 4

<table>
<thead>
<tr>
<th>$t$</th>
<th>$q$ generic</th>
<th>$q^{12} = 1$</th>
<th>$q^{10} = 1$</th>
<th>$q^6 = 1$</th>
<th>$q^8 = 1$</th>
<th>$q^4 = 1$</th>
<th>$q = -1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_{1,1}$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>1.1, 1.1.2.2</td>
</tr>
<tr>
<td>$t_{1,-1}$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>1.2.2</td>
<td>3.3.3</td>
</tr>
<tr>
<td>$t_{1/1,1}$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>N/A</td>
<td>12</td>
<td>3.3.6</td>
<td></td>
</tr>
<tr>
<td>$t_{1,q^2}$</td>
<td>1.1, 2, 3, 3</td>
<td>1.1, 2, 3, 3</td>
<td>1.1, 2, 3, 3</td>
<td>1.1, 1.3, 3</td>
<td>N/A</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>$t_{1,q^2,q}$</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>$t_{1,z}$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>6.6</td>
<td></td>
</tr>
<tr>
<td>$t_{q^2,1}$</td>
<td>6,6</td>
<td>6,6</td>
<td>1.1, 2, 3, 3</td>
<td>6,6</td>
<td>6,6</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>$t_{q^2,2}$</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>$t_{q^2,1}$</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>2.4,4</td>
<td></td>
</tr>
<tr>
<td>$t_q$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>6.6</td>
<td></td>
</tr>
<tr>
<td>$t_{1,1/1,q^2}$</td>
<td>3,3,3,3</td>
<td>N/A</td>
<td>3,3,3,3</td>
<td>N/A</td>
<td>3,3,3,3</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>$t_{q^2,q^2,q^2}$</td>
<td>2.2, 4,4</td>
<td>N/A</td>
<td>2.2, 4,4</td>
<td>N/A</td>
<td>N/A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_{q^2,q^2,q^2}$</td>
<td>1.1, 5.5</td>
<td>1.1, 2, 3, 3</td>
<td>N/A</td>
<td>N/A</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$t_{w,z}$</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>$t_{w,q}$</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>6,6</td>
<td>N/A</td>
<td></td>
</tr>
<tr>
<td>$t_{w,q^2}$</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

If $q^6 = 1$, the entries for the central characters $t_{\pm q,1}$ and $t_{1, \pm q}$ only apply to the characters $t_{-q^{-2},1}$ and $t_{1,-q^{-2}}$ (depending on whether $q^3 = 1$ or $-1$). Then we note that $t_{1/1,1} = t_{q^{-2},1}$, and $t_{q^{-2},1}$ is in the same orbit as $t_{q^{-2},1}$. Also, $t_{1,q^{-2}} = w_0 t_{1,q^2}$, and $t_{q^{-2},1} = w_0 t_{q^2,1}$. Finally, $t_{1/1,q^2} = t_{q^{-2},q^2}$, but $s_1 q^2 q^2 = q^{-2} q^2 = s_2 t_{1,q^{-2}}$ and so both are in the same orbit as $t_{q^2,q^2}$. When $q^2 = -1$, a number of characters change from the general case. Now, $t_{1,-1} = t_{1,q^2}$, which is in the same orbit as $t_{q^{-2},q^{-2}}$, $t_{q^2,q^2}$ and $t_{q^2,1}$. Similarly, $t_{1/1,q^2}$ is in the same orbit as $t_{q^{2/3},1}$.

Acknowledgments

The author would like to thank Arun Ram for his continued guidance and many helpful conversations regarding this work, and an anonymous referee, whose comments helped streamline and improve an earlier version of this work.

References