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Representations of rank two affine Hecke algebras at roots of unity

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ABSTRACT

In this paper, we will fully describe the irreducible representations of the crystallographic rank two affine Hecke algebras using algebraic and combinatorial methods, for all possible values of q. The focus is on the case when q is a root of unity of small order.

1. Introduction

The affine Hecke algebra $\mathcal{H}$ was introduced by Iwahori and Matsumoto [3]. Knowing the representations of $\mathcal{H}$ gives a substantial amount of information about the representations of a closely related $\mathfrak{p}$-adic group. The definition of $\mathcal{H}$ involves a parameter q which can have a large effect on the structure of the algebra. In this paper, we will fully describe the irreducible representations of the affine Hecke algebras of type C_2 and G_2, for all possible values of q. The methods are essentially those introduced in [9], with the modifications required to deal with q being a root of unity.

The representations in type A were described in the non-root of unity case by Zelevinsky, in terms of combinatorial objects called multisegments (see [1] and [14]). In the root of unity case, these representations are indexed by the aperiodic multisegments (see the appendix of [7] for an argument relying on the results of [8]). The representations of $\mathcal{H}$ in all types have been classified geometrically by Kazhdan and Lusztig [6] in the non-root of unity case, and studied in the root of unity case by Grojnowski [2] and N. Xi [12,13], among others. In the root of unity case, Grojnowksi gives a simple description of a geometric indexing set [2, Theorem 2] only in type A. However, Theorem 1 of [2] does not apply in all cases (see the remark on p. 524 of [12]). And, to this author, at least, it is not obvious how to turn the statement of Theorem 1 into Theorem 2. One hopes that a better understanding of the representations of $\mathcal{H}$ in some small cases will help clarify these issues.

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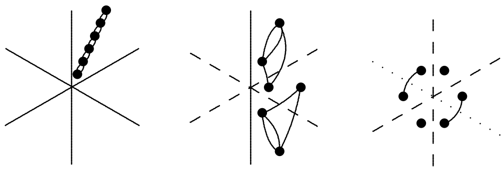
We begin by defining the affine Hecke algebra $\mathcal{H}$, and recalling some basic facts about the representations of $\mathcal{H}$. We will make extensive use of $\mathbb{C}[X]$, a large commutative subalgebra of $\mathcal{H}$, and *weights*, elements of $\mathrm{Hom}(\mathbb{C}[X], \mathbb{C})$, which describe the simple representations of $\mathbb{C}[X]$. An $\mathcal{H}$ module M can be described in part by which weights appear, i.e. which simple $\mathbb{C}[X]$ modules are composition factors of it. The most important construction we will use is that of the *principal series module* $M(t)$ which can be constructed from any weight $t \in T$, since every simple $\mathcal{H}$-module is a quotient of $M(t)$ for an appropriate choice of a weight t (Proposition 1(c)). We also recall from [9] several facts needed to analyze the modules $M(t)$, with some adaptations as necessary to deal with the root of unity case.

The main goal of the paper is to describe a way of visualizing and describing the composition factors of $M(t)$ directly from the combinatorial data of the weight t. This can be done with particular pictures based on the root system underlying $\mathcal{H}$. The following are examples in the type A_2 case.

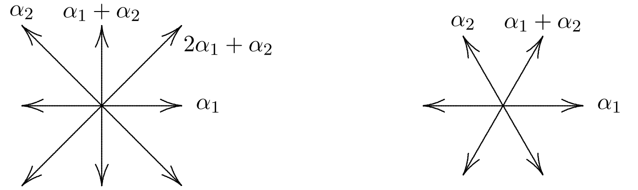


The lines in this picture represent the hyperplanes perpendicular to the roots α, and are drawn as solid, shaded, or dotted based on the values of the weight t on the elements $X^\alpha \in \mathcal{H}$. Each dot in the picture represents one dimension of the module $M(t)$, and dots are connected if a single composition factor of $M(t)$ contains both of these basis elements. The general goal is to determine a few rules that determine which of these lines should be drawn. That is, we hope to find a few algebraic statements that describe how $M(t)$ breaks down into composition factors which can be translated into these pictures. Essentially, Theorem 3(b), Proposition 4, and Theorem 5 below are sufficient to complete the classification in the rank two cases, for all values of q. These pictures provide a very straightforward way of determining the composition factors of $M(t)$, without relying on heavy computations. One also hopes that a complete classification of the rank two crystallographic cases will facilitate a greater understanding of the representation theory of $\mathcal{H}$ in all types.

2. Definitions

In this section, we introduce the needed definitions and several preliminary results about the affine Hecke algebra. Proofs of most previously known results will not be given.

The affine Hecke algebra. Let R be a root system in $\mathbb{R}^n$ with simple roots $\alpha_1, \dots \alpha_n$. Let R^+ be the set of positive roots and R^- the set of negative roots. We define the *rank* of R to be the number of simple roots n.



$R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2), \pm(2\alpha_1+\alpha_2)\}$ $\qquad$ $R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$

Two examples of root systems in $\mathbb{R}^2$

The reflection through H_α will be denoted by s_α, or s_i for the reflection through H_{α_i}. If π/m_{ij} is the angle between H_{α_i} and H_{α_j}, then $m_{ij} \in \{2, 3, 4, 6\}$ for $1 \leqslant i, j \leqslant n$, and the *Weyl Group* W_0 has presentation

$$W_0 = \langle s_1, \ldots s_n \mid s_i^2 = 1, \underbrace{s_i s_j s_i \ldots}_{m_{ij}\text{ factors}} = \underbrace{s_j s_i s_j \ldots}_{m_{ij}\text{ factors}}, \text{ for } 1 \leqslant i, j \leqslant n \rangle.$$

Let P be the *weight lattice*, spanned by the elements ω_i satisfying

$$\langle \omega_i, \alpha_j \rangle = \delta_{ij} \cdot \frac{1}{2} \langle \alpha_j, \alpha_j \rangle,$$

for α_i and α_j simple roots. Let Q be the lattice spanned by the simple roots α_i. Let

$$X = \{X^\lambda \mid \lambda \in P\}, \quad \text{with } X^\lambda \cdot X^\mu = X^{\lambda+\mu} \text{ for } \lambda, \mu \in P. \tag{1}$$

Then W_0 acts on X by

$$w \cdot X^\lambda = X^{w \cdot \lambda},$$

and this action extends linearly to an action of W_0 on the group algebra $\mathbb{C}[X]$.

The *affine Hecke algebra* $\mathcal{H}$ is the $\mathbb{C}$-algebra generated by $\{T_i \mid i \in I\}$ and $\{X^\lambda \mid \lambda \in P\}$, where $\mathbb{C}[X]$ is a subalgebra of $\mathcal{H}$, and subject to the relations

$$T_i^2 = (q - q^{-1}) T_i + 1, \quad \text{for } i = 1, 2, \ldots n, \tag{2}$$

$$\underbrace{T_i T_j T_i \ldots}_{m_{ij}\text{ factors}} = \underbrace{T_j T_i T_j \ldots}_{m_{ij}\text{ factors}} \quad \text{for } i \neq j, \quad \text{and} \tag{3}$$

$$X^\lambda T_{s_i} = T_{s_i} X^{s_i \cdot \lambda} + (q - q^{-1}) \frac{X^\lambda - X^{s_i \lambda}}{1 - X^{-\alpha_i}}, \quad \text{for } \lambda \in P,\ 1 \leqslant i \leqslant n. \tag{4}$$

The *rank* of $\tilde{H}$ is defined to be the rank of the underlying root system R. For $w \in W_0$, let

$$T_w = T_{i_1} T_{i_2} \ldots T_{i_k}$$

for a reduced word $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ in W_0. Then $\{T_w X^\lambda \mid w \in W_0, \lambda \in P\}$ is a $\mathbb{C}$-basis for $\mathcal{H}$.

Weights. Let $T = \mathrm{Hom}(X, \mathbb{C}^\times)$ be the set of group homomorphisms from X to $\mathbb{C}^\times$. Then T is an abelian group with W_0-action given by

$$w \cdot t(X^\lambda) = t(X^{w^{-1} \cdot \lambda}) \quad \text{for } t \in T,\ w \in W_0,\ \lambda \in P.$$

An element of T is called a *weight*. For a weight t, the subgroup of W_0 that fixes t under this action is generated by $\{s_i \mid t(X^{\alpha_i}) = 1\}$. (This relies on the fact that we chose P rather than Q to build $\mathcal{H}$. See [11], 3.15, 4.2, and 5.3).

For any finite-dimensional $\mathcal{H}$-module M, define the *t-weight space* and the *generalized t-weight space* of M by

$$M_t = \{m \in M \mid X^\lambda \cdot m = t(X^\lambda) m \text{ for all } X^\lambda \in X\}, \quad \text{and}$$

$$M_t^{\text{gen}} = \{m \in M \mid \text{for all } X^\lambda \in X,\ (X^\lambda - t(X^\lambda))^k m = 0 \text{ for some } k \in \mathbb{Z}_{>0}\},$$

respectively. Then $M=\bigoplus_{t\in T} M_t^{\mathrm{gen}}$ is a decomposition of M into Jordan blocks for the action of $\mathbb{C}[X]$. An element $t \in T$ is a *weight of* M if $M_t^{\mathrm{gen}} \neq 0$.

Induced modules and intertwining operators. If $I \subseteq \{1,\dots n\}$, define $W_I = \langle s_i \mid i \in I\rangle$ and

$$\mathcal{H}_I = \left\{T_w X^\lambda \;\middle|\; \lambda \in P,\ w \in W_I\right\}.$$

For example, $\mathcal{H}_\emptyset = \mathbb{C}[X]$, while $\mathcal{H}_{\{i\}}$ is the subalgebra of $\mathcal{H}_{\{i\}}$ generated by $\mathbb{C}[X]$ and T_i. Then for $t \in T$ such that $t(X^{\alpha_i}) = q^2$ for $i \in I$, define $\mathbb{C}v_t$ to be the one-dimensional $\mathcal{H}_I$-module spanned by v_t, with $\mathcal{H}_I$ action given by

$$T_i \cdot v_t = q v_t \quad \text{and} \quad X^\lambda \cdot v_t = t\big(X^\lambda\big) v_t, \quad \text{for } X^\lambda \in X.$$

Proposition 1. *(See [9], Lemma 1.17.) Let $\mathbb{C}v_t$ be defined as above, and let $M = \mathrm{Ind}_{\mathcal{H}_I}^{\mathcal{H}} \mathbb{C}v_t$. Let $W_I = \langle s_i \mid i \in I\rangle$, and let W_0/W_I be a set of minimal length coset representatives of W_I-cosets in W_0.*

(a) *Then the weights of M are $\{wt \mid w \in W_0/W_I\}$, and*

$$\dim\big(M_{wt}^{\mathrm{gen}}\big) = (\#\ \textit{of}\ v \in W_0/W_I \ \textit{with}\ vt = wt).$$

(b) *There is a basis of M consisting of elements of the form*

$$m_w = T_w v_t + \sum_{u<w,\, u\in W_0/W_I} p_{w,u} T_u v_t,$$

for $w \in W_0/W_I$, such that $m_w \in M_{wt}$.

(c) *If t is a weight of an irreducible $\mathcal{H}$-module N and $I=\emptyset$, then N is a quotient of M. In fact, if $v \in N$ is a non-zero vector in N_t, then*

$$\phi : M \to N,$$
$$v_t \mapsto v$$

extends to a surjective $\mathcal{H}$-module homomorphism.

In particular, if $I=\emptyset$, then we call

$$M(t) = \mathcal{H} \otimes_{\mathbb{C}[X]} \mathbb{C}v_t = \mathrm{span}\{T_w v_t \mid w \in W_0\}$$

the *principal series module* for t.

Part (c) of this lemma implies that the weights of a single simple finite-dimensional module M lie in a single orbit Wt. We call this orbit (and, by abuse of terminology, any element of the orbit) the *central character* of M. In fact, $\mathcal{H}$ has finite dimension over its center, and thus all simple $\mathcal{H}$-modules are finite-dimensional (see [9], Section 2.3). Thus, this proposition tells us that understanding the composition factors of all the principal series modules $M(t)$ is sufficient for understanding all the simple $\tilde{H}$-modules.

For a weight t with $t(X^{\alpha_i}) \neq 1$ and an $\mathcal{H}$-module M, define a $\mathbb{C}$-linear operator $\tau_i : M_t^{\mathrm{gen}} \to M$ by

$$\tau_i(m) = \left(T_i - \frac{q-q^{-1}}{1-X^{-\alpha_i}}\right)\cdot m. \tag{5}$$

Theorem 2. *(See [9], Proposition 1.18.)*

(a) $1 - X^{-\alpha_i}$ *is invertible as an operator on* M_t^{gen}*, so that* $\tau_i : M_t^{\text{gen}} \to M$ *is well defined.*
(b) *As operators on* M_t^{gen}, $X^\lambda \tau_i = \tau_i X^{s_i\lambda}$ *for all* $X^\lambda \in X$*, so that* $\tau_i(M_t^{\text{gen}}) \subseteq M_{s_it}^{\text{gen}}$.
(c) *As operators on* M_t^{gen},

$$\tau_i\tau_i = \frac{(q - q^{-1}X^{\alpha_i})(q - q^{-1}X^{-\alpha_i})}{(1 - X^{\alpha_i})(1 - X^{-\alpha_i})}.$$

(d) *The maps* $\tau_i : M_t^{\text{gen}} \to M_{s_it}^{\text{gen}}$ *and* $\tau_i : M_{s_it}^{\text{gen}} \to M_t^{\text{gen}}$ *are both invertible if and only if* $t(X^{\alpha_i}) \neq q^{\pm 2}$.
(e) *If* $i \neq j$ *and* m_{ij} *is defined as in* (3)*, then* $\underbrace{\tau_i\tau_j\tau_i\ldots}_{m_{ij}\text{ factors}} = \underbrace{\tau_j\tau_i\tau_j\ldots}_{m_{ij}\text{ factors}}$*, whenever both sides are well-defined operators.*

For $t \in T$, the *calibration graph* of t is the graph with vertices labeled by the elements of the orbit W_0t and edges (wt, s_iwt) if $(wt)(X^{\alpha_i}) \neq q^{\pm 2}$. The τ operators are used to prove the following.

Theorem 3.

(a) *(See [10], Proposition 2.3.) If* $w \in W_0$ *and* $t \in T$ *then* $M(t)$ *and* $M(wt)$ *have the same composition factors.*
(b) *(See [9], Proposition 1.6.) Let* M *be a finite-dimensional* $\mathcal{H}$*-module, and let* t *and* wt *be two elements of* W_0t *in the same connected component of the calibration graph for* t*. Then*

$$\dim(M_t^{\text{gen}}) = \dim(M_{wt}^{\text{gen}}).$$

(c) *(See [4].)* $M(t)$ *is irreducible if and only if* $P(t) := \{\alpha \in R^+ \mid t(X^\alpha) = q^{\pm 2}\} = \emptyset$.

The structure of modules. Theorem 3(b) shows that the connected components of the calibration graph encode certain sets of weights whose corresponding weight spaces M_t^{gen} must have the same dimension in any irreducible $\tilde{H}$-module M. These ideas lead us to the following propositions which will be fundamental in our later classification.

Proposition 4. *Let* M *be an irreducible* 2*-dimensional* $\mathcal{H}$*-module and assume* $q^2 \neq 1$.

(a) *If* M *has two different weight spaces* M_t *and* $M_{t'}$*, then* $t' = s_it$ *for some* i*, and* $t(X^{\alpha_i}) \neq q^{\pm 2}$ *or* 1*, but* $t(X^{\alpha_j}) = q^{\pm 2}$ *and* $s_it(X^{\alpha_j}) = q^{\pm 2}$ *for* $j \neq i$*. Moreover, there is a unique* 2*-dimensional module* (*up to isomorphism*) *containing these two weight spaces.*
(b) *If* M *has only one weight space* M_t^{gen}*, then* $t(X^{\alpha_i}) = 1$ *for some* i*, and for* $j \neq i$, $t(X^{\alpha_j}) = q^2$*, and either* $\langle \alpha_j, \alpha_i^\vee \rangle = 0$ *or else* $q^2 = -1$ *and it is not the case that* $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ *and* $\langle \alpha_j, \alpha_i^\vee \rangle = -2$.

Proof. (a) If $t(X^{\alpha_i}) = 1$ for some i, then consider M as an $\mathcal{H}_{\{i\}}$-module. By Kato's criterion (Theorem 3(c)), the fact that $q^2 \neq 1$, and Proposition 1(c), there is only one irreducible $\mathcal{H}_{\{i\}}$-module N with central character t, where $t(X^{\alpha_i}) = 1$. This module is 2-dimensional with $\dim N_t^{\text{gen}} = 2$. Thus $M \cong N$ as $\mathcal{H}_{\{1\}}$-modules and $t = t'$.

Assume M has two different weight spaces M_t and $M_{t'}$. Then since M is irreducible, some τ_i must be non-zero on M_t, and $t' = s_it$. Then τ_i must also be non-zero on M_{s_it}, and $t(X^{\alpha_i}) \neq q^{\pm 2}$. Since $M_{s_jt} = 0 = M_{s_jt'}$ for $j \neq i$, $t(X^{\alpha_j}) = q^{\pm 2}$ and $s_it(X^{\alpha_j}) = q^{\pm 2}$. The weight structure determines the action of $\mathbb{C}[X]$ on M, and since we know how the operators τ_i act, the actions of the T_i are determined as well, so that the module structure of M is determined by its weight structure.

(b) Assume M consists of one generalized weight space M_t^{gen}, with $v_t \in M_t$. If all the operators τ_i were defined on M_t, then $\tau_i(v_t) = 0$ for all i. Hence $T_iv_t \in \mathbb{C}v_t$ for all i and v_t would span a submodule of M, a contradiction. Thus some τ_i is not well defined and $t(X^{\alpha_i}) = 1$.

If $t(X^{\alpha_j}) = 1$ for any $j \neq i$, then $t(X^{\beta}) = 1$ for any β in the span of α_i and α_j. Then M as an $\mathcal{H}_{\{i,j\}}$-module contains a principal series module, and must have dimension at least as large as the number of roots in the root subsystem generated by α_i and α_j, which is a contradiction since this number will be greater than 2. Then $t(X^{\alpha_j}) \neq 1$ for all $j \neq i$.

Then consider M as an $\mathcal{H}_{\{i\}}$ module, which is irreducible by Theorem 3(c). The action of $\mathcal{H}$ on the basis $\{v_t, T_i v_t\}$ is given by

$$M(T_i) = \begin{bmatrix} 0 & 1 \\ 1 & q - q^{-1} \end{bmatrix}, \quad \text{and} \quad M\big(X^{\alpha_j}\big) = t\big(X^{\alpha_j}\big) \begin{bmatrix} 1 & (q - q^{-1})\langle \alpha_j, \alpha_i^\vee \rangle \\ 0 & 1 \end{bmatrix}.$$

Then since τ_j (which is well defined since $t(X^{\alpha_j}) \neq 1$) must be the zero map on M_t^{gen}, we have $M(T_j) = M(\frac{q-q^{-1}}{1-X^{-\alpha_j}})$, and

$$M(T_j) = \big(q - q^{-1}\big)\left(\frac{1}{1 - t(X^{-\alpha_j})}\right) \begin{bmatrix} 1 & \frac{(q-q^{-1})t(X^{-\alpha_j})}{1-t(X^{-\alpha_j})} \langle -\alpha_j, \alpha_i^\vee \rangle \\ 0 & 1 \end{bmatrix}.$$

However, since the relation (2) can be written $(T_j - q)(T_j + q^{-1}) = 0$, $M(T_j)$ must have eigenvalues q or $-q^{-1}$ and $t(X^{-\alpha_j}) = q^{\pm 2}$.

If $q^2 \neq -1$, so that $q \neq -q^{-1}$, then either $M(T_j) - qI$ or $M(T_j) + q^{-1}I$ is invertible, so that the other must actually be zero and so the off-diagonal term must be zero. The only way this can occur is if $\langle \alpha_j, \alpha_i^\vee \rangle = 0$.

If $q^2 = -1$ then

$$M(T_j) = \begin{bmatrix} q & \langle \alpha_j, \alpha_i^\vee \rangle \\ 0 & q \end{bmatrix}.$$

Then $M(T_i)$ and $M(T_j)$ must satisfy the same braid relation as T_i and T_j, which is determined by the type of root system spanned by α_i and α_j. A check of the possible root systems ($A_1 \times A_1$, A_2, C_2, and G_2) shows that the braid relation is satisfied unless $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ and $\langle \alpha_j, \alpha_i^\vee \rangle = -2$. □

Theorem 5. *(See [9], Lemma 1.19.) Assume $q^2 \neq 1$. Let $t \in T$ such that $t(X^{\alpha_i}) = 1$ and suppose that M is an $\mathcal{H}(q)$-module such that $M_t^{\text{gen}} \neq 0$. Let W_t be the stabilizer of t under the action of W_0 on T. Assume that $\overline{w} \in W_0/W_t$ is such that t and $\overline{w}t$ are in the same connected component of the calibration graph for t, and let w be a minimal length coset representative for $\overline{w}$. Then*

(a) $\dim(M_{wt}^{\text{gen}}) \geqslant 2$ *and* $\dim M_{wt}^{\text{gen}} > \dim M_{wt}$.
(b) *If* $M_{s_j wt}^{\text{gen}} = 0$ *then* $(\overline{w}t)(X^{\alpha_j}) = q^{\pm 2}$ *and if, in addition,* $q^2 \neq -1$, *then* $\langle w^{-1}\alpha_j, \alpha_i^\vee \rangle = 0$.

Visualizing modules. For $t \in T$, define

$$Z(t) = \big\{\alpha \in R^+ \;\big|\; t\big(X^\alpha\big) = 1\big\} \quad \text{and} \quad P(t) = \big\{\alpha \in R^+ \;\big|\; t\big(X^\alpha\big) = q^{\pm 2}\big\}.$$

Notice that $|Z(t)|$ and $|P(t)|$ are constant on orbits $W_0 t$, since the action of W_0 permutes the multiset $\{t(X^\alpha) \mid \alpha \in R\}$.

The τ operators and the sets $Z(t)$ and $P(t)$ provide extensive information about the structure (and sometimes the composition factors) of $M(t)$. Let H_α be the hyperplane fixed by s_α for $\alpha \in R$. A *chamber* is a connected component of $\mathbb{R}^n \setminus \cup_{\alpha \in R^+} H_\alpha$, and W_0 acts faithfully and simply transitively on the set of chambers. Choose a fundamental chamber C and define the positive side of a hyperplane H_α to be the side on which C lies. The map

$$\{\text{chambers}\} \leftrightarrow W_0,$$
$$w^{-1}C \mapsto w \tag{6}$$

is a bijection.

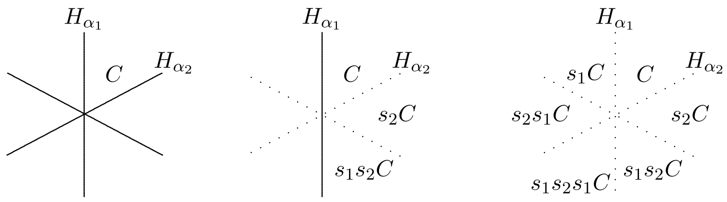
By 3.15, 4.2, and 5.3 of Steinberg [11] the stabilizer of t is

$$W_t = \langle s_\alpha \mid \alpha \in Z(t) \rangle.$$

Thus, if W_0/W_t is a set of minimal length coset representatives of W_t-cosets in W_0, then

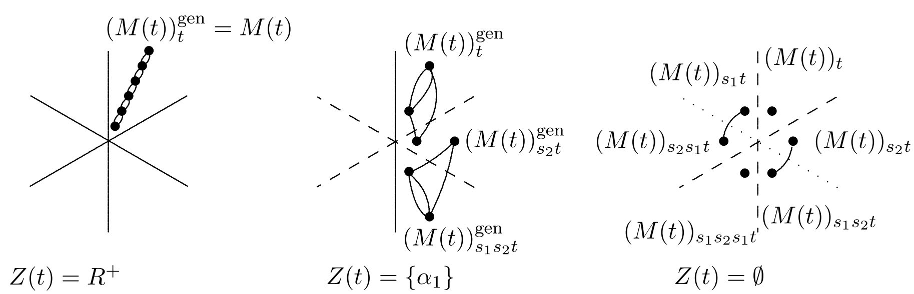
$$W_0/W_t \leftrightarrow W_0 t \leftrightarrow \{\text{chambers on the positive side of all } H_\alpha,\ \alpha \in Z(t)\},$$
$$w \mapsto wt \mapsto w^{-1}C, \quad \text{for } w \in W_0/W_t \tag{7}$$

are bijections. Again using type A_2 as an example, each W_0-orbit in T has a representative such that the bijection (7) is illustrated by one of the following pictures.



$Z(t) = R^+$ $W_0/W_t = \{1\}$ $Z(t) = \{\alpha_1\}$ $W_0/W_t = \{1, s_2, s_1s_2\}$ $Z(t) = \emptyset$ $W_0/W_t = W_0$

The bijection (7) shows that the weights of $M(t)$ are in bijection with the chambers on the positive side of the H_α with $\alpha \in Z(t)$, so that $M(t)$ can be visualized within those chambers. Recall that the elements of the orbit $W_0 t$ are the vertices of the calibration graph. The hyperplanes H_α for $\alpha \in P(t)$ (which are drawn as dashed lines) divide the chambers into subsets corresponding to the components of the calibration graph. To visualize $M(t)$ in the picture of the chambers, we draw a number of dots in each chamber equal to the dimension of the corresponding weight space. (See Fig. 1.) Then the behavior of the τ operators between two weight spaces is also encoded in the lines between the corresponding chambers – solid, dashed, and dotted hyperplanes correspond to τ operators that are, respectively, undefined, defined but not invertible in both directions, or defined and invertible in both directions. All this information can be combined to visualize the composition factors of $M(t)$, which is the main goal of this paper.



$Z(t) = R^+$ $P(t) = \emptyset$ $Z(t) = \{\alpha_1\}$ $P(t) = \{\alpha_2, \alpha_1 + \alpha_2\}$ $Z(t) = \emptyset$ $P(t) = \{\alpha_1, \alpha_2\}$

Fig. 1. Visualizing modules in type A_2.

The structural theorems above tell us which dots to connect together in these drawings, and the resulting picture describes the composition factors of $M(t)$ – dots are connected via some path exactly when the corresponding basis vectors lie in the same composition factor. In particular, the third drawing gives a good picture of Theorem 3(b). If two chambers have a common boundary that is a dotted hyperplane, then the τ operator between the two corresponding weight spaces will be invertible in both directions, and those weight spaces must have the same dimension in any irreducible $\mathcal{H}$-module with the central character shown in the picture. Thus in this case, the basis vectors with those weights lie in the same composition factor and are connected above. Similarly, the second drawing demonstrates Theorem 5. For this central character, Theorem 5 implies that the t-weight space of an irreducible $\mathcal{H}$-module must have dimension 0 or 2, and an irreducible containing a 2-dimensional generalized t weight space must also have a non-zero s_2t weight space. Thus, in the picture, the dots in the t chamber are connected, and are jointly connected to a dot in the s_2t chamber, since the corresponding basis vectors must lie in a 3-dimensional composition factor. In the first picture, $M(t)$ is irreducible by Theorem 3(c), so the dots are all connected.

Calibrated modules and weights. A weight t is defined to be *regular* if W_t, the subgroup of the Weyl group that fixes t, is trivial. Then a weight t is regular if and only if $Z(t) = \emptyset$.

A representation M is *calibrated* if

$$M_t^{\text{gen}} = M_t$$

for all weights t, i.e. the subalgebra $\mathbb{C}[X] \subseteq \mathcal{H}$ acts diagonally on M.

Proposition 6. *(See [9], Proposition 1.10.)*

(a) *If $q^2 \neq 1$, an irreducible $\mathcal{H}$-module is calibrated if and only if*

$$\dim\bigl(M_t^{\text{gen}}\bigr) = 1$$

for all weights t of M.

(b) *If M is an $\mathcal{H}$-module with regular central character, then M is calibrated.*

When $q^2 = 1$, all irreducible modules are calibrated, as will be shown by Theorem 9.

Calibrated modules with regular central character.

Theorem 7. *(See [9], Proposition 1.11.) Assume $q^2 \neq 1$. Let t be a regular central character, and let G be a component of the calibration graph. Define*

$$\mathcal{H}^{(t,G)} = \mathbb{C}\text{-}span\{v_w \mid wt \in G\}.$$

Then the vector space $\mathcal{H}^{(t,G)}$ is an irreducible calibrated $\mathcal{H}$-module with action

$$X^\lambda \cdot v_w = (wt)\bigl(X^\lambda\bigr)v_w \quad \text{for } X^\lambda \in X,\ w \in W_0, \quad \text{and}$$

$$T_i \cdot v_w = (T_i)_w v_w + \bigl(q^{-1} + (T_i)_w\bigr)v_{s_i w} \quad \text{for } 1 \leqslant i \leqslant n,\ w \in W_0,$$

where $(T_i)_w = \frac{q-q^{-1}}{1-wt(X^{-\alpha_i})}$, and $v_{s_i w} = 0$ if $s_i wt \notin G$.

Note that since t is a regular central character, $wt(X^{\alpha_i}) \neq 1$ for $w \in W_0$ and $i = 1, \ldots, n$. Hence $(T_i)_w$ is always well defined. The most difficult part of this theorem is checking that the given

$\mathcal{H}$-module structure satisfies the braid relation. Since t is assumed to be regular, this essentially follows from the braid relation on the τ_i. (See [9] for details.) In fact, more is true.

Theorem 8. *(See [9], Proposition 1.11.) Assume $q^2 \neq 1$, and let M be an irreducible $\mathcal{H}$-module with regular central character t. (M is therefore calibrated). Then if wt is a weight of M, let G be the component of the calibration graph containing wt. Then the weights of M are exactly the vertices in G. In addition, M is isomorphic to the module $\mathcal{H}^{(t,G)}$ given in Theorem 7.*

Clifford theory when $q^2 = 1$. Let $q^2 = 1$. Then we can identify the subalgebra H spanned by $\{T_w \mid w \in W_0\}$ with $\mathbb{C}[W_0]$, so that

$$\mathcal{H} = \operatorname{span}\{wX^\lambda \mid w \in W_0,\ \lambda \in P\}.$$

Let M be a finite-dimensional simple $\mathcal{H}$-module and let $t \in T$ such that $M_t \neq 0$. Let W_t be the stabilizer of t in W_0. As vector spaces, $M_t \cong M_{wt}$ via the map $m \mapsto wm$, and

$$M = \bigoplus_{w \in W_0/W_t} M_{wt},$$

since M is simple and the right-hand side is a submodule of M. (This implies that all $\mathcal{H}$ modules are calibrated.)

Theorem 9. *(See also [5].) Let $q^2 = 1$ and let M be an irreducible $\mathcal{H}$-module. Let $t \in T$ be such that $M_t \neq 0$. Define $\mathcal{H}_{W_t} = \mathbb{C}$-Span$(\{wX^\lambda \mid w \in W_t, \lambda \in P\})$, a subalgebra of $\mathcal{H}$. Then*

(a) *M_t is an irreducible W_t-module.*
(b) *M_t is an $\mathcal{H}_{W_t}$-module and*

$$M \cong \mathcal{H} \otimes_{\mathcal{H}_{W_t}} M_t.$$

Thus, when $q^2 = 1$, the standard conclusions of Clifford Theory completely describe the irreducible representations of $\mathcal{H}$.

3. Type A_1

We begin with the type A_1 affine Hecke algebra. The results here are known, but this section serves as a model for the other types.

The affine Hecke algebra. The type A_1 affine Hecke algebra is built on the root data of SL_2. Let

$$R = \mathbb{Z}\alpha_1, \qquad P = \mathbb{Z}\omega_1 \quad \text{and} \quad X = \{X^{k\omega_1} \mid k \in \mathbb{Z}\}$$

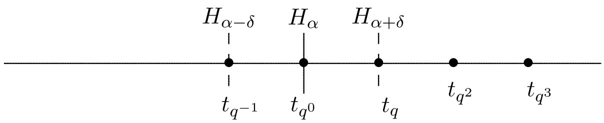
so that X is the group generated by X^{ω_1} and is isomorphic to P. The Weyl group is $W_0 = \{1, s_1\}$ with $s_1^2 = 1$, and setting $s_1 X^{\omega_1} = X^{-\omega_1}$ defines an action of W_0 on X. Let $q \in \mathbb{C}^\times$. The *affine Hecke algebra* of type A_1 is defined as in Section 2. We let t_z denote the weight given by $t_z(X^{\omega_1}) = z$.

Proposition 10. *Let $M(t_z)$ denote the principal series module for a weight t_z.*

(a) *If $z \neq \pm q^{\pm 1}$, then $M(t_z)$ is irreducible.*
(b) *If $z = \pm q^{\pm 1}$, then $M(t_z)$ has two 1-dimensional composition factors.*
(c) *If $q^2 = 1$ and $z = \pm q$, $M(t_z)$ is a direct sum of its composition factors.*

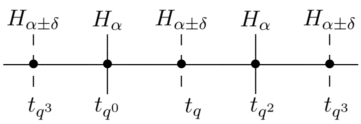
Proof. Part (a) is given by Kato's criterion (Theorem 3(c)), which also shows that $M(t)$ must be reducible if $t(X^{\omega_1}) = \pm q^{\pm 1}$. In parts (b) and (c), it is straightforward to explicitly calculate the action of $\mathcal{H}$ on the basis $\{v_t, T_1 v_t\}$. □

To visualize this classification, identify $\{t_{q^x} \mid x \in \mathbb{R}\}$ with the real line. In this picture, the hyperplane H_{α_1} is marked with a solid line, while $H_{\alpha_1 \pm \delta}$ are denoted by dashed lines.

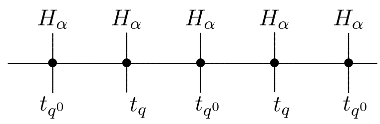


Characters t_{q^x}, generic q

If q is a primitive 2ℓth root of unity then $\{t_{q^x} \mid x \in \mathbb{R}\}$ is identified with $\mathbb{R}/2\ell\mathbb{Z}$ and $H_\alpha = \{k\ell \mid k \in \mathbb{Z}\}$. The following picture shows the specific case $\ell = 2$, so that $t_1 = t_{q^2} = \cdots$, and $\ell = 1$, in which case $t_1 = t_q = t_{q^2} = \cdots$. The periodicity is evident in the picture.

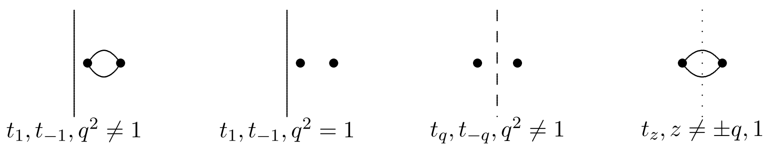


Characters t_{q^x}, $q^4 = 1$



Characters t_{q^x}, $q^2 = 1$

The following pictures show the chambers around t as in Fig. 1, which give a picture of $M(t)$ and its composition factors.



The visualization is not as clear in this case as in others, since it is the smallest example of the affine Hecke algebra, but the essential ingredients are present. The chamber pictures should be interpreted as those in Fig. 1. The weights of the $M(t)$ are all displayed, as are the actions of the τ operators that determine the composition factors of $M(t)$. Notice also the connection to the drawings of central characters above. The pictures of the $M(t_{q^x})$ are a picture of a small open neighborhood around the point t_{q^x} in the picture of the characters. The complete classification of $\mathcal{H}$ modules is summarized in the following tables (see Table 1).

Table 1
Table of possible central characters in type A_1.

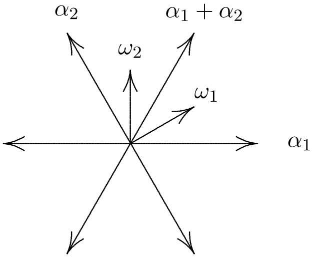
	Dims. of irreds. by weight				
	t_1	t_{-1}	t_q	t_{-q}	t_z, $z \neq \pm 1$ or $\pm q$
$q^4 \neq 1$	2	2	1, 1	1, 1	2
$q^2 = -1$	2	2	1, 1	N/A	2
$q = -1$	1, 1	1, 1	N/A	N/A	2

The way that the representation theory of $\mathcal{H}$ varies with q can be seen through a number of different lenses. The structure of $M(t)$ is controlled by the τ operators, which in turn are largely controlled by the sets $P(t)$ and $Z(t)$. In the picture of the characters, we see that the hyperplanes

H_α and $H_{\alpha\pm\delta}$ are distinct *unless* $q^2 = \pm 1$. When these hyperplanes coincide, the sets $P(t)$ and $Z(t)$ change, changing the structure of the corresponding modules. Similar interpretations of course hold in all types.

4. Type A_2

The type A_2 root system is $R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$, where $\langle \alpha_1, \alpha_2^\vee \rangle = -1 = \langle \alpha_2, \alpha_1^\vee \rangle$, with Weyl group $W_0 \cong S_3$. The simple roots are α_1 and α_2, and $\alpha_1 + \alpha_2$ is the only other positive root.



The type A_2 root system

In this picture, s_i is reflection through the hyperplane perpendicular to α_i. The fundamental weights satisfy

$$\omega_1 = \frac{1}{3}(2\alpha_1 + \alpha_2), \qquad \alpha_1 = 2\omega_1 - \omega_2,$$
$$\omega_2 = \frac{1}{3}(2\alpha_2 + \alpha_1), \qquad \alpha_2 = 2\omega_2 - \omega_1.$$

Let

$$P = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\} \quad \text{and} \quad Q = \mathbb{Z}\text{-span}(R)$$

be the *weight lattice* and *root lattice* of R, respectively.

The affine Hecke algebra $\mathcal{H}$ is defined as in Section 2. Let

$$\mathbb{C}[Q] = \{X^\lambda \mid \lambda \in Q\} \quad \text{and} \quad T_Q = \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[Q], \mathbb{C}).$$

Define

$$t_{z,w} : \mathbb{C}[Q] \to \mathbb{C} \quad \text{by } t_{z,w}(X^{\alpha_1}) = z \quad \text{and} \quad t_{z,w}(X^{\alpha_2}) = w.$$

For each $t_{z,w} \in T_Q$, there are 3 elements $t \in T$ with $t|_Q = t_{z,w}$, determined by

$$t(X^{\omega_1})^3 = z^2 w \quad \text{and} \quad t(X^{\omega_2}) = t(X^{-\omega_1}) \cdot zw.$$

The dimension of the simple modules with central character t and the submodule structure of $M(t)$ depends only on $t|_Q$. Thus we begin by examining the W_0-orbits in T_Q. For a generic weight t, $P(t)$ and $Z(t)$ are empty so that $M(t)$ is irreducible by Theorem 3(c). Thus we examine only the non-generic orbits.

Proposition 11. *If $t \in T_Q$, and $P(t) \cup Z(t) \neq \emptyset$, then t is in the W_0-orbit of one of the following weights*:

$$t_{1,1}, t_{1,q^2}, t_{q^2,1}, t_{q^2,q^2}, \quad \{t_{1,z} \mid z \in \mathbb{C}^\times z \neq 1,\ q^{\pm 2}\},$$
$$\text{or} \quad \{t_{q^2,z} \mid z \in \mathbb{C}^\times z \neq 1,\ q^{\pm 2},\ q^{-4}\}.$$

Proof. The proof consists of exhausting all possibilities for $Z(t)$ and $P(t)$, up to the action of W_0.

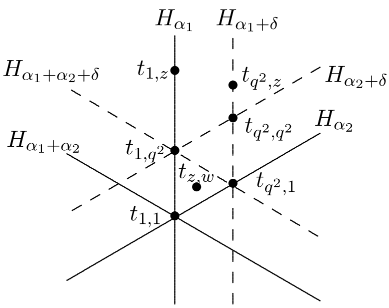
Case 1: If $Z(t)$ contains two positive roots, then it must contain the third. This implies $t = t_{1,1}$.

Case 2: If $Z(t)$ contains only one root, by applying an element of W_0, assume that it is α_1. Then $t(X^{\alpha_2}) = t(X^{\alpha_1+\alpha_2})$, so either $P(t) = \emptyset$ or $P(t) = \{\alpha_2, \alpha_1 + \alpha_2\}$. The first central character is $t_{1,z}$ for some $z \neq 1$ or $q^{\pm 2}$. (If $z = 1$ or $z = q^{\pm 2}$, either $P(t)$ or $Z(t)$ would be larger.) For the second case, there are two potential choices for the orbit, arising from choosing $t(X^{\alpha_2}) = q^2$ or q^{-2}. However, $t_{1,q^{-2}}$ is in the same orbit as $t_{q^2,1}$.

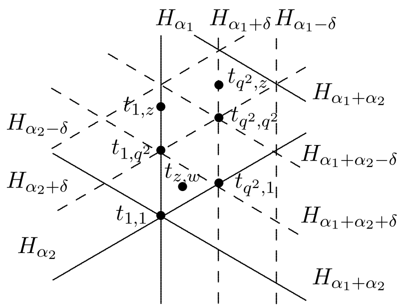
Case 3: Now assume that $Z(t) = \emptyset$. If $P(t)$ is not empty, assume that $\alpha_1 \in P(t)$ and $t(X^{\alpha_1}) = q^2$. Then $t(X^{\alpha_2}) \neq q^{-2}$ by assumption on $Z(t)$. Then it is possible that $\alpha_2 \in P(t)$, in which case $t = t_{q^2,q^2}$. If $\alpha_1 + \alpha_2 \in P(t)$, then $t(X^{\alpha_2}) = q^{-4}$ and $t = t_{q^2,q^{-4}} = s_2 s_1 t_{q^2,q^2}$. Otherwise, $t = t_{q^2,z}$ for some $z \neq 1, q^{\pm 2}, q^{-4}$. □

Remark. Note that if $q^2 = -1$, then t_{1,q^2}, $t_{q^2,1}$, and t_{q^2,q^2} are all in the same W_0-orbit. If $q^2 = 1$, then $t_{1,1} = t_{q^2,1} = t_{1,q^2} = t_{q^2,q^2}$, and $t_{1,z} = t_{q^2,z}$. Also, for every generic weight $t_{z,w}$, there are six weights in its W_0-orbit, all of which are generic.

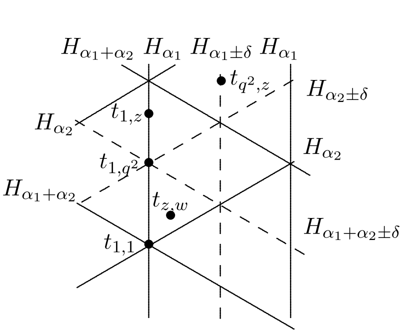
It is helpful to draw a picture of the weights $\{t_{q^x,q^y} \mid x, y \in \mathbb{R}\}$ for various values of q. Solid lines in these pictures show the hyperplanes H_α, while dashed lines denote hyperplanes $H_{\alpha\pm\delta}$, for $\alpha \in R^+$. The weight t_{q^x,q^y} is the point x units away from H_{α_1} and y units away from H_{α_2}.



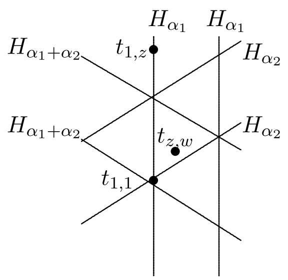
Central characters with general q



Characters with q^2 a third root of unity



Central characters with $q^2 = -1$



Central characters with $q^2 = 1$

Analysis of the characters.

Proposition 12. *There are six* 1*-dimensional representations of* $\mathcal{H}$*. Three of these representations are given by the three weights* t *with* $t|_Q = t_{q^2,q^2}$*, with each* T_i *acting with eigenvalue* q*. The other three are given by the three weights* t *with* $t|_Q = t_{q^{-2},q^{-2}}$*, with each* T_i *acting with eigenvalue* $-q^{-1}$*.*

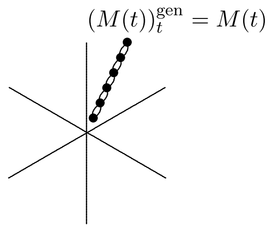
Proof. The relation (2) determines the two possible eigenvalues for the action of T_1 on a 1-dimensional module. The relation in (4) relates the eigenvalues for X^{ω_1} and T_1. □

Principal series modules. We now examine the pictures of the chambers around a weight $t|_Q$, as a way of visualizing $M(t)$. The solid, dashed and dotted hyperplanes hold the same interpretation as in Fig. 1. These hyperplanes encode the action of the τ operators, which largely determine the composition factors of $M(t)$. Assume for now that $q^2 \neq 1$.

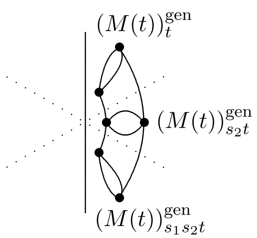
Case 1: $P(t)$ empty. By Theorem 3(c), if $P(t) = \{\alpha \in R^+ | t(X^{\alpha_1}) = 1\}$ is empty, then $M(t)$ is irreducible and is the only irreducible module with central character t. This case includes the central characters $t_{1,1}$, $t_{1,z}$, and $t_{z,w}$ for generic z, w – that is, any z and w for which $P(t_{z,w}) = Z(t_{z,w}) = \emptyset$.

Case 2: $Z(t) = \emptyset$, $P(t) \neq \emptyset$. This case includes the central characters t_{q^2,q^2}, and $t_{q^2,z}$. If $Z(t)$ is empty, then $M(t)$ is calibrated and the irreducible modules with central character t are in one-to-one correspondence with the components of the calibration graph.

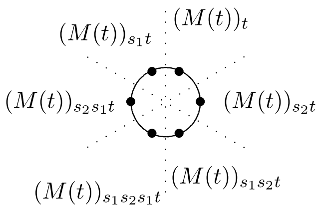
Case 1:



$t_{1,1}$, $q^2 \neq 1$

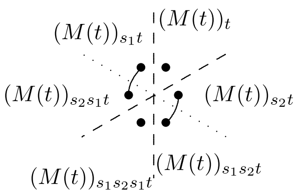


$t_{1,z}$, $q^2 \neq 1$

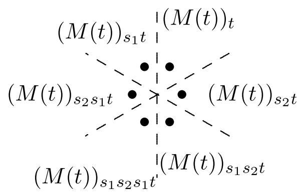


$t_{z,w}$

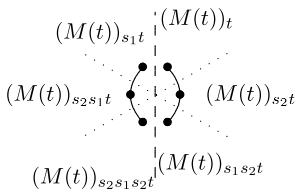
Case 2:



$t_{q^2,q^2}, q^4 \neq 1, q^6 \neq 1$



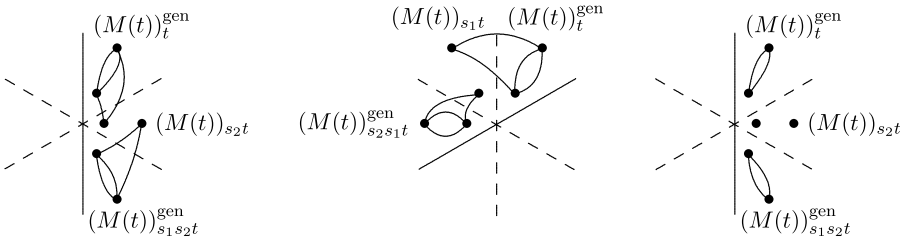
t_{q^2,q^2}, q^2 a primitive third root of unity



$t_{q^2,z}, q^2 \neq 1$

Case 3: $P(t) \neq \emptyset$, $Z(t) \neq \emptyset$. The only central characters with both $Z(t)$ and $P(t)$ non-empty are $t_{1,1}$ and $t_{1,z}$ when $q^2 = 1$, and $t_{q^2,1}$ and t_{1,q^2} in all cases. If $q^4 = 1$, then $t_{q^2,1}$ and t_{1,q^2} are in the same orbit, and are in the same orbit as t_{q^2,q^2}. If $q^2 = 1$, then $t_{q^2,1} = t_{1,q^2} = t_{1,1}$.

If $q^2 \neq \pm 1$ and $t|_Q = t_{1,q^2}$ or $t_{q^2,1}$, then Theorem 5 shows that $M(t)$ has two 3-dimensional composition factors. When $q^2 = -1$ and $t|_Q = t_{1,q^2}$, Proposition 4 shows the two-dimensional weight space $M(t)^{\mathrm{gen}}_t$ makes up an entire composition factor, as does $M(t)^{\mathrm{gen}}_{s_1s_2t}$. The remaining composition factors are two copies of the 1-dimensional module with weight s_2t.



$t_{1,q^2}, q^4 \neq 1$ $\qquad$ $t_{q^2,1}, q^4 \neq 1$ $\qquad$ $t_{1,q^2}, q^2 = -1$

Explicitly, let $\mathbb{C}_{q^2,1}$ be the 1-dimensional $\mathcal{H}_{\{1\}}$-module spanned by v_t and let $\mathbb{C}_{1,q^{-2}}$ be the 1-dimensional $\mathcal{H}_{\{2\}}$-module spanned by $v_{s_2s_1t}$, given by

$$X^\lambda v_t = t(X^\lambda) v_t \quad \text{and} \quad T_1 v_t = q v_t, \quad \text{and}$$
$$X^\lambda v_{s_2s_1t} = (s_2s_1t)(X^\lambda) v_{s_2s_1t} \quad \text{and} \quad T_1 v_{s_2s_1t} = -q^{-1} v_{s_2s_1t}.$$

Then

$$M = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}_{q^2,1} \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{1,q^{-2}}$$

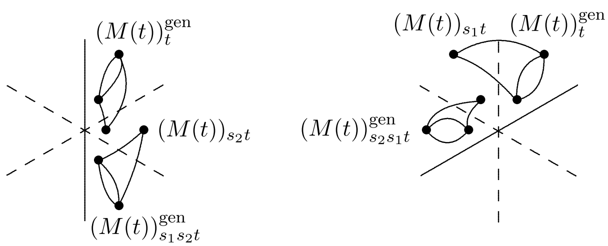
are 3-dimensional $\mathcal{H}$-modules with central character $t_{q^2,1}$.

Proposition 13. *Let $M = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}_{q^2,1}$ and $N = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{1,q^{-2}}$.*

(a) *If $q^4 \neq 1$ then M and N are irreducible.*
(b) *If $q^2 = -1$ then M_{s_1t} is an irreducible submodule of M and N_{s_1t} is an irreducible submodule of N. The quotients N/N_{s_1t} and M/M_{s_1t} are irreducible.*

Proof. (a) Assume $q^4 \neq 1$. If either M or N were reducible, it would have a 1-dimensional submodule or quotient, which cannot happen since the 1-dimensional modules have central character t_{q^2,q^2}. Thus both M and N are reducible.

(b) If $t|_Q = t_{q^2,1}$, then the action of τ_1 is non-zero on M_t^{gen} by Proposition 1, and M_t^{gen} is not a submodule of M. But M is not irreducible, and the only possible remaining submodule is M_{s_1t}. A similar argument shows the result for N as well. □



$t_{1,q^2}, q^4 \neq 1$ $\qquad$ $t_{q^2,1}, q^4 \neq 1$

The modules with central character t such that $t|_Q = t_{1,q^2}$ can be constructed in an entirely analogous fashion, for $q^2 \neq 1$. Finally, if $q^2 = 1$, then Theorem 9 suffices to classify the representations of $\mathcal{H}$ with central characters $t_{1,1}$ and $t_{1,z}$ for $z \neq q^{\pm 2}$.

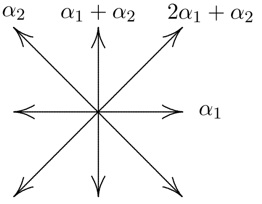
Summary. Table 2 summarizes the classification.

Table 2
Table of possible central characters in type A_2.

	Dims. of irreds. by weight						
	$t_{1,1}$	$t_{1,z}$	t_{1,q^2}	$t_{q^2,1}$	t_{q^2,q^2}	$t_{q^2,z}$	$t_{z,w}$, $z, w \neq \pm 1$ or $q^{\pm 2}$
$q^6 \neq 1$, $q^4 \neq 1$	6	6	3, 3	3, 3	1, 1, 2, 2	3, 3	6
$q^6 = 1$	6	6	3, 3	3, 3	1, 1, 1, 1, 1, 1	3, 3	6
$q^2 = -1$	6	6	1, 2, 2	N/A	N/A	3, 3	6
$q = -1$	1, 1, 2	3, 3	1, 2, 2	N/A	N/A	N/A	6

5. Type C_2

The type C_2 root system is $R = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2)\}$, where $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ and $\langle \alpha_2, \alpha_1^\vee \rangle = -2$. Then the Weyl group is $W_0 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 \rangle$, which is isomorphic to the dihedral group of order 8. The simple roots are α_1 and α_2, with additional positive roots $\alpha_1 + \alpha_2$ and $2\alpha_1 + \alpha_2$. Then the action of W_0 on R can be seen in the following picture, where s_i acts by reflection through H_{α_i}, the hyperplane perpendicular to α_i.



The type C_2 root system

The fundamental weights satisfy

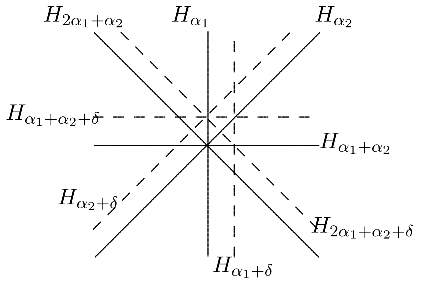
$$\omega_1 = \alpha_1 + \frac{1}{2}\alpha_2, \qquad \alpha_1 = 2\omega_1 - \omega_2,$$
$$\omega_2 = \alpha_1 + \alpha_2, \qquad \alpha_2 = 2\omega_2 - 2\omega_1.$$

Let

$$P = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\} \quad \text{and} \quad Q = \mathbb{Z}\text{-span}(R)$$

be the weight lattice and root lattice of R, respectively.

Then the affine Hecke algebra $\mathcal{H}$ is defined as in Section 2.



The torus T_Q

For all weights $t_{z,w} \in T_Q$, there are 2 elements $t \in T$ with $t|_Q = t_{z,w}$, determined by

$$t(X^{\omega_1})^2 = z^2 w \quad \text{and} \quad t(X^{\omega_2}) = zw.$$

We denote these two elements as $t_{z,w,1}$ and $t_{z,w,2}$. Which particular weight $t_{z,w,i}$ is which is unimportant since we will always be examining them together. And in fact, most of the time, we will only refer to the restricted weight $t_{z,w}$, since the dimension of the modules with central character t depends only on $t|_Q$. One important remark, though, is that if $t(X^{\alpha_1}) = -1$, then the two weights t with $t|_Q = t_{-1,w}$ are in the same W_0-orbit and represent the same central character.

We begin by examining the W_0-orbits in T_Q. The structure of the modules with weight t depends virtually exclusively on $P(t) = \{\alpha \in R^+ \mid t(X^\alpha) = q^{\pm 2}\}$ and $Z(t) = \{\alpha \in R^+ \mid t(X^\alpha) = 1\}$. For a generic weight t, $P(t)$ and $Z(t)$ are empty, so we examine only the non-generic orbits.

Proposition 14. *Let q be generic. If $t \in T_Q$, and $P(t) \cup Z(t) \neq \emptyset$, then t is in the W_0-orbit of one of the following weights*:

$$t_{1,1},\ t_{-1,1},\ t_{1,q^2},\ t_{q^2,1},\ t_{\pm q,1},\ t_{q^2,q^2},\ t_{-1,q^2},\quad \{t_{1,z} \mid z \neq 1,\ q^{\pm 2}\},\quad \{t_{z,1} \mid z \neq \pm 1,\ q^{\pm 2},\ \pm q^{\pm 1}\},$$
$$\{t_{q^2,z} \mid z \neq 1,\ q^{\pm 2},\ q^{-4},\ q^{-6}\},\quad \text{or} \quad \{t_{z,q^2} \mid z \neq \pm 1,\ q^{\pm 2},\ -q^{-2},\ q^{-4},\ \pm q^{-1}\}.$$

Proof. The proof consists of exhausting all possibilities for $Z(t)$ and $P(t)$, up to the action of W_0. In the following, we refer to α_1 and $\alpha_1 + \alpha_2$ as "short" roots, and α_2 and $2\alpha_1 + \alpha_2$ as "long" roots.

Case 1: $|Z(t)| \geqslant 2$.

If $Z(t)$ contains a short root and any other root, then $t = t_{1,1}$. If $Z(t)$ contains two long roots, then $t = t_{-1,1}$.

Case 2: $|Z(t)| = 1$.

If $Z(t)$ contains exactly one root, we may assume it is either α_1 or α_2. If $t(X^{\alpha_1}) = 1$, then $t(X^{\alpha_2}) = t(X^{\alpha_1+\alpha_2}) = t(X^{2\alpha_1+\alpha_2})$. Thus either $P(t) = \{\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$, or $P(t) = \emptyset$. That is, t is in the orbit of t_{1,q^2} or $t_{1,z}$ for some $z \neq q^{\pm 2}, 1$. If $t(X^{\alpha_2}) = 1$, then $t(X^{\alpha_1}) = t(X^{\alpha_1+\alpha_2})$. Then either $P(t) = \{\alpha_1, \alpha_1 + \alpha_2\}$, $P(t) = \{2\alpha_1 + \alpha_2\}$, or $P(t) = \emptyset$. These are the orbits of $t_{q^2,1}$, $t_{\pm q,1}$, and $t_{z,1}$, respectively, where $z \neq q^{\pm 2}, \pm q^{\pm 1}$, or 1.

Case 3: $Z(t) = \emptyset$.

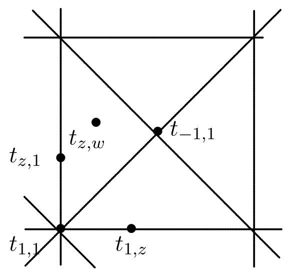
Now assume that $Z(t)$ is empty, and $P(t)$ is not empty. First assume that $P(t)$ contains at least one short root. We can apply an element of w to assume that $\alpha_1 \in P(t)$ and $t(X^{\alpha_1}) = q^2$. Then if $\alpha_2 \in P(t)$, we must have $t(X^{\alpha_2}) = q^2$ or else $Z(t)$ would be non-empty. Thus $t = t_{q^2,q^2}$. If $\alpha_1 + \alpha_2 \in P(t)$, then $t(X^{\alpha_1+\alpha_2}) = q^{\pm 2}$, so that either $t(X^{\alpha_2}) = 1$ or $t(X^{2\alpha_1+\alpha_2}) = 1$. If $2\alpha_1 + \alpha_2 \in P(t)$, then $t(X^{2\alpha_1+\alpha_2}) = q^{-2}$ or else $\alpha_1 + \alpha_2 \in Z(t)$. Hence $t(X^{\alpha_2}) = q^{-6}$. But then $s_2 s_1 s_2 t = t_{q^2,q^2}$. If $P(t) = \{\alpha_1\}$, then $t = t_{q^2,z}$ for some $z \neq 1, q^{\pm 2}, q^{-4}, q^{-6}$.

Now, assume that $P(t)$ contains a long root but no short roots. Then we may assume that $t(X^{\alpha_2}) = q^2$. If $t(X^{2\alpha_1+\alpha_2}) = q^2$ then $t(X^{\alpha_1}) = -1$. If $t(X^{2\alpha_1+\alpha_2}) = q^{-2}$ then $t(X^{\alpha_1}) = -q^{-2}$. However, $s_1 s_2 s_1 t_{-q^{-2},q^2} = t_{-1,q^2}$. Thus t is in the same orbit as t_{-1,q^2} or t_{z,q^2} for $z \neq \pm 1, q^{\pm 2}, q^{-4}, -q^{-2}, \pm q^{-1}$. $\square$

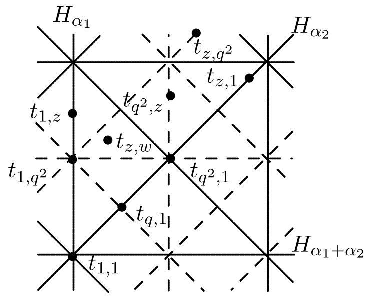
Remark. If q^2 is a root of unity of order less than or equal to 4, there is redundancy in the list of characters given above. Essentially, this is a result of the periodicity in T_Q when q^2 is an ℓth root of unity. If q^2 is a primitive fourth root of unity, then $t_{q^2,q^2} = s_1 s_2 s_1 t_{-1,q^2}$.

If q^2 is a primitive third root of unity, $t_{q^2,q^2} = s_2 s_1 s_2 t_{q^2,1}$. Also, one of $t_{q,1}$ and $t_{-q,1}$ is equal to $t_{q^{-2},1}$ and is in the same orbit as $t_{q^2,1}$. (Which one it is depends on whether $q^3 = 1$ or -1. In either case, $t_{q^2+1,1}$ is in a different orbit than $t_{q^2,1}$, so $t_{q^2+1,1}$ is our preferred notation for this character.)

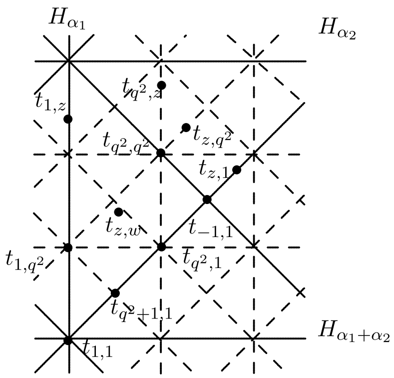
If $q^2 = -1$, then $t_{-1,1} = t_{q^2,1}$, and t_{1,q^2} is in the same orbit as $t_{q^2,q^2} = t_{-1,q^2}$. Also in this case, $t_{z,q^2} = t_{z,-1} = s_2 t_{-z,-1}$. Finally, if $q = -1$, we have $t_{1,1} = t_{q^2,1} = t_{1,q^2} = t_{q^2,q^2} = t_{-q,1}$. Also, $t_{-1,1} = t_{-1,q^2}$, while $t_{q^2,z} = t_{1,z}$ and $t_{z,q^2} = t_{z,1}$.



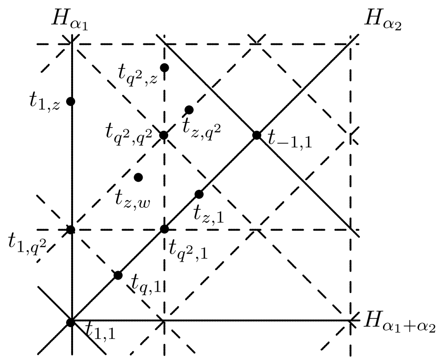
$q^2 = 1$



$q^2 = -1$



q^2 a third root of unity



q^2 a fourth root of unity

Analysis of the characters.

Proposition 15. *There are eight 1-dimensional representations of $\mathcal{H}$, one for each weight t with $t|_Q = t_{q^{\pm 2},q^{\pm 2}}$. In each of these representations, T_i acts with eigenvalue q or $-q^{-1}$ when $t(X^{\alpha_i}) = q^2$ or q^{-2}, respectively.*

Proof. As in Proposition 12. □

Remark. We will use the notation $L_{z,w,i}$ to denote the 1-dimensional representation with weight $t_{z^2,w^2,i}$, where each of z and w is either q or $-q^{-1}$. Note that if q is a primitive fourth root of unity, then $L_{q,q,i} \cong L_{q,-q^{-1},3-i} \cong L_{-q^{-1},q,i} \cong L_{-q^{-1},-q^{-1},3-i}$, for $i = 1$ or 2.

Principal series modules. A weight $t|_Q$ corresponds to a point in the root lattice Q as described above. The composition structure of the principal series module $M(t)$ is largely determined by the structure of the operators τ_i, which can be encoded in the following pictures of small neighborhoods of the various points t in Q. The solid lines are the hyperplanes H_{α_i}, while the dashed lines represent the hyperplanes $H_{\alpha_i \pm \delta}$. Thus the hyperplanes in the picture of the neighborhood of t show which τ operators are invertible and which are not (or are not well defined). In most cases, this is enough to determine the exact composition factors of $M(t)$.

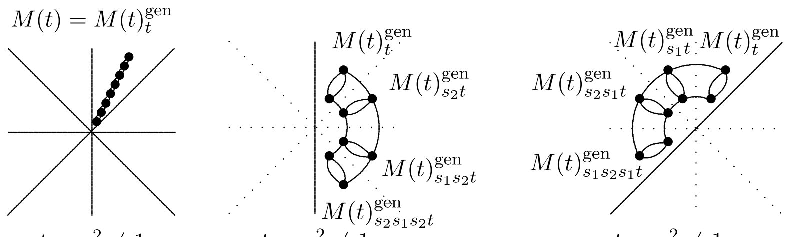
Case 1: $P(t) = \emptyset$.

If $P(t) = \emptyset$, then by Kato's criterion (Theorem 3(c)), $M(t)$ is irreducible and is the only irreducible module with central character t. The weights of $M(t)$ are in bijection with W_0/W_t, the cosets of the centralizer of t in W, and $\dim(M(t)_{wt}) = |W_t|$. If w and $s_i w$ are distinct weights in $M(t)$, then $\tau_i : M(t)_t \to M(t)_{s_i t}$ is a bijection.

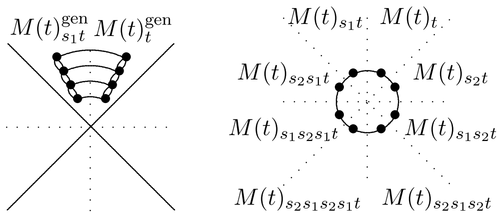
Case 2: $Z(t) = \emptyset$, but $P(t) \neq \emptyset$.

If $Z(t) = \emptyset$ then t is a regular central character. Then the irreducibles with central character t are in bijection with the connected components of the calibration graph for t, and can be constructed using Theorem 7.

Case 1:

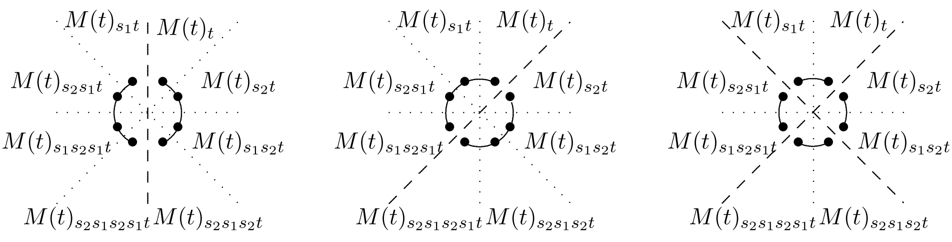


$t_{1,1}, q^2 \neq 1$ $t_{1,z}, q^2 \neq 1$ $t_{z,1}, q^2 \neq 1$

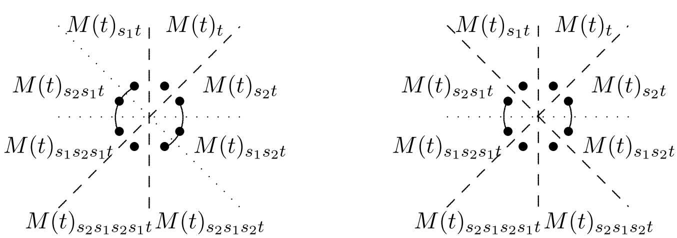


$t_{-1,1}, q \neq \pm i$ $t_{z,w}$

Case 2:



$t_{q^2,z}, q^2 \neq 1$ $t_{z,q^2}, q^2 \neq 1$ $t_{-1,q^2}, q^4 \neq 1, q^8 \neq 1$



t_{q^2,q^2}, q generic t_{q^2,q^2}, q a primitive eighth root of unity

Case 3: $Z(t) \neq \emptyset$, $P(t) \neq \emptyset$.

The only central characters not covered in Cases 1 and 2 are those in the orbits of t_{1,q^2}, $t_{q^2,1}$, and $t_{\pm q,1}$.

$t|_Q = t_{q^2,1}$.

If $q^2 = 1$, then Theorem 9 shows that $\mathcal{H}$ has five irreducible representations – four of them 1-dimensional, and one 2-dimensional.

Assume $q^2 \neq 1$ and let

$$w_1 = \begin{cases} s_1 & \text{if } q^2 = -1, \\ s_1 s_2 s_1 & \text{if } q^2 \neq -1. \end{cases}$$

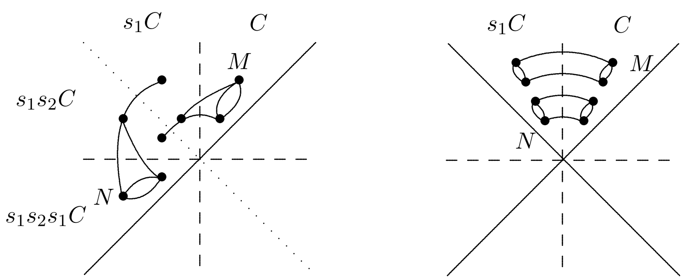
Then let $\mathbb{C}_{q^2,1}$ and $\mathbb{C}_{q^{-2},1}$ be the 1-dimensional $\mathcal{H}_{\{1\}}$-modules spanned by v_t and $v_{w_1 t}$, respectively, given by

$$X^\lambda v_t = t\big(X^\lambda\big) v_t \quad \text{and} \quad T_1 v_t = q v_t, \quad \text{and}$$
$$X^\lambda v_{w_1 t} = (w_1 t)\big(X^\lambda\big) v_{w_1 t} \quad \text{and} \quad T_1 v_{w_1 t} = -q^{-1} v_{w_1 t}.$$

Then

$$M = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}_{q^2,1} \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}_{q^{-2},1}$$

are 4-dimensional $\mathcal{H}$-modules.



$t_{q^2,1}$, $q^2 \neq \pm 1$ $\qquad$ $t_{q^2,1}$, q a primitive fourth root of unity

Proposition 16. *If* $q^2 = -1$ *and* $M = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}_{q^2,1}$ *and* $N = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}_{q^{-2},1}$ *then*

(a) *M is irreducible, and*

(b) *The map*

$$\phi : N \to M,$$
$$h v_{wt} \mapsto h v, \quad \textit{for } h \in \mathcal{H}$$

is an $\mathcal{H}$-module isomorphism, where $v = T_1 T_2 v_t - q T_2 v_t - v_t \in M$, *and*

(c) *Any irreducible $\mathcal{H}$-module L with central character t is isomorphic to M.*

Proof. (a) If q is a primitive fourth root of unity, then M has weight spaces M_t^{gen}, and $M_{s_1 t}^{\mathrm{gen}}$, each of which is 2-dimensional. By Theorem 5 and Proposition 4, M is irreducible.

(b) Let $v = T_1T_2v_t - qT_2v_t - v_t$. Then a straightforward computation using equation (4) shows that v spans a 1-dimensional $\mathcal{H}_{\{1\}}$-submodule of M, given by

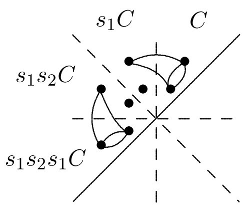
$$T_1v = qv, \quad \text{and} \quad X^\lambda v = s_1t\big(X^\lambda\big)v.$$

Then the $\mathcal{H}_{\{1\}}$-module map given by $v_{wt} \mapsto v$ corresponds to ϕ under the adjunction

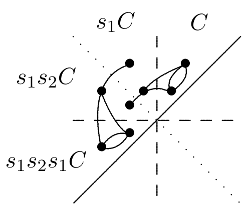
$$\mathrm{Hom}_{\mathcal{H}}(\mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}_{q^{-2},1}, M) = \mathrm{Hom}_{\mathcal{H}_{\{1\}}}(\mathbb{C}_{q^{-2},1}, M|_{\mathcal{H}_{\{1\}}}).$$

Thus ϕ is an $\mathcal{H}$-module map and since M is irreducible, the map is surjective. Since M and N have the same dimension, ϕ is an isomorphism.

(c) Let L be an irreducible $\mathcal{H}$-module with central character t, which must have weights t and s_1t. Then, viewing L as an $\mathcal{H}_{\{1\}}$-module, it must have all 1-dimensional composition factors, and it must have a 1-dimensional $\mathcal{H}_{\{1\}}$-submodule, with weight t or s_1t. Then the same argument as in part (b) gives an isomorphism from M to L or from N to L. □



$t_{q^2,1}$, q^2 a primitive third root of unity



$t_{q^2,1}$, q generic

Proposition 17.

(a) *If q^2 is a primitive third root of unity then $M_{s_2s_1t}$ is a submodule of M isomorphic to $L_{-q^{-1},q,\pm 1}$ and $M/M_{s_2s_1t}$ is irreducible. In addition, N_{s_1t} is a submodule of N isomorphic to $L_{q,-q^{-1}}$, and N/N_{s_1t} is irreducible.*

(b) *If q^2 is not ± 1 or a primitive third root of unity then M and N are irreducible and nonisomorphic.*

Proof. (a) Assume q^2 is a primitive third root of unity. Then Proposition 1 shows that $\tau_2 : M_{s_1t} \to M_{s_2s_1t}$ is non-zero. But $s_2s_1t(X^{\alpha_2}) = q^2$ so that $\tau_2 : M_{s_2s_1t} \to M_{s_1t}$ is the zero map by Theorem 2, and $M_{s_1s_2t}$ is a submodule of M. By Theorem 5, $M/M_{s_1s_2t}$ is irreducible. A parallel argument shows that N_{s_1t} is a submodule of N, with N/N_{s_1t} irreducible.

(b) If $q^4 \neq 1$ and $q^6 \neq 1$, then $P(t) = \{\alpha_1, \alpha_1 + \alpha_2\}$. Then Theorem 5 shows that the composition factor M' of M with $(M')_t \neq 0$ has $\dim(M')_t^{\mathrm{gen}} \geqslant 2$ and $(M')_{s_1t} \neq 0$. Then by Theorem 3(b), $(M')_{s_2s_1t} \neq 0$, so that $M' = M$. Similarly, Theorem 5 and Theorem 3(b) show that N is irreducible. Since they have different weight spaces, they are not isomorphic. □

$\underline{t|_Q = t_{1,q^2}}$.

Note that if $q^2 = 1$, then $t_{1,q^2} = t_{q^2,1} = t_{1,1}$, so this case has already been addressed.

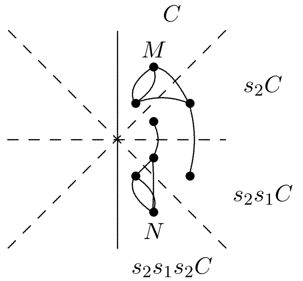
Let $\mathbb{C}_{1,q^2}$ and $\mathbb{C}_{1,q^{-2}}$ be the 1-dimensional $\mathcal{H}_{\{2\}}$-modules spanned by v_t and v_{w_0t}, respectively, and given by

$$T_2v_t = qv_t \quad \text{and} \quad X^\lambda v_t = t\big(X^\lambda\big)v_t, \quad \text{and}$$
$$T_2v_{w_0t} = -q^{-1}v_{w_0t} \quad \text{and} \quad X^\lambda v_{w_0t} = w_0t\big(X^\lambda\big)v_{w_0t}.$$

Then

$$M = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{1,q^2} \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{1,q^{-2}}$$

are 4-dimensional $\mathcal{H}$-modules.



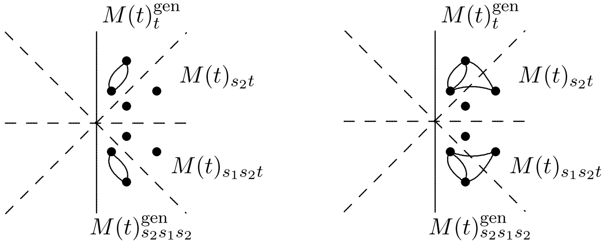
t_{1,q^2}, q generic

Proposition 18. *Assume $q^2 = -1$ and let $M = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{1,q^2}$ and $N = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{1,q^{-2}}$. Then*

(a) *$M_{s_1s_2t}$ is a submodule of M, and the image of M_{s_2t} is a submodule of $M/M_{s_1s_2t}$. The resulting 2-dimensional quotient of M is irreducible. Also, N_{s_2t} is a submodule of N and the image of $N_{s_1s_2t}$ in N/N_{s_2t} is a submodule of N/N_{s_2t}. The resulting 2-dimensional quotient of N is irreducible, and*
(b) *Any composition factor of $M(t)$ is a composition factor of either M or N.*

Proof. (a) If $q^2 = -1$, then M has weight spaces M_t^{gen}, which is two-dimensional, and M_{s_2t} and $M_{s_1s_2t}$, both of which are 1-dimensional. Proposition 1 and Theorem 2 show that τ_1 is non-zero on M_{s_2t}, but zero on $M_{s_1s_2t}$, so that $M_{s_1s_2t}$ is a submodule of M. The resulting quotient must be reducible, but since v_t generates all of M, the generalized t weight space cannot be a submodule. Thus the s_2t weight space is the submodule, and its quotient must be the 2-dimensional module constructed in Proposition 4, since it accounts for the entire t weight space of $M(t)$. A similar argument shows the result for N.

(b) By counting dimensions of weight spaces, the remaining composition factor(s) of $M(t)$ must have weights s_2t and s_1s_2t. If there were only one composition factor L left, it would contain both weight spaces which would each have dimension 1, which is impossible by Proposition 4. Thus the remaining composition factors are more copies of the 1-dimensional modules. □



t_{1,q^2}, q a primitive fourth root of unity $\qquad$ t_{1,q^2}, q generic

Proposition 19. *Assume $q^2 \neq \pm 1$. Then $M_{s_1s_2t}$ is a submodule of M and $M/M_{s_1s_2t}$ is irreducible. Similarly, N_{s_2t} is a submodule of N and N/N_{s_2t} is irreducible.*

Proof. If $q^4 \neq 1$, then by the same reasoning as in Proposition 18, $M_{s_1 s_2 t}$ must be a submodule of M. Similarly, $N_{s_2 t}$ is a submodule of N. Then Theorem 5 shows that the resulting 3-dimensional quotients of M and N are irreducible. □

If $q^2 \neq 1$, the composition factors of M and N account for all 8 dimensions of $M(t)$.

$\underline{t|_Q = t_{\pm q,1}}$.

Let $\mathbb{C}_{\pm q,1}$ and $\mathbb{C}_{\pm q^{-1},1}$ be the 1-dimensional $\mathcal{H}_{\{2\}}$-modules spanned by $v_{s_1 t}$ and $v_{s_2 s_1 t}$, respectively, and given by

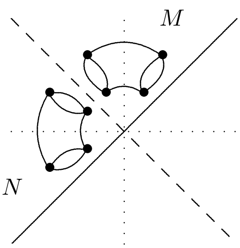
$$T_2 v_{s_1 t} = q v_{s_1 t} \quad \text{and} \quad X^\lambda v_{s_1 t} = s_1 t\big(X^\lambda\big) v_{s_1 t}, \quad \text{and}$$
$$T_2 v_{s_2 s_1 t} = -q^{-1} v_{s_2 s_1 t} \quad \text{and} \quad X^\lambda v_{s_2 s_1 t} = s_2 s_1 t\big(X^\lambda\big) v_{s_2 s_1 t}.$$

Then

$$M = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{\pm q,1} \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{\pm q^{-1},1}$$

are 4-dimensional $\mathcal{H}$-modules.

If $t|_Q = t_{-q,1}$ and q is a primitive sixth root of unity or if $t|_Q = t_{q,1}$ and q is a primitive third root of unity, then $t|_Q = t_{q^{-2},1}$, which is in the same orbit as $t_{q^2,1}$, and the irreducibles with central character t have already been analyzed.



$t_{\pm q,1}, q^2 \neq 1$
(excluding $t_{-q,1}$ when q is a primitive sixth root of unity, and
$t_{q,1}$ when q is a primitive third root of unity)

Proposition 20. *Let $M = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{\pm q,1}$ and $N = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}_{\pm q^{-1},1}$. Unless $t|_Q = t_{-q,1}$ and q is a primitive sixth root of unity or $t|_Q = t_{q,1}$ and q is a primitive third root of unity, M and N are irreducible.*

Proof. By assumption, $P(t) = \{2\alpha_1 + \alpha_2\}$. Then the claim follows from Theorem 5. □

Since they have different weight spaces and are thus not isomorphic, M and N are the only two irreducibles with central character t.

Summary. Table 3 summarizes the classification. Note that in this table, the entries for $t_{\pm q,1}$ assume that $\pm q \neq q^{-2}$, as described above.

6. Type G_2

The type G_2 root system is

$$R = \big\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1 + \alpha_2), \pm(2\alpha_1 + \alpha_2), \pm(3\alpha_1 + \alpha_2), \pm(3\alpha_1 + 2\alpha_2)\big\},$$

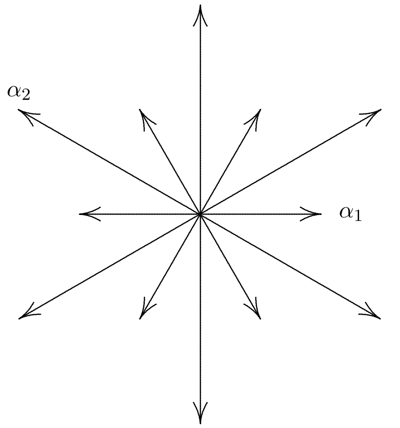
Table 3
Table of possible central characters in type C_2, with varying values of q.

t	Dims. of irreds.				
	$q^8 \neq 1$, $q^6 \neq 1$	$q^8 = 1$, $q^4 \neq 1$	$q^6 = 1$	$q^2 = -1$	$q = -1$
$t_{1,1}$	8	8	8	8	1, 1, 1, 1, 2
$t_{-1,1}$	8	8	8	4	2, 2, 2, 2
$t_{1,z}$	8	8	8	8	4, 4
t_{1,q^2}	1, 1, 3, 3	1, 1, 3, 3	1, 1, 3, 3	1, 1, 2, 2	N/A
$t_{q^2,1}$	4, 4	4, 4	1, 1, 3, 3	N/A	N/A
$t_{q,1}$	4, 4	4, 4	4, 4	4, 4	N/A
$t_{-q,1}$	4, 4	4, 4	4, 4	N/A	N/A
$t_{z,1}$	8	8	8	8	4, 4
t_{q^2,q^2}	1, 1, 3, 3	1, 1, 1, 1, 2, 2	N/A	N/A	N/A
$t_{q^2,z}$	4, 4	4, 4	4, 4	4, 4	N/A
t_{-1,q^2}	2, 2, 2, 2	N/A	2, 2, 2, 2	N/A	N/A
t_{z,q^2}	4, 4	4, 4	4, 4	4, 4	N/A
$t_{z,w}$	8	8	8	8	8

with $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ and $\langle \alpha_2, \alpha_1^\vee \rangle = -3$. Then the Weyl group is

$$W_0 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1 \rangle,$$

isomorphic to the dihedral group of order 12. The simple roots are α_1 and α_2, and $\alpha_1, \alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$ will be referred to as *short* roots, while $\alpha_2, 2\alpha_1 + \alpha_2$, and $3\alpha_1 + 2\alpha_2$ will be referred to as *long* roots.



The type G_2 root system

The fundamental weights satisfy

$$\omega_1 = 2\alpha_1 + \alpha_2, \qquad \alpha_1 = 2\omega_1 - \omega_2,$$
$$\omega_2 = 3\alpha_1 + 2\alpha_2, \qquad \alpha_2 = 2\omega_2 - 3\omega_1.$$

Let

$$P = \mathbb{Z}\text{-span}\{\omega_1, \omega_2\}.$$

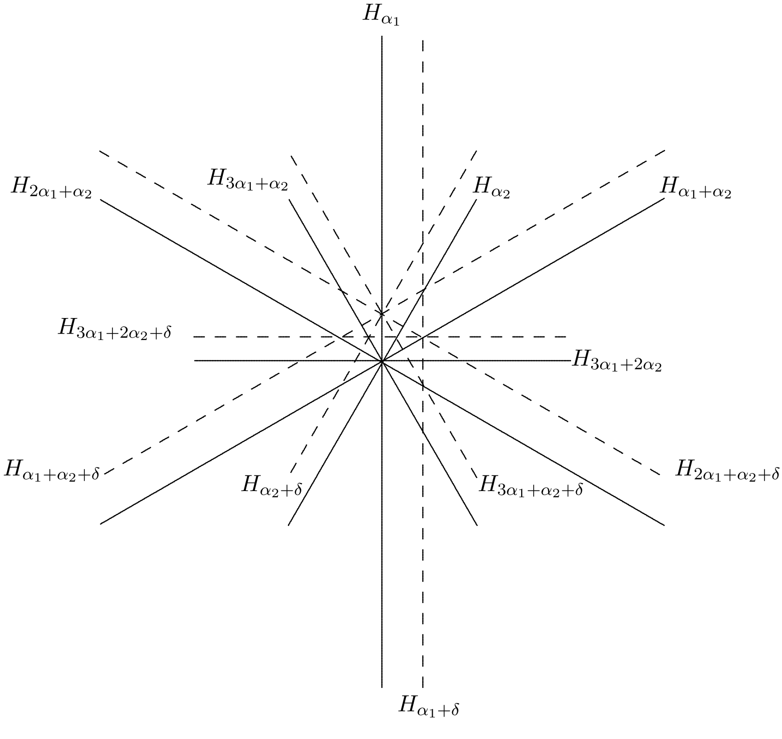
This is the same lattice spanned by α_1 and α_2. Then W_0 acts on X by

$$s_1 \cdot X^{\omega_1} = X^{\omega_2-\omega_1},$$
$$s_1 \cdot X^{\omega_2} = X^{\omega_2},$$
$$s_2 \cdot X^{\omega_1} = X^{\omega_1}, \quad \text{and}$$
$$s_2 \cdot X^{\omega_2} = X^{3\omega_1-\omega_2}.$$

The affine Hecke algebra $\mathcal{H}$ of type G_2 is defined as in Section 2.

Let $T = \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[X], \mathbb{C})$ and define

$$t_{z,w} : T \to \mathbb{C} \quad \text{by } t_{z,w}\big(X^{\alpha_1}\big) = z \quad \text{and} \quad t_{z,w}\big(X^{\alpha_2}\big) = w.$$



The structure of the modules with weight t depends virtually exclusively on $P(t)$ and $Z(t)$. For a generic weight t, $P(t)$ and $Z(t)$ are empty, so we examine only the non-generic orbits.

Theorem 21. *If q^2 is not a primitive ℓth root of unity for $\ell \leqslant 6$ and $Z(t) \cup P(t) \neq \emptyset$, then t is in the same W_0-orbits as one of the following weights.*

$$t_{1,1}, t_{1,-1}, t_{1^{1/3},1}, t_{1,q^2}, t_{1,\pm q}, t_{q^2,1}, t_{\pm q,1}, t_{q^{2/3},1}, t_{q^2,-q^{-2}}, t_{1^{1/3},q^2}, t_{q^2,q^2},$$
$$\big\{t_{1,z} \,\big|\, z \in \mathbb{C}^\times,\ z \neq \pm 1,\ q^{\pm 2},\ \pm q^{\pm 1}\big\}, \quad \big\{t_{z,1} \,\big|\, z \in \mathbb{C}^\times,\ z \neq \pm 1,\ 1^{1/3},\ q^{\pm 2},\ \pm q^{\pm 1},\ q^{\pm 2/3}\big\},$$

$$\{t_{q^2,z} \mid z \in \mathbb{C}^\times,\ \{1, q^2, q^{-2}\} \cap \{z, q^2z, q^4z, q^6z, q^6z^2\} = \emptyset\}, \quad \textit{or}$$

$$\{t_{z,q^2} \mid z \in \mathbb{C}^\times,\ \{1, q^2, q^{-2}\} \cap \{z, q^2z, q^2z^2, q^2z^3, q^4z^3\} = \emptyset\}.$$

Proof. In general, the third roots of unity in this theorem are assumed to be primitive, so that there are two different weights that we call $t_{1^{1/3},1}$ and $t_{1^{1/3},q^2}$. Similarly, $t_{q^{2/3},1}$ typically refers to one of three different characters, corresponding to the three third roots of q^2. We refer to α_1, $\alpha_1+\alpha_2$, and $2\alpha_1+\alpha_2$ as "short" roots. The other roots are referred to as "long" roots.

Case 1: $|Z(t)| \geqslant 2$.

If $Z(t)$ contains at least two roots, and one of them is short, then $Z(wt)$ contains α_1 for some $w \in W_0$. If $Z(wt)$ also contains any of $\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2$, or $3\alpha_1+\alpha_2$, then it contains both simple roots and thus $wt = t_{1,1}$. It is also possible that $Z(wt) = \{\alpha_1, 3\alpha_1+2\alpha_2\}$, in which case $wt(X^{\alpha_1+\alpha_2}) = -1$, and $wt(X^{\alpha_2}) = -1$, so that $wt = t_{1,-1}$.

If $Z(t)$ contains no short roots, it contains two of α_2, $3\alpha_1+\alpha_2$, and $3\alpha_1+2\alpha_2$. But then it must also contain the third, and $Z(t) = \{\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}$. In this case, $wt(X^{\alpha_2}) = 1$, but $wt(X^{\alpha_1})$ is a third root of unity, so that $wt = t_{1^{1/3},1}$.

Case 2: $|Z(t)| = 1$.

If $Z(t)$ has exactly one root, then there is some $w \in W_0$ with $Z(wt) = \{\alpha_1\}$ or $Z(wt) = \{\alpha_2\}$.

If $Z(wt) = \{\alpha_1\}$, then $P(t)$ either contains all of $\alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2$, and $3\alpha_1+\alpha_2$, or it contains none of them. If it contains all of them, $wt(X^{\alpha_2}) = q^{\pm 2}$, and t is in the same W_0-orbit as t_{1,q^2}. If $3\alpha_1+2\alpha_2 \in P(wt)$, then $wt(X^{\alpha_2}) = \pm q^{\pm 1}$, and t is in the same orbit as $t_{1,\pm q}$. Otherwise, $wt = t_{1,z}$ for some z besides $\pm q^{\pm 1}$ and $q^{\pm 2}$. Also then, $z \neq \pm 1$ by assumption on $Z(t)$.

If $Z(wt) = \{\alpha_2\}$, then any two roots that differ by a multiple of α_2 are either both in $P(wt)$ or both not in $P(wt)$. By applying w_0 if necessary, we can assume that $wt(X^{\alpha}) = q^2$ for the α that are in $P(wt)$. If $\alpha_1 \in P(wt)$, then $wt(X^{\alpha_1}) = q^2$, and $wt = t_{q^2,1}$. If $2\alpha_1+\alpha_2 \in P(wt)$, then $wt = t_{\pm q,1}$. If $3\alpha_1+\alpha_2 \in P(wt)$, then $wt(X^{\alpha_1})$ is a third root of q^2 and $wt = t_{1,q^{2/3}}$. Otherwise, $wt = t_{z,1}$ for some z so that none of z, z^2, z^3 is equal to $q^{\pm 2}$ or 1. That is, $z \neq \pm 1, 1^{1/3}, q^{\pm 2}, \pm q^{\pm 1}, q^{\pm 2/3}$.

Case 3: $|Z(t)| = \emptyset$.

If $Z(t)$ is empty but $P(t)$ contains a short root, then $\alpha_1 \in P(wt)$ for some $w \in W_0$. If $P(wt)$ contains another short root, then we can apply s_1 if necessary so that $P(t)$ contains α_1 and $\alpha_1+\alpha_2$. Then either $wt(X^{\alpha_1}) = wt(X^{\alpha_1+\alpha_2})$ so that $wt(X^{\alpha_2}) = 1$, or $wt(X^{\alpha_1})$ and $wt(X^{\alpha_1+\alpha_2})$ are q^2 and q^{-2} in some order, so that $wt(X^{2\alpha_1+\alpha_2}) = 1$. Thus $P(wt)$ contains at most one short root. If $P(wt)$ also contains a long root, then applying s_1 if necessary, we can assume $P(wt)$ contains either α_2 or $3\alpha_1+2\alpha_2$. If $P(wt)$ contains α_1 and α_2, then we can apply w_0 to assume that $wt(X\alpha_1) = q^2$. If $wt(X^{\alpha_2}) = q^{-2}$, then $\alpha_1+\alpha_2 \in Z(wt)$. Then $wt(X^{\alpha_2}) = q^2$ and $wt = t_{q^2,q^2}$. If $P(wt)$ contains α_1 and $3\alpha_1+2\alpha_2$, then since α_1 is perpendicular to $3\alpha_1+2\alpha_2$, we can apply s_1 and/or $s_{3\alpha_1+2\alpha_2}$ to assume $wt(X^{\alpha_1}) = q^2 = wt(X^{3\alpha_1+2\alpha_2})$. Hence $wt(X^{2\alpha_2}) = q^{-4}$ and by assumption, $wt = t_{q^2,-q^{-2}}$. If $P(wt) = \{\alpha_1\}$, then $wt = t_{q^2,z}$ does not take the value 1 or $q^{\pm 2}$ on any other positive root. Then $\{1, q^2, q^{-2}\} \cap \{z, q^2z, q^4z, q^6z, q^6z^2\} = \emptyset$.

If $P(t)$ contains no short roots, but at least two long roots, then $wt(X^{\alpha_2}) = q^2 = wt(X^{3\alpha_1+\alpha_2})$ for some $w \in W_0$. (If $wt(X^{\alpha_2}) = q^{-2}$, then $wt(X^{3\alpha_1+2\alpha_2}) = 1$, a contradiction.) Hence $wt(X^{\alpha_1})$ is a primitive third root of unity and $wt = t_{1^{1/3},q^2}$. If $P(t)$ contains exactly one long root, then $wt = t_{z,q^2}$ for some $z \in \mathbb{C}^\times$ so that wt does not take the value 1 or $q^{\pm 2}$ on any other positive root. Thus $\{1, q^2, q^{-2}\} \cap \{z, q^2z, q^2z^2, q^2z^3, q^4z^3\} = \emptyset$. □

Remark. There are some redundancies in this list for specific values of q. If q^2 is a primitive fifth root of unity, then q and $-q$ are equal to q^{-4} and $-q^{-4}$ in some order depending on whether $q^5 = 1$ or -1. Then t_{q^2,q^2} is in the same orbit as $t_{q^{-4},1}$, which is equal to either $t_{q,1}$ or $t_{-q,1}$.

If q^2 is a primitive fourth root of unity, then one note is necessary on the weight $t_{q^{2/3},1}$. Since q^{-2} is a third root of q^2, we take $q^{2/3}$ to mean a different third root of q^2 so that $t_{q^{2/3},1}$ and $t_{q^2,1}$ are in different orbits. In addition, $t_{q^2,-q^{-2}} = t_{q^2,q^2}$, which is in the same orbit as $t_{q^2,1}$.

If q^2 is a primitive third root of unity, then $1^{1/3} = q^2, q^{-2}$, or 1. Then $t_{1^{1/3},1}$ is in the same orbit as $t_{q^2,1}$ or $t_{1,1}$. Also, $t_{1^{1/3},q^2}$ is in the same orbit as t_{q^2,q^2}, which is in turn in the same orbit as t_{1,q^2}. In addition q and $-q$ are equal to q^{-2} and $-q^{-2}$ in some order depending on whether q^3 is 1 or -1. Then t_{1,q^2} is in the same orbit as either $t_{1,q}$ or $t_{1,-q}$, and $t_{q^2,1}$ is in the same orbit as either $t_{q,1}$ or $t_{-q,1}$.

If $q^2 = -1$, then $t_{q^2,-q^{-2}} = t_{q^2,1} = t_{-1,1}$, which is in the same orbit as t_{q^2,q^2}, while $t_{1,q^2} = t_{1,-1}$. In fact, $t_{-1,1} = s_1 s_2 t_{1,-1}$. Also, since $q = -q^{-1}$, the weights $t_{1,\pm q}$ are in the same orbit as each other, as are the weights $t_{\pm q,1}$. Finally, $t_{1^{1/3},q^2}$ is in the same orbit as $t_{q^{2/3},1}$.

Finally, if $q = -1$, then $t_{1,1} = t_{q^2,1} = t_{1,q^2} = t_{q^2,q^2} = t_{1,-q} = t_{-q,1}$. Also, $t_{q,1} = t_{-1,1}$, which is in the same orbit as $t_{1,-1} = t_{1,q} = t_{q^2,-q^{-2}}$. Finally, $t_{1^{1/3},1} = t_{q^{2/3},1} = t_{1^{1/3},q^2}$, while $t_{1,z} = t_{q^2,z}$ and $t_{z,1} = t_{z,q^2}$.

Analysis of the characters.

Proposition 22. *There are four 1-dimensional representations of $\mathcal{H}$, one for each weight $t_{q^{\pm 2},q^{\pm 2}}$. In these modules, T_i acts with eigenvalue q or $-q^{-1}$ when $t(X^{\alpha_i}) = q^2$ or q^{-2}, respectively.*

Proof. As in Proposition 12. □

Remark. We will use the notation $L_{z,w}$ to denote the 1-dimensional representation with weight t_{z^2,w^2}, where each of z and w is either q or $-q^{-1}$. Note that if q is a primitive fourth root of unity, then all four 1-dimensional representations are isomorphic.

Principal series modules. We now examine the principal series modules $M(t)$ for all the possible central characters above.

Case 1: $P(t) = \emptyset$.

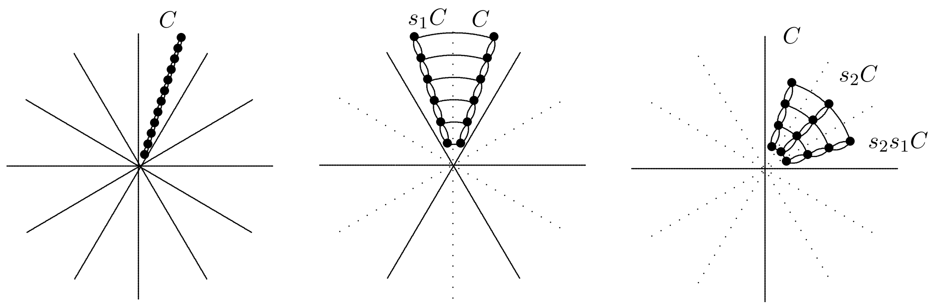
If $P(t) = \emptyset$ then by Kato's criterion, Theorem 3(c), $M(t)$ is irreducible and thus is the only irreducible module with central character t.

Case 2: $Z(t) = \emptyset$.

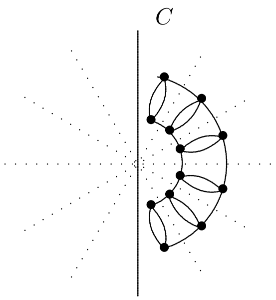
If $Z(t) = \emptyset$ then t is a regular central character. Then the irreducibles with central character t are in bijection with the connected components of the calibration graph for t, and can be constructed using Theorem 7.

The following graphs show the pictures of the central characters found in Cases 1 and 2 of Theorem 21, for the particular values of q for which either $Z(t)$ or $P(t)$ is empty. The remark after Theorem 21 details these values of q.

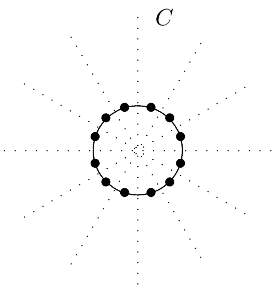
Case 1:



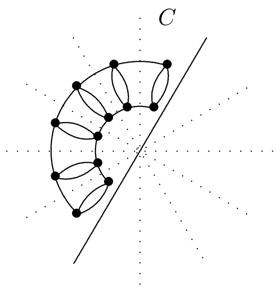
$t_{1,1},\ q^2 \neq 1$ $\quad$ $t_{\sqrt[3]{1},1}, q^2 \neq 1,\ q^6 \neq 1$ $\quad$ $t_{1,-1},\ q^2 \neq \pm 1$



$t_{1,z},\ q^2 \neq 1,\ z$ generic

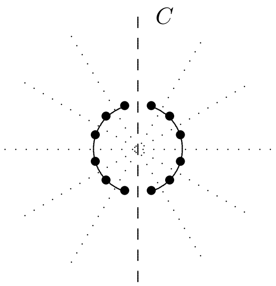


$t_{z,w},\ z, w$ generic

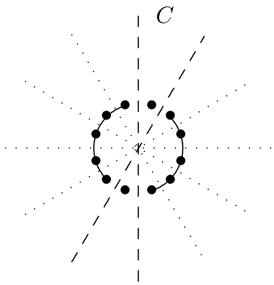


$t_{z,1},\ q^2 \neq 1,\ z$ generic

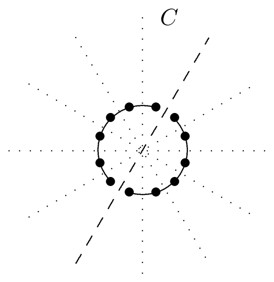
Case 2:



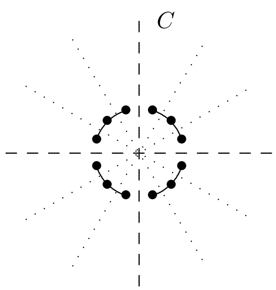
$t_{q^2,z},\ q^2 \neq 1,\ z$ generic



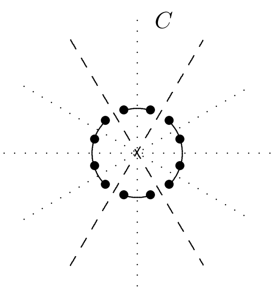
$t_{q^2,q^2},\ q$ generic



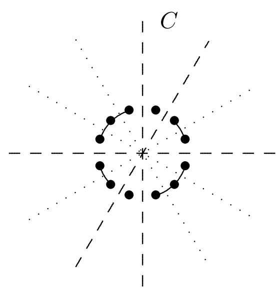
$t_{z,q^2},\ q^2 \neq 1,\ z$ generic



$t_{q^2,-q^{-2}},\ q$ generic or q^2 a primitive third or fifth root of unity



$t_{\sqrt[3]{1},q^2},\ q^2 \neq 1,\ q$ generic or q^2 a primitive fourth or fifth root of unity



$t_{q^2,q^2},\ q$ a primitive twelfth root of unity

Case 3: $Z(t), P(t) \neq \emptyset$.

For these central characters, rather than analyzing $M(t)$ directly, it is easier to construct several irreducible $\mathcal{H}$-modules and show that they include all the composition factors of $M(t)$.

Case 3a: t_{1,q^2}.

Assume $\alpha_1 \in Z(t)$ and $\alpha_2 \in P(t)$. Then $t = t_{1,q^{\pm 2}}$, but $s_2 s_1 s_2 s_1 s_2 t_{1,q^{-2}} = t_{1,q^2}$, so that analyzing $M(t_{1,q^2})$ is sufficient. Then let $t = t_{1,q^2}$. We have $Z(t) = \{\alpha_1\}$ and $P(t) = \{\alpha_2\}$ unless $q^2 = \pm 1$. Hence the cases $q^2 = 1$ and $q^2 = -1$ will be treated separately.

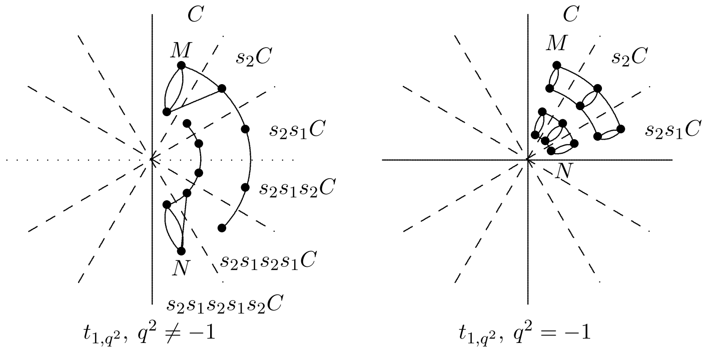
If $q^2 = 1$, then $Z(t) = P(t) = R^+$, and the irreducibles with this central character can be constructed using Theorem 9.

If $q^2 \neq 1$, let $\mathbb{C}v_t$ and $\mathbb{C}v_{w_0 t}$ be the 1-dimensional $\mathcal{H}_{\{2\}}$-modules spanned by v_t and $v_{w_0 t}$, respectively, and given by

$$T_2 v_t = q v_t, \qquad X^\lambda v_t = t(X^\lambda) v_t,$$
$$T_2 v_{w_0 t} = -q^{-1} v_{w_0 t}, \quad \text{and} \quad X^\lambda v_{w_0 t} = w_0 t(X^\lambda) v_{w_0 t}.$$

Then define

$$M = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C} v_t \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C} v_{w_0 t}.$$



Proposition 23. *Assume $q^2 \neq \pm 1$. Let $M = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C} v_t$ and $N = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C} v_{w_0 t}$, where $t = t_{1,q^2}$.*

(a) *$M_{s_1 s_2 s_1 s_2 t}$ is a 1-dimensional submodule of M. M', the image of the weight spaces $M_{s_2 s_1 s_2 t}$ and $M_{s_1 s_2 t}$ in $M/M_{s_1 s_2 s_1 s_2 t}$, is a submodule of $M/M_{s_1 s_2 s_1 s_2 t}$. The resulting quotient of M is irreducible.*
(b) *If q^2 is not a primitive third root of unity, then M' is irreducible.*
(c) *If q^2 is a primitive third root of unity, then $(M')_{s_2 s_1 s_2 t}$ is a submodule of M'.*
(d) *$N_{s_2 t}$ is a 1-dimensional submodule of N. N', the image of the weight spaces $N_{s_1 s_2 t}$ and $N_{s_2 s_1 s_2 t}$ in $N/N_{s_2 t}$, is a submodule of $N/N_{s_2 t}$. The resulting quotient of N is irreducible.*
(e) *If q^2 is not a primitive third root of unity, then N' is irreducible.*
(f) *If q^2 is a primitive third root of unity, then $(N')_{s_1 s_2 t}$ is a submodule of N'.*

Proof. Assume $q^2 \neq -1$. Then $Z(t) = \{\alpha_1\}$ and $P(t)$ contains $\alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2$, and $3\alpha_1 + \alpha_2$. If q^2 is a primitive third root of unity, then $P(t)$ also contains $3\alpha_1 + 2\alpha_2$. Then M has one 2-dimensional weight space M_t^{gen} and four 1-dimensional weight spaces $M_{s_2 t}$, $M_{s_1 s_2 t}$, $M_{s_2 s_1 s_2 t}$, and $M_{s_1 s_2 s_1 s_2 t}$. For $w \in \{s_2, s_1 s_2, s_2 s_1 s_2, s_1 s_2 s_1 s_2\}$, let m_{wt} be a non-zero vector in M_{wt}. By a calculation as in Proposition 1(b),

$$m_{wt} = T_w T_1 v_t + \sum_{w' < w} a_{w,w'} T_{w'} T_1 v_t,$$

for $w \in \{s_2, s_1 s_2, s_2 s_1 s_2, s_1 s_2 s_1 s_2\}$, where $a_{w,w'} \in \mathbb{C}$. Then if $s_i w > w$, $\tau_i m_{wt} \neq 0$ for $w \in \{s_2, s_1 s_2, s_2 s_1 s_2\}$, since the term $T_i T_w T_1$ cannot be canceled by any other term in $\tau_i m_{wt}$.

Thus $\tau_1 : M_{s_1 s_2 s_1 s_2 t} \to M_{s_2 s_1 s_2 t}$ is the zero map since, by Theorem 2, $\tau_1^2 : M_{s_2 s_1 s_2} t \to M_{s_2 s_1 s_2 t}$ is the zero map. Hence $M_{s_1 s_2 s_1 s_2 t}$ is a submodule of M. Similarly, $\tau_1 : M_{s_1 s_2 t} \to M_{s_2 t}$ must be the zero map since, by Theorem 2, $\tau_1^2 : M_{s_2 t} \to M_{s_2 t}$ is the zero map. Let $M_1 = M/M_{s_1 s_2 s_1 s_2 t}$. Then M', the subspace spanned by $\overline{m_{s_1 s_2 t}}$ and $\overline{m_{s_2 s_1 s_2 t}}$ in M_1, is a submodule of M_1. Theorem 5 shows that $M_2 = M_1/M'$ is irreducible.

(b) If q^2 is not a primitive third root of unity, $\tau_2^2 : (M')_{s_1 s_2 t} \to (M')_{s_1 s_2 t}$ is invertible, so that M' is irreducible.

(c) If q^2 is a primitive third root of unity, then $\tau_2 : (M'_1)_{s_2 s_1 s_2 t} \to (M'_1)_{s_1 s_2 t}$ is the zero map and $(M'_1)_{s_2 s_1 s_2 t}$ is a 1-dimensional submodule of M'_1, and $M'_1/(M'_1)_{s_2 s_1 s_2 t}$ is 1-dimensional as well.

(d)–(f) The same argument used in (a)–(c) applies, with each weight space M_{wt} replaced by $N_{w w_0 t}$. $\square$

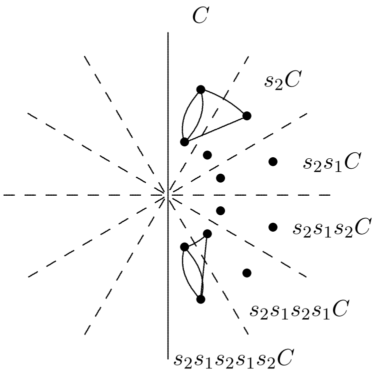
However, the composition factors of M and N are not distinct. If q^2 is not a primitive third root of unity, then M' and N' are irreducible 2-dimensional modules with the same weight space structure. Then Proposition 4 shows that $M' \cong N'$. If q^2 is a primitive third root of unity, then note that two 1-dimensional modules $\mathbb{C}v_t$ and $\mathbb{C}v_{t'}$ are isomorphic if and only if they have the same weight. Then M'_1 and N'_1 have the same composition factors. In any case, the 3-dimensional modules are different since their weight space structures are different.

Proposition 24. *If $q^2 \neq \pm 1$ then the composition factors of M and N are the only irreducibles with central character t_{1,q^2}.*

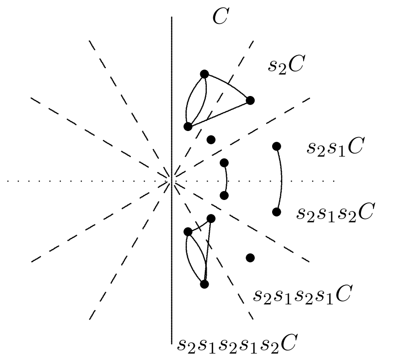
Proof. Counting multiplicities of weight spaces in $M(t)$ and the distinct composition factors of M and N shows that the remaining composition factor(s) of $M(t)$ must contain an s_1s_2t weight space and an $s_2s_1s_2t$ weight space, each of dimension 1.

If q^2 is not a primitive third root of unity then Theorem 3(b) shows that there must be one remaining composition factor with an s_1s_2t weight space and an $s_2s_1s_2t$ weight space, and Proposition 4 shows that it is isomorphic to M_1.

If q^2 is a primitive third root of unity then $\tau_2^2 M_{s_1s_2t} \to M_{s_1s_2t}$ is not invertible. Hence there cannot be an irreducible module consisting of an s_1s_2t weight space and an $s_2s_1s_2$ weight space, and the remaining composition factors of $M(t)$ are 1-dimensional. □

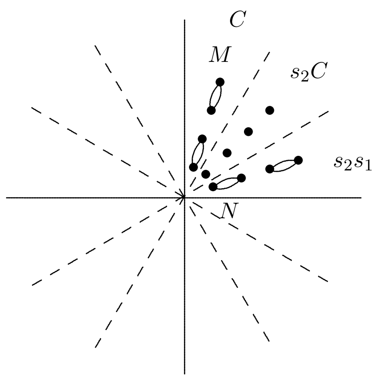


t_{1,q^2}, q^2 a primitive third root of unity



t_{1,q^2}, q^2 not a primitive third root of unity, $q^2 \neq -1$

If $q^2 = -1$, then $\dim M_t^{\text{gen}} = \dim M_{s_2t}^{\text{gen}} = \dim M_{s_1s_2t}^{\text{gen}} = 2$ and $\dim N_t^{\text{gen}} = \dim N_{s_2t}^{\text{gen}} = \dim N_{s_1s_2t}^{\text{gen}} = 2$.



$t_{1,q^2}, q^2 = -1$

Proposition 25. *Assume $q^2 = -1$ and $t = t_{1,q^2}$. Let $M = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}v_t$ and $N = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}v_{s_1 s_2 t}$.*

(a) *M and N each have two 1-dimensional modules and two 2-dimensional modules as composition factors.*
(b) *The composition factors of M and N are the only irreducible modules with central character t.*

Proof. By Proposition 4, there is a 2-dimensional module P with $P = P_t^{\text{gen}}$. Let $v \in P_t$ be non-zero. The map

$$\mathbb{C}v_t \to P,$$
$$v_t \mapsto v$$

is an $\mathcal{H}_{\{2\}}$-module homomorphism. Since

$$\operatorname{Hom}_{\mathcal{H}}(M, P) = \operatorname{Hom}_{\mathcal{H}_{\{2\}}}(\mathbb{C}v_t, P),$$

there is a non-zero map from M to P. Since P is irreducible, this map is surjective and P is a quotient of M. The kernel of any map from M to P must be

$$M_1 = M_{s_2 t}^{\text{gen}} \oplus M_{s_1 s_2 t}^{\text{gen}},$$

which is then a submodule of M.

Then we note that $m = T_1T_2T_1T_2T_1v - qT_2T_1T_2T_1v - T_1T_2T_1v + qT_2T_1v + T_1v - qv$ spans a 1-dimensional submodule of M_1. Then let $M_2 = M_1/m$, so that $T_1T_2T_1T_2T_1v = qT_2T_1T_2T_1v + T_1T_2T_1v - qT_2T_1v - T_1v + qv$ in M_2.

Then by a calculation as in Proposition 1(b), M_2 contains an element $m' = T_2T_1v - qT_1v - 3v \in M_{s_2 t}$. Then m', $\tau_1(m')$ and $T_2 \cdot \tau_1(m')$ are linearly independent (since their leading terms cannot be canceled) and span M_2. However, $M_3 = \langle \tau_1(m'), T_2 \cdot \tau_1(m')\rangle$ is clearly closed under the action of T_2. Also, $\tau_1(m') \in M_{s_1 s_2 t}$, so that

$$X^{\lambda} \cdot T_2\tau_1(m') = T_2 X^{s_2\lambda}\tau_1(m') + (q - q^{-1})\frac{X^{\lambda} - X^{s_2\lambda}}{1 - X^{-\alpha_2}}\tau_1(m'),$$

which again lies in M_3. Finally, one can compute that

$$T_1 \cdot \tau_1(m') = -q^{-1}\tau_1(m'), \quad \text{and} \quad T_1 \cdot T_2\tau_1(m') = q(\tau_1(m')) + T_2\tau_1(m').$$

Thus M_3 is a submodule of M_2. By Theorem 5, M_3 is irreducible, and M_2/M_3 is a 1-dimensional module which is isomorphic to the 1-dimensional module spanned by m.

An analogous argument proves the same result for N. Let Q be the 2-dimensional module with $Q = Q_{s_1 s_2 t}^{\text{gen}}$. Then there is a surjection from N to Q, and the kernel of this map, N_1, consists of the t and $s_2 t$ weight spaces of N. Then $n = T_2T_1T_2T_1T_2v - qT_1T_2T_1T_2v - T_2T_1T_2v + qT_1T_2v + T_2v - qv$ spans a 1-dimensional submodule of N_1. Let $N_2 = N_1/\mathbb{C}n$.

Then $N_{s_2 t}$ contains a non-zero element n', and $n', \tau_2(n')$, and $T_1\tau_2(n')$ are linearly independent and span N_2. But $\tau_2(n')$ and $T_1\tau_2(n')$ span a submodule of N_2, which is irreducible by Theorem 5.

(b) Let $\mathbb{C}v_{s_2 t}$ be the one-dimensional $\mathcal{H}_{\{1\}}$-module with weight $s_2 t$, and define $L = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}v_{s_2 t}$. We claim that the composition factors of L are the same as those of M. First, note that the one-dimensional $\mathcal{H}$-module $L_{q,q}$ restricted to $\mathcal{H}_{\{1\}}$ is $\mathbb{C}v_{s_2 t}$. Then there is an $\mathcal{H}_{\{1\}}$-module map from $\mathbb{C}v_{s_2 t}$ to $L_{q,q}$, and thus there is a map from L to $L_{q,q}$. Let L_1 be the kernel of this map. Then L_1 has a 1-dimensional $s_2 t$ weight space, and 2-dimensional generalized t and $s_1 s_2 t$ weight spaces. Also, L_1 contains $l = \tau_2(v_{s_2 t}) = T_2 v_{s_2 t} - q v_{s_2 t}$, an element of the t weight space of L_1. Then we note that $T_2 \cdot$

$(T_2 v_{s_2t} - q v_{s_2t}) = (q - q^{-1})T_2 v_{s_2t} + v_{s_2t} - qT_2 v_{s_2t} = q(T_2 v_{s_2t} - q v_{s_2t})$, so that l spans a 1-dimensional $\mathcal{H}_{\{2\}}$-submodule of L_1, with weight t.

Thus, there is an $\mathcal{H}_2$-module map from $\mathbb{C}v_t$ to L_1, and thus an $\mathcal{H}$-module map from M to L. This map is surjective since $l, T_2l, T_1T_2l, T_2T_1T_2l$, and $T_1T_2T_1T_2l$ are linearly independent and span L_1. Then L_1 is a quotient of M and its composition factors are composition factors of M.

Now, let P be any irreducible $\mathcal{H}$-module with central character t_{1,q^2}. If P is not a composition factor of M or N, then P must be in the kernel of the (surjective) map from $M(t)$ to M. Hence P is at most 6-dimensional, and each of its generalized weight spaces is at most 2-dimensional. If $P = P_t^{\mathrm{gen}}$, then P is 2-dimensional and must be the module described in Proposition 4. Otherwise, we note that $P_{s_2t}^{\mathrm{gen}} \oplus P_{s_1s_2t}^{\mathrm{gen}}$ is an $\mathcal{H}_{\{1\}}$-submodule of P, since the action of τ_1 fixes this subspace of P. Thus $P_{s_2t}^{\mathrm{gen}} \oplus P_{s_1s_2t}^{\mathrm{gen}}$ contains an irreducible $\mathcal{H}_{\{1\}}$-submodule. This subspace must be either $P_{s_1s_2t}^{\mathrm{gen}}$ or a 1-dimensional module with weight s_2t. Hence P is a quotient of either L or M and is isomorphic to a composition factor of M. □

Case 3b: $t_{1,\pm q}$.

Let $t' \in T$ and assume $\alpha_1 \in Z(t')$ but $\alpha_2 \notin P(t')$, so that none of $\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2$, or $3\alpha_1 + \alpha_2$ are in $P(t')$. Since $P(t') \neq \emptyset$, $3\alpha_1 + 2\alpha_2 \in P(t')$, so that $t'(X^{2\alpha_2}) = q^{\pm 2}$ and $t'(X^{\alpha_2}) = \pm q^{\pm 1}$. By applying w_0 if necessary, we may assume $t'(X^{\alpha_2}) = \pm q$. Thus we will analyze the weights $t_{1,\pm q}$ together, except in one case. If q is a primitive third root of unity then $q = q^{-2}$ and $q^{-1} = q^2$, so that $t_{1,q} = t_{1,q^{-2}}$ was analyzed in Case 3a. If q is a primitive sixth root of unity then $-q = q^{-2}$ and $-q^{-1} = q^2$ so that $t_{1,-q} = t_{1,q^{-2}}$ was analyzed in Case 3a. Thus these cases are excluded from the following analysis by simply assuming that $t'(X^{\alpha_2}) \neq q^{-2}$.

If $q^2 = 1$, then $Z(t') = P(t') = \{\alpha_1, 3\alpha_1 + 2\alpha_2\}$, and the irreducibles with central character t can be constructed using Theorem 9. Specifically, there are four 3-dimensional modules with central character t'.

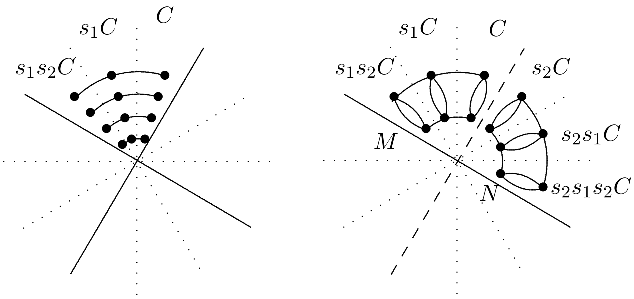
If $q^2 \neq 1$, then $s_1s_2t'(X^{\alpha_1}) = t'(X^{-\alpha_1-\alpha_2}) = \pm q^{\mp 1}$ and $s_1s_2t'(X^{\alpha_2}) = t'(X^{3\alpha_1+2\alpha_2}) = q^{\pm 2}$. Then by Theorem 3, $M(t')$ and $M(t)$ have the same composition factors, where $t = s_1s_2t'$. Also by assuming that $t'(X^{\alpha_2}) \neq q^{-2}$, we have $Z(t) = \{2\alpha_1 + \alpha_2\}$ and $P(t) = \{\alpha_2\}$. Let $\mathbb{C}v_t$ and $\mathbb{C}v_{w_0t}$ be the 1-dimensional $\mathcal{H}_{\{2\}}$-modules spanned by v_t and v_{w_0t}, respectively, and given by

$$T_2 v_t = q v_t, \qquad X^\lambda v_t = t(X^\lambda) v_t,$$

$$T_2 v_{w_0t} = -q^{-1} v_{w_0t}, \quad \text{and} \quad X^\lambda v_{w_0t} = w_0t(X^\lambda) v_{w_0t}.$$

Then define

$$M = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}v_t \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{2\}}} \mathbb{C}v_{w_0t}.$$



$t_{-1,1}, q^2 = 1$ $\qquad$ $t_{\pm q^{-1},q^2}, q^2 \neq 1$

Proposition 26. *Assume $q^2 \neq 1$. Let $t' = t_{1,\pm q}$, and define M and N as above. Assume that it is not true that $t'(X^{\alpha_2}) = q^{-2}$. Then M and N are irreducible.*

Let $t = s_1 s_2 t$. Under the assumptions, $Z(t) = \{2\alpha_1 + \alpha_2\}$ and $P(t) = \{\alpha_2\}$. Then $\dim M_t^{\text{gen}} = \dim M_{s_1 t}^{\text{gen}} = \dim M_{s_2 s_1 t}^{\text{gen}} = 2$. By Theorem 5, M has some composition factor M' with $\dim M_{s_2 s_1 t}^{\text{gen}} = 2$, and by Theorem 3(b), $M' = M$ and M is irreducible. Similarly, Theorem 5 and Theorem 3(b) show that N is irreducible. □

Under the assumptions of this theorem, since M and N are each 6-dimensional, they must be the only composition factors of $M(t)$. If $t'(X^{\alpha_2}) = q^{-2}$, then $w_0 t' = t_{1,q^2}$, which was discussed in the case above.

These two cases are the only weights t with $P(t)$ non-empty and $Z(t)$ containing a short root. Specifically, if t is any weight such that $Z(t)$ contains $\alpha_1, \alpha_1 + \alpha_2$, or $2\alpha_1 + \alpha_2$, there exists $w \in W_0$ so that $\alpha_1 \in Z(wt)$. Then t is in the orbit of one of the weights in the previous cases. Then for the following cases, assume $\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2 \notin Z(t)$.

Case 3c: $t_{q^2,1}$.

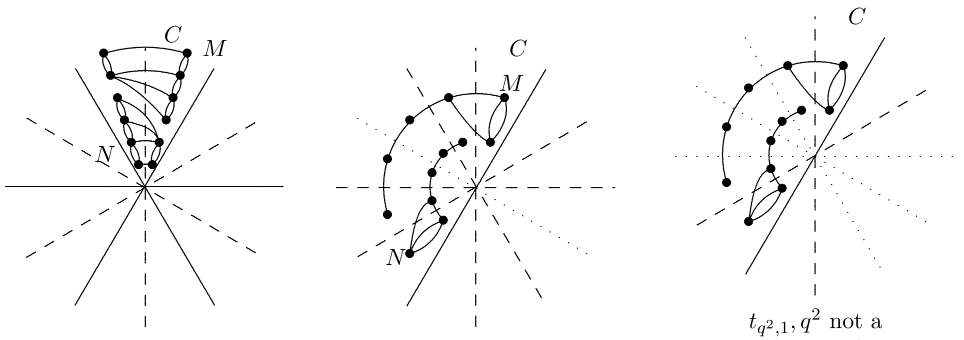
If $\alpha_2 \in Z(t)$ and $\alpha_1 \in P(t)$, then $\alpha_1 + \alpha_2 \in P(t)$ as well, and $t = t_{q^{\pm 2},1}$. These weights are in the same orbit, so we examine $M(t_{q^2,1})$. If $q^2 = -1$ then $t(X^{2\alpha_1 + \alpha_2}) = 1$, so that t is in the orbit of one of the weights considered in Cases 3a and 3b. If $q^2 = 1$, then $t = t_{1,1}$ which has also already been considered. Then we assume $q^2 \neq \pm 1$.

Let $\mathbb{C}v_t$ and $\mathbb{C}v_{w_0 t}$ be the 1-dimensional $\mathcal{H}_{\{1\}}$-modules spanned by v_t and v_{wt}, respectively, and given by

$$T_1 v_t = q v_t, \qquad X^\lambda v_t = t\big(X^\lambda\big) v_t,$$
$$T_1 v_{w_0 t} = -q^{-1} v_{w_0 t}, \quad \text{and} \quad X^\lambda v_{w_0 t} = w_0 t\big(X^\lambda\big) v_{w_0 t}.$$

Then define

$$M = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}v_t \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}v_{w_0 t}.$$



$t_{q^2,1}, q^2$ a primitive third root of unity

$t_{q^2,1}, q^2$ a primitive fourth root of unity

$t_{q^2,1}, q^2$ not a primitive third or fourth root of unity and $q^2 \neq \pm 1$

Proposition 27. *If q^2 is a primitive third root of unity, then M and N are irreducible.*

Proof. If q^2 is a primitive third root of unity, then $Z(t) = \{\alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$ and $P(t) = \{\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$.

Then $\dim M_t^{\mathrm{gen}} = 4$ and $\dim M_{s_1t}^{\mathrm{gen}} = 2$. By Theorem 5, if $M' \subseteq M$ is a submodule of M, then $\dim(M')_t^{\mathrm{gen}} \geqslant 2$ and $\dim(M')_{s_1t}^{\mathrm{gen}} \geqslant 2$. Then $\dim(M/M')_t^{\mathrm{gen}} \leqslant 2$, but $\dim(M/M')_{s_1t}^{\mathrm{gen}} = 0$, so that Theorem 5 implies that $(M/M')_t = 0$. Thus $M' = M$ and M is irreducible. Theorem 5 similarly implies that N is irreducible. □

Then since M and N have different weight spaces, they are not isomorphic and are the only irreducibles with central character t.

Proposition 28. *Assume $q^2 \neq \pm 1$ and that q^2 is not a primitive third root of unity.*

(a) *If q^2 is a primitive fourth root of unity then $M_{s_2s_1s_2s_1t}$ is a 1-dimensional submodule of M, and M', the image of the weight spaces $M_{s_1s_2s_1t}$ and $M_{s_2s_1t}$ in $M/M_{s_1s_2s_1t}$, is an irreducible submodule of $M/M_{s_1s_2s_1t}$. The resulting quotient of M is irreducible.*
(b) *If q^2 is a primitive fourth root of unity then N_{s_1t} is a 1-dimensional submodule of N, and N', the image of the weight spaces $N_{s_2s_1t}$ and $N_{s_1s_2s_1t}$ in N/N_{s_1t}, is an irreducible submodule of N/N_{s_1t}. The resulting quotient of N is irreducible.*
(c) *The composition factors of M and N are the only composition factors of $M(t)$.*
(d) *If q^2 is not a primitive third or fourth root of unity then M and N are irreducible, and are the only irreducible modules with central character t.*

Proof. If q^2 is not ± 1 or a primitive third root of unity, $Z(t) = \{\alpha_2\}$, so that M has one 2-dimensional weight space M_t^{gen} and four 1-dimensional weight spaces M_{s_1t}, $M_{s_2s_1t}$, $M_{s_1s_2s_1t}$, and $M_{s_2s_1s_2s_1t}$.

(a) If q^2 is a primitive fourth root of unity, then $P(t) = \{\alpha_1, \alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$. For $w \in \{s_1, s_2s_1, s_1s_2s_1, s_2s_1s_2s_1\}$, let m_{wt} be a non-zero vector in M_{wt}. By Proposition 1(b),

$$m_{wt} = T_w T_2 v_t + \sum_{w'<w} a_{w,w'} T_{w'} T_2 v_t,$$

for $w \in \{s_1, s_2s_1, s_1s_2s_1, s_2s_1s_2s_1\}$, where $a_{w,w'} \in \mathbb{C}$. Then if $s_i w > w$,

$$\tau_i m_{wt} \neq 0$$

for $w \in \{s_1, s_2s_1, s_1s_2s_1\}$, since the term $T_i T_w T_2$ cannot be canceled by any other term in $\tau_i m_{wt}$.

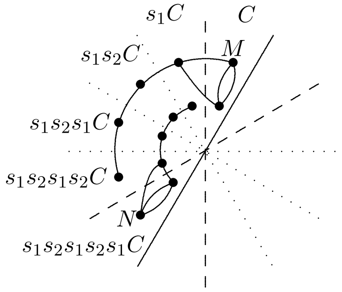
Thus $\tau_2 : M_{s_2s_1s_2s_1t} \to M_{s_1s_2s_1t}$ is the zero map since, by Theorem 2, $\tau_2^2 : M_{s_1s_2s_1}t \to M_{s_1s_2s_1t}$ is the zero map. Hence $M_{s_2s_1s_2s_1t}$ is a submodule of M. Let $M_1 = M/M_{s_2s_1s_2s_1t}$. Similarly, $\tau_2 : M_{s_2s_1t} \to M_{s_1t}$ must be the zero map since, by Theorem 2, $\tau_2^2 : M_{s_1t} \to M_{s_1t}$ is the zero map. Then M_1', the subspace spanned by $\overline{m_{s_2s_1t}}$ and $\overline{m_{s_1s_2s_1t}}$ in M_1, is a submodule of M_1. Since $\tau_1^2 : (M_1')_{s_2s_1t} \to (M_1')_{s_2s_1t}$ is invertible, M_1' is irreducible, and Theorem 5 shows that $M_2 = M_1/M_1'$ is irreducible.

(b) Replacing t by w_0t in this argument shows that N also has three composition factors. The weight space N_{s_1t} is a submodule of N, and $N_1 = N/N_{s_1t}$ has an irreducible 2-dimensional submodule N_1', consisting of the image of $N_{s_2s_1t}$ and $N_{s_1s_2s_1t}$ in N_1. Theorem 5 shows that N_1/N_1' is irreducible.

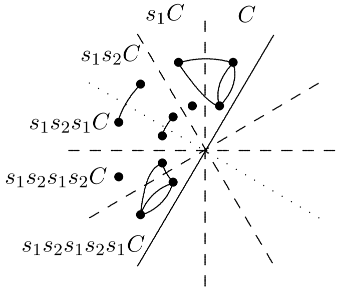
(c) The composition factors of M and N are not distinct, since M_1' and N_1' are irreducible 2-dimensional modules with the same weight spaces, and Proposition 4 shows that $M_1' \cong N_1'$. The 1-dimensional composition factors of M and N are not isomorphic since they have different weights, and the 3-dimensional modules are different since their weight space structures are different.

Counting multiplicities of weight spaces in M, N, and $M(t)$ shows that the remaining composition factor(s) of $M(t)$ must contain an s_2s_1t weight space and an $s_1s_2s_1t$ weight space, each of dimension 1. But Theorem 3(b) shows that there must be one remaining composition factor, and Proposition 4 shows that it is isomorphic to M_1. Then the composition factors of M and N are all the composition factors of $M(t)$.

(d) Theorems 5 and 3(b) show that both M and N are irreducible if q^2 is not a primitive third or fourth root of unity. Since M and N are not isomorphic and are each 6-dimensional, they are the only composition factors of $M(t)$. □



$t_{q^2,1}, q^2$ not a primitive third or fourth root of unity



$t_{q^2,1}, q^2$ a primitive fourth root of unity

Case 3d: $t_{\pm q,1}$.

If $\alpha_2 \in Z(t')$ and $2\alpha_1 + \alpha_2 \in P(t')$, then $t'(X^{2\alpha_1}) = q^{\pm 2}$ and $t'(X^{\alpha_1}) = \pm q^{\pm 1}$. By replacing t' by $w_0 t'$ if necessary, it suffices to assume that $t'(X^{\alpha_1}) = \pm q$. If $t'(X^{\alpha_1}) = q^{-2}$, then t' was analyzed in Case 3c. This occurs when $q^3 = 1$ and $t' = t_{q,1}$, or when $q^3 = -1$ and $t' = t_{-q,1}$. Thus the following analysis will apply to $t_{q,1}$ except if $q^3 = 1$, and $t_{-q,1}$ except for when $q^3 = -1$. (This is tantamount to assuming that $P(t)$ and $Z(t)$ each contain exactly one element for this t.)

Also, if $t'(X^{3\alpha_1}) = q^{-2}$, then $P(t)$ also contains $3\alpha_1 + \alpha_2$ and $3\alpha_1 + 2\alpha_2$. This occurs when $q^5 = 1$ and $t'(X^{\alpha_1}) = q$ or when $q^5 = -1$ and $t'(X^{\alpha_1}) = -q$. When either of these hold, t' is the same orbit as t_{q^2,q^2}. This case (which was specifically not addressed in Case 2 above) will be treated separately below.

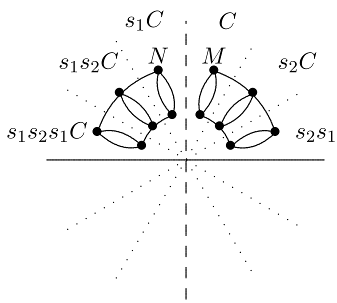
Define $t = s_2 s_1 t'$ so that $t(X^{\alpha_1}) = q^2$ and $t(X^{\alpha_2}) = \pm q^{-3}$. Let $\mathbb{C}v_t$ and $\mathbb{C}v_{w_0t}$ be the 1-dimensional $\mathcal{H}_{\{1\}}$-modules spanned by v_t and v_{wt}, respectively, and given by

$$T_1 v_t = q v_t, \qquad X^\lambda v_t = t\big(X^\lambda\big) v_t,$$

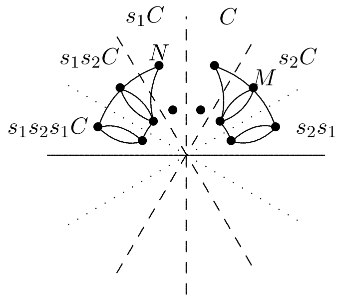
$$T_1 v_{w_0t} = -q^{-1} v_{w_0t}, \quad \text{and} \quad X^\lambda v_{w_0t} = w_0 t\big(X^\lambda\big) v_{w_0t}.$$

Then define

$$M = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}v_t \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}v_{w_0t}.$$



$t_{q^2,\pm q^{-3}}$, $P(t) = \{\alpha_1\}$, $Z(t) = \{3\alpha_1 + 2\alpha_2\}$



$t_{q^2,q^2}, q^{10} = 1$

Proposition 29. *Let $t' = t_{\pm q,1}$. Assume that $t'(X^{\alpha_1}) \neq q^{-2}$ and $t'(X^{3\alpha_1}) \neq q^{-2}$. Then M and N are irreducible.*

Proof. Let $t = s_2 s_1 t'$. Under the assumptions, $Z(t) = \{3\alpha_1 + 2\alpha_2\}$ and $P(t) = \{\alpha_1\}$. Then $\dim M_t^{\text{gen}} = \dim M_{s_2 t}^{\text{gen}} = \dim M_{s_1 s_2 t}^{\text{gen}} = 2$. Then Theorem 5 and Theorem 3(b) show that M is irreducible. N is also irreducible by the same reasoning. □

Under the assumption of the theorem, since M and N are not isomorphic and are each 6-dimensional, they are the only composition factors of $M(t)$. Note that $t'(X^{\alpha_1}) = q^{-2}$ exactly if $q^3 = 1$ or -1 and $t'(X^{\alpha_1}) = q$ or $-q$, respectively. In this case, the central character t' has been analyzed above (Case 3c) Also, $t'(X^{\alpha_1}) = q^{-4}$ exactly if $q^5 = 1$ or -1 and $t'(X^{\alpha_1}) = q$ or $-q$, respectively. In this case, t' is in the same orbit as t_{q^2,q^2}.

Proposition 30. *If $t = t_{q^2,q^2}$ and q^2 is a fifth root of unity, then*

(a) *M has a 5-dimensional irreducible submodule M' and*
(b) *N has a 5-dimensional irreducible submodule N'.*

Proof. Given these assumptions, $Z(t) = \{3\alpha_1 + 2\alpha_2\}$ and $P(t) = \{\alpha_1, \alpha_2, 3\alpha_1 + \alpha_2\}$. Then $\dim M_t^{\text{gen}} = \dim M_{s_2 t}^{\text{gen}} = \dim M_{s_1 s_2 t}^{\text{gen}} = 2$. Let $L_{q,q} = \mathbb{C}v$ be the 1-dimensional $\mathcal{H}$ module given by

$$T_i v = qv, \qquad X^{\alpha_i} = q^2 v, \quad \text{for } i = 1, 2.$$

Since

$$\operatorname{Hom}_{\mathcal{H}}(\mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}v_t, L_{q,q}) = \operatorname{Hom}_{\mathcal{H}_{\{1\}}}(\mathbb{C}v_t, L_{q,q}|_{\mathcal{H}_{\{1\}}})$$

and

$$\phi : \mathbb{C}v_t \to L_{q,q},$$
$$v_t \mapsto v$$

is a map of $\mathcal{H}_{\{1\}}$-modules, there is a non-zero map $\theta : M \to L_{q,q}$. Then let M_1 be the kernel of θ, which is 5-dimensional. Similarly, there is a map $\rho : N \to L_{q^{-1},q^{-1}}$, where $L_{q^{-1},q^{-1}} = \mathbb{C}v$ is given by

$$T_i = -q^{-1} v, \qquad X^{\alpha_i} = q^{-2} v, \quad \text{for } i = 1, 2.$$

Then if N_1 is the 5-dimensional kernel of ρ, Theorem 5 and Theorem 3(b) show that M_1 and N_1 are both irreducible. □

These two 5-dimensional modules, plus the 1-dimensional modules $L_{q,q}$ and $L_{-q^{-1},-q^{-1}}$ account for all the composition factors of $M(t)$.

Case 3e: $t_{q^{2/3},1}$.

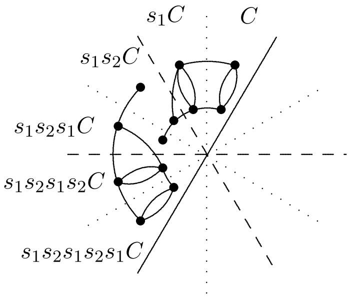
If $\alpha_2 \in Z(t)$ and $3\alpha_1 + \alpha_2 \in P(t)$, then $3\alpha_1 + 2\alpha_2 \in P(t)$ as well. If $t(X^{3\alpha_1+\alpha_2}) = q^{-2}$, then $w_0 t(X^{\alpha_2}) = 1$ and $w_0 t(X^{3\alpha_1+\alpha_2}) = q^2$, so by replacing t with $w_0 t$ if necessary, assume that $t(X^{\alpha_1})^3 = q^2$. If $\alpha_1 \in P(t)$, then this weight was analyzed in Case 3c, and if $2\alpha_1 + \alpha_2 \in P(t)$, then this weight was analyzed in Case 3d.

Then we assume $Z(t) = \{\alpha_2\}$ and $P(t) = \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$. Let $t' = s_1 t$ so that $Z(t') = \{3\alpha_1 + \alpha_2\}$ and $P(t') = \{\alpha_2, 3\alpha_1 + 2\alpha_2\}$. Let $\mathbb{C}v_t$ and $\mathbb{C}v_{w_0 t'}$ be the 1-dimensional $\mathcal{H}_{\{1\}}$-modules spanned by $v_{t'}$ and $v_{w_0 t'}$, respectively, and given by

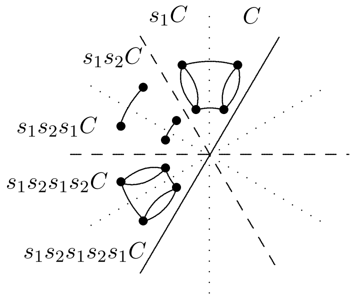
$$T_1 v_{t'} = q v_{t'}, \qquad X^\lambda v_{t'} = t'\bigl(X^\lambda\bigr) v_{t'},$$
$$T_1 v_{w_0 t'} = -q^{-1} v_{w_0 t'}, \quad \text{and} \quad X^\lambda v_{w_0 t'} = w_0 t'\bigl(X^\lambda\bigr) v_{w_0 t'}.$$

Then define

$$M = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}v_{t'} \quad \text{and} \quad N = \mathcal{H} \otimes_{\mathcal{H}_{\{1\}}} \mathbb{C}v_{w_0 t'}.$$



$t_{q^{2/3},1}$, $P(t) = \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$, $q^2 \neq -1$



$t_{q^{2/3},1}$, $P(t) = \{3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$, $q^2 = -1$

Proposition 31. *Assume $t = t_{q^{2/3},1}$, where $q^{2/3}$ is a third root of q^2 not equal to $q^{\pm 2}$ or $\pm q^{\pm 1}$, and that $q^2 \neq \pm 1$. Then M and N are irreducible.*

Proof. Under the assumptions, $Z(t') = \{3\alpha_1 + \alpha_2\}$ and $P(t) = \{\alpha_2, 3\alpha_1 + 2\alpha_2\}$, so that $\dim M_t^{\mathrm{gen}} = \dim M_{s_1 t}^{\mathrm{gen}} = 2$ while $\dim M_{s_2 s_1} = \dim M_{s_1 s_2 s_1 t} = 1$. Then Theorem 5 and Theorem 3(b) show that M is irreducible. Similarly, N is irreducible by the same reasoning, so that M and N are the only irreducible modules with central character t. $\square$

Proposition 32. *Assume $q^2 = -1$ and that $t = t_{q^{2/3},1}$, where $q^{2/3}$ is a third root of q^2 not equal to $q^{\pm 2}$ or $\pm q^{\pm 1}$. Then M and N each have an irreducible 2-dimensional submodule consisting of their $s_2 s_1 t$ and $s_1 s_2 s_1 t$ weight spaces. The resulting quotients are irreducible.*

Proof. Let $X = \{e, s_1, s_2 s_1, s_1 s_2 s_1\}$. By a calculation analogous to that in Proposition 1(b), the generalized t' weight space of M is generated by $v_{t'}$ and a vector v of the form $\sum_{x \in X} a_x T_x v_{t'}$, where the a_x are in $\mathbb{C}$ and $a_{s_1 s_2 s_1} \neq 0$. Then $\tau_2(v) \neq 0$, since it contains a non-zero $T_2 T_1 T_2 T_1 v_t$ term. But then $\tau_2 \tau_2(v) = 0$ by Theorem 2(c), so that the space $M_1 = M_{s_2 t'} \oplus M_{s_1 s_2 t'}$ is actually a submodule of M. The resulting quotient M/M_1 is irreducible by Theorem 3. A similar argument shows the same for N. $\square$

Note that Proposition 4 shows that the 2-dimensional composition factors of M and N are isomorphic, and this proposition implies that when $q^2 = -1$, the composition factors of M and N are the only irreducibles with this central character. Counting dimensions of the weight spaces of these irreducibles shows that the final composition factor of $M(t)$ must be 2-dimensional with weights $s_2 t'$ and $s_1 s_2 t'$, since there are no 1-dimensional modules with this central character. So the last composition factor of $M(t)$ must also be isomorphic to the 2-dimensional submodule of M.

Summary. We summarize the results of the previous theorems, including our choices of representatives for the various central characters, in Table 4. Some notes are necessary about Table 4. An entry of "N/A" means that the given central character is in the same orbit as a previous character for that particular value of q, as described after Theorem 21.

If $q^{10} = 1$, we are assuming that $q^5 = -1$, so that $t_{\pm q,1} = t_{\pm q^{-4},1}$.

If $q^8 = 1$, then only the central characters $t_{q^2,1}$, $t_{q^2,-q^{-2}}$, and t_{q^2,q^2} change from the generic case. All three of these characters are now in the same orbit. Also, we assume that for the central character $t_{q^{2/3},1}$, we choose a cube root of q^2 besides q^{-2}.

Table 4

t	Dims. of irreds.						
	q generic	$q^{12}=1$	$q^{10}=1$	$q^8=1$	$q^6=1$	$q^4=1$	$q=-1$
$t_{1,1}$	12	12	12	12	12	12	1, 1, 1, 1, 2, 2
$t_{1,-1}$	12	12	12	12	12	1, 2, 2	3, 3, 3, 3
$t_{1^{1/3},1}$	12	12	12	12	N/A	12	3, 3, 6
t_{1,q^2}	1, 1, 2, 3, 3	1, 1, 2, 3, 3	1, 1, 2, 3, 3	1, 1, 2, 3, 3	1, 1, 1, 1, 3, 3	N/A	N/A
$t_{1,\pm q}$	6, 6	6, 6	6, 6	6, 6	6, 6	6, 6	N/A
$t_{1,z}$	12	12	12	12	12	12	6, 6
$t_{q^2,1}$	6, 6	6, 6	6, 6	1, 1, 2, 3, 3	6, 6	N/A	N/A
$t_{q,1}$	6, 6	6, 6	1, 1, 5, 5	6, 6	6, 6	6, 6	N/A
$t_{-q,1}$	6, 6	6, 6	6, 6	6, 6	6, 6	N/A	N/A
$t_{q^{2/3},1}$	6, 6	6, 6	6, 6	6, 6	6, 6	2, 4, 4	N/A
$t_{z,1}$	12	12	12	12	12	12	6, 6
$t_{1^{1/3},q^2}$	3, 3, 3, 3	N/A	3, 3, 3, 3	N/A	3, 3, 3, 3	N/A	N/A
$t_{q^2,-q^{-2}}$	2, 2, 4, 4	N/A	2, 2, 4, 4	2, 2, 4, 4	N/A	N/A	N/A
t_{q^2,q^2}	1, 1, 5, 5	1, 1, 2, 2, 3, 3	N/A	N/A	N/A	N/A	N/A
$t_{q^2,z}$	6, 6	6, 6	6, 6	6, 6	6, 6	6, 6	N/A
t_{z,q^2}	6, 6	6, 6	6, 6	6, 6	6, 6	6, 6	N/A
$t_{z,w}$	12	12	12	12	12	12	12

If $q^6=1$, the entries for the central characters $t_{\pm q,1}$ and $t_{1,\pm q}$ only apply to the characters $t_{-q^{-2},1}$ and $t_{1,-q^{-2}}$ (depending on whether q^3 is 1 or -1). Then we note that $t_{1^{1/3},1}=t_{q^{\pm 2},1}$, and $t_{q^{-2},1}$ is in the same orbit as $t_{q^2,1}$. Also, $t_{1,q^{-2}}=w_0 t_{1,q^2}$, and $t_{q^{-2},1}=w_0 t_{q^2,1}$. Finally, $t_{1^{1/3},q^2}=t_{q^{\pm 2},q^2}$, but $s_1 t_{q^2,q^2}=t_{q^{-2},q^2}=s_2 t_{1,q^{-2}}$ and so both are in the same orbit as t_{1,q^2}.

When $q^2=-1$, a number of characters change from the general case. Now, $t_{1,-1}=t_{1,q^2}$, which is in the same orbit as $t_{q^2,-q^{-2}}$, t_{q^2,q^2} and $t_{q^2,1}$. Similarly, $t_{1^{1/3},q^2}$ is in the same orbit as $t_{q^{2/3},1}$.

When $q^2=1$, $Z(t)=P(t)$ for all $t\in T$.

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