Topology and its Applications 156 (2009) 2357–2363

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

On continuously irreducible continua ☆

Sergio Macías

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, Ciudad Universitaria, México D.F., C.P. 04510, Mexico

ARTICLE INFO

Article history:
Received 21 May 2008
Accepted 6 January 2009

Dedicated to Professor Charles P. Hagopian

MSC:
54C60
54B20

Keywords:
Continuous decomposition
Continuously irreducible continuum
Continuum
Homogeneous continuum
Hyperspace
Idempotent
Irreducible continuum
Point $\mathcal{T}$-symmetric
Set function $\mathcal{T}$
$\mathcal{T}$-additive
Type λ continuum
Upper semicontinuous decomposition
Z-set

ABSTRACT

We study continuously irreducible continua and characterize them as those continua of type λ for which the set function $\mathcal{T}$ is continuous. Using results by Mohler and Oversteegen, we present a new family of one-dimensional continua for which the set function $\mathcal{T}$ is continuous and no element of the family contains a pseudo-arc. We study the hyperspaces of these continua.

1. Introduction

The purpose of the paper is to study continuously irreducible continua. The paper consists of six sections. After the Introduction and Definitions, Section 3 is devoted to obtaining general properties of this class of continua, in particular we characterize them as those continua of type λ for which the set function $\mathcal{T}$ is continuous (Theorem 3.2) and we show that such continua cannot be homogeneous (Theorem 3.5). In Section 4, using results by Mohler and Oversteegen [15], we present a new family of one-dimensional continua for which the set function $\mathcal{T}$ is continuous and no element of the family contains a pseudo-arc (Theorem 4.4). In Section 5 we study maps of a continuously irreducible continuum into itself, we prove that if f is such a map and its image is not contained in a layer, then the image of f is a subcontinuum which is also continuously irreducible (Theorem 5.1) and we prove that if f is a map of a continuously irreducible continuum into itself with connected fibres, then f has a fixed point (Theorem 5.4). In Section 6 we study the hyperspace of subcontinua of a continuously irreducible continuum, we show that if X is such a continuum, then its hyperspace of subcontinua is

☆ This research was partially supported by the grant 42602 of CONACyT.
E-mail address: macias@servidor.unam.mx.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.01.024

locally a 2-cell at the top (Theorem 6.1) and $\mathcal{F}_n(X)$ is a Z-set in the hyperspaces 2^X and $\mathcal{C}_n(X)$ for any positive integer n (Theorem 6.4). We also show that if X is a continuously irreducible continuum with nondegenerate layers, then its hyperspace of subcontinua admits a Whitney map whose levels admit a continuous decomposition into subcontinua such that the quotient space is $[0, 1]$ (Theorem 6.2).

2. Definitions

If (Z, d) is a metric space, then given $A \subset Z$ and $\varepsilon > 0$, the open ball about A of radius ε is denoted by $\mathcal{V}^d_\varepsilon(A)$, the interior of A is denoted by $Int_Z(A)$, and the closure of A is denoted by $Cl_Z(A)$. The power set of Z is denoted by $\mathcal{P}(Z)$.

A *map* means a continuous function. A surjective map $f : X \to Y$ between metric spaces is said to be:

- *monotone* if $f^{-1}(B)$ is connected for each connected subset B of Y;
- *open* provided that $f(U)$ is open in Y for every open subset U of X;
- *closed* if $f(C)$ is closed in Y for every closed subset C of X;
- *atomic* provided that for each subcontinuum L of X such that $f(L)$ is nondegenerate, we have that $L = f^{-1}(f(L))$.

Given a metric space Z, a *decomposition* of Z is a family $\mathcal{G}$ of nonempty and mutually disjoint subsets of Z such that $\bigcup \mathcal{G} = Z$. A decomposition $\mathcal{G}$ of a metric space Z is said to be *upper semicontinuous* if the quotient map $q : Z \twoheadrightarrow Z/\mathcal{G}$ is closed. The decomposition is *continuous* provided that the quotient map is both closed and open.

Let Y be a metric space. A closed subset A of Y is a *Z-set of Y* provided that for each $\varepsilon > 0$, there exists a map $f : Y \to Y \setminus A$ such that $d(x, f(x)) < \varepsilon$, where d is the metric of Y.

A *continuum* is a nonempty compact, connected metric space. A *subcontinuum* is a continuum contained in a space Z. A continuum X is *decomposable* if there exist two proper subcontinua A and B of X such that $X = A \cup B$. A continuum is *indecomposable* if it is not decomposable. The continuum X is *unicoherent* provided that every time $X = A \cup B$, where A and B are subcontinua of X, $A \cap B$ is connected. A subcontinuum Y of a continuum X is *terminal* provided that if K is a subcontinuum of X and $K \cap Y \neq \emptyset$, then either $K \subset Y$ or $Y \subset K$.

An *arc* is any space homeomorphic to $[0, 1]$.

A continuum X is *irreducible* if there are two points p and q of X such that no proper subcontinuum of X contains both p and q. A continuum X is of *type* λ provided that X is irreducible and each indecomposable subcontinuum of X has empty interior. By [18, Theorem 10, p. 15], a continuum X is of type λ if and only if admits a finest monotone upper semicontinuous decomposition $\mathcal{G}$ such that each element of $\mathcal{G}$ is nowhere dense and $X/\mathcal{G}$ is an arc. Each element of $\mathcal{G}$ is called a *layer* of X. Following [15], we say that a continuum X of type λ for which $\mathcal{G}$ is continuous, is a *continuously irreducible continuum*.

Given a continuum X, we define its *hyperspaces* as the following sets:

$$2^X = \{A \subset X \mid A \text{ is closed and nonempty}\};$$

$$\mathcal{C}(X) = \{A \in 2^X \mid A \text{ is connected}\};$$

and given a positive integer n:

$$\mathcal{C}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ components}\};$$

$$\mathcal{F}_n(X) = \{A \in 2^X \mid A \text{ has at most } n \text{ points}\}.$$

All these sets are topologized with the Vietoris topology [16, (0.11)] or the Hausdorff metric [16, (0.1)].

If X is a continuum, then $\mathcal{C}(X)$ is *locally a 2-cell at the top* provided that there exists a 2-cell $\mathcal{E}$ in $\mathcal{C}(X)$ such that $X \in Int_{\mathcal{C}(X)}(\mathcal{E})$.

Given a continuum X, a *Whitney map* for 2^X is a map $\mu : 2^X \to [0, 1]$ such that $\mu(X) = 1$, $\mu(\{x\}) = 0$ for each $x \in X$, and if $A, B \in 2^X$ and $A \subsetneq B$, then $\mu(A) < \mu(B)$.

If $f : X \to Y$ is a map, then $2^f : 2^X \to 2^Y$ and $\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(Y)$, given by $2^f(A) = f(A)$ and $\mathcal{C}(f)(A) = f(A)$, respectively, are maps [16, (1.168)].

Given a continuum X, we define the set function $\mathcal{T} : \mathcal{P}(X) \to \mathcal{P}(X)$ as follows: if $A \in \mathcal{P}(X)$ then

$$\mathcal{T}(A) = X \setminus \{x \in X \mid \text{there exists a subcontinuum } W \text{ of } X \text{ such that } x \in Int_X(W) \subset W \subset X \setminus A\}.$$

We write $\mathcal{T}_X$ if there is a possibility of confusion. Let us observe that for any subset A of X, $\mathcal{T}(A)$ is a closed subset of X and $A \subset \mathcal{T}(A)$. A continuum X is *aposyndetic* provided that $\mathcal{T}(\{p\}) = \{p\}$ for every $p \in X$.

A continuum X is *$\mathcal{T}$-additive* provided that $\mathcal{T}(A \cup B) = \mathcal{T}(A) \cup \mathcal{T}(B)$ for each pair of nonempty closed subsets A and B of X. We say that X is *point $\mathcal{T}$-symmetric* if for any two points p and q of X, $p \in \mathcal{T}(\{q\})$ if and only if $q \in \mathcal{T}(\{p\})$. The set function $\mathcal{T}$ is *idempotent on* X provided that $\mathcal{T}^2(A) = \mathcal{T}(A)$ for each subset A of X, where $\mathcal{T}^2 = \mathcal{T} \circ \mathcal{T}$.

We say that $\mathcal{T}$ *is continuous for a continuum* X provided that $\mathcal{T} : 2^X \to 2^X$ is continuous.

3. General properties

Our first property is due to G.R. Gordh Jr. [6, Theorem 3.2].

3.1. Theorem. *If X is a continuously irreducible continuum, then X is unicoherent.*

3.2. Theorem. *Let X be a type λ continuum. Then X is a continuously irreducible continuum if and only if the set function $\mathcal{T}_X$ is continuous for X. Moreover, $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is the finest continuous monotone decomposition of X such that $X/\mathcal{G}$ is an arc.*

Proof. Suppose X is a continuously irreducible continuum. Let $\mathbb{G}$ be the finest continuous monotone decomposition of X such that $X/\mathbb{G}$ is an arc, and let $q : X \to [0,1]$ be the quotient map. Then q is monotone and open. Since X is irreducible, if Z is a proper subcontinuum of X, then $q(Z)$ is a proper subcontinuum of $[0,1]$. Hence, by [2, Theorem 5, p. 9], $\mathcal{T}_X$ is continuous. In fact, $\mathcal{T}_X(A) = q^{-1}(q(A))$ for every subset A of X. In particular, $\mathcal{T}_X(\{x\}) = q^{-1}(q(x))$ for each $x \in X$, and $\mathbb{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$.

Now suppose X is a continuum of type λ for which $\mathcal{T}_X$ is continuous. Let $q : X \to [0,1]$ be the quotient map given by the finest upper semicontinuous monotone decomposition of X. Let $x, z \in X$ such that $q(x) \neq q(z)$. Then there exists a subinterval A of $[0,1]$ such that $q(z) \in Int_{[0,1]}(A)$ and $q(x) \notin A$. Hence, $z \in Int_X(q^{-1}(A))$ and $x \notin q^{-1}(A)$. Thus, $z \notin \mathcal{T}_X(\{x\})$. Therefore, $\mathcal{T}_X(\{x\}) \subset q^{-1}(q(x))$. Let $z \in q^{-1}(q(x))$, and suppose W is a subcontinuum of X such that $z \in Int_X(W)$. Then $q^{-1}(q(x)) \subset W$ [18, Theorem 5, p. 10]. Hence, $z \in \mathcal{T}_X(\{x\})$ and $\mathcal{T}_X(\{x\}) = q^{-1}(q(x))$. Thus, $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is the finest upper semicontinuous monotone decomposition of X such that $X/\mathcal{G}$ is an arc. Since $\mathcal{T}_X$ is continuous, $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is continuous. Therefore, X is a continuously irreducible continuum. □

3.3. Lemma. *Let X be a continuously irreducible continuum, and let $\mathcal{G}$ be the finest continuous monotone decomposition of X such that $X/\mathcal{G}$ is an arc. If $q : X \to [0,1]$ is the quotient map, then q is atomic and $q^{-1}(t)$ is a terminal subcontinuum of X for every $t \in [0,1]$.*

Proof. Let X be a continuously irreducible continuum, and let $q : X \to [0,1]$ be the quotient map obtained from the finest continuous monotone decomposition of X. Let K be a subcontinuum of X such that $q(K)$ is nondegenerate. It is always true that $K \subset q^{-1}(q(K))$. Let $x \in q^{-1}(q(K))$. Then there exists $y \in K$ such that $q(y) = q(x)$. Hence, by [18, Theorem 5, p. 10], $q^{-1}(q(x)) \subset K$. Therefore, $K = q^{-1}(q(K))$, and q is atomic.

Let $t \in [0,1]$. Since q is an atomic map, by [14, (1.2)], $q^{-1}(t)$ is a terminal subcontinuum of X. □

As a consequence of Theorem 3.2 and Lemma 3.3, we obtain:

3.4. Corollary. *Let X be a continuously irreducible continuum. If $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$, then $\mathcal{G}$ is the finest continuous decomposition of X such that $X/\mathcal{G}$ is an arc, and $\mathcal{T}_X(\{x\})$ is a terminal subcontinuum of X for every $x \in X$.*

With what we have done, we provide a different proof of the following result:

3.5. Theorem. *If X is a continuously irreducible continuum, then X is not homogeneous.*

Proof. Suppose X is a continuously irreducible homogeneous continuum. Then $\mathcal{T}_X$ is continuous for X, by Theorem 3.2. Hence, by [10, 3.6], we have three possibilities, namely:

(1) X is indecomposable;
(2) X is not aposyndetic and $X/\mathcal{G}$ is homeomorphic to the unit circle $\mathcal{S}^1$, or to the Menger universal curve $\mathcal{M}$, where $\mathcal{G} = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$;
(3) X is locally connected.

Note that if X is as in (1) or (2), then X is not a continuum of type λ. Since the only locally connected irreducible continuum is an arc, X is not homogeneous. Thus, in any case, we obtain a contradiction. Therefore, X is not homogeneous. □

The following result is used in the proof of Theorems 6.1 and 6.2.

3.6. Lemma. *Let X be a continuously irreducible continuum. If D is a subcontinuum of X such that either $D = \mathcal{T}_X(\{x\})$ for some $x \in X$ or $D \not\subset \mathcal{T}_X(\{x\})$ for any $x \in X$, then $\mathcal{T}_X(D) = D$.*

Proof. Let X be a continuously irreducible continuum and let D be a subcontinuum of X. If $D = \mathcal{T}_X(\{x\})$ for some $x \in X$, then since $\mathcal{T}_X$ is continuous (Theorem 3.2), $\mathcal{T}_X$ is idempotent [1, Lemma 3]. Hence, $\mathcal{T}_X(D) = \mathcal{T}_X(\mathcal{T}_X(\{x\})) = \mathcal{T}_X(\{x\}) = D$.

Next, suppose that $D \not\subset \mathcal{T}_X(\{x\})$ for any $x \in X$. Then, by Corollary 3.4, $\mathcal{T}_X(\{x\}) \subset D$ for each $x \in D$. Since $\{\mathcal{T}_X(\{x\}) \mid x \in X\}$ is a decomposition, Theorem 3.2, X is point $\mathcal{T}_X$-symmetric. Hence, by [1, Lemma 9], X is $\mathcal{T}_X$-additive ($\mathcal{T}_X$ is continuous, by Theorem 3.2). Thus, $\mathcal{T}_X(D) = \bigcup\{\mathcal{T}_X(\{x\}) \mid x \in D\}$ [3, Theorem B]. Therefore, $\mathcal{T}_X(D) = D$. $\square$

4. New examples of continua for which $\mathcal{T}$ is continuous

All the known examples of decomposable non-locally connected continua X for which the set function $\mathcal{T}$ is continuous have the property that there exist many points $x \in X$ such that $\mathcal{T}(\{x\})$ is a pseudo-arc [9] and [10]. In Theorem 4.4 we present a new family of one-dimensional continua for which the set function $\mathcal{T}$ is continuous which do not contain pseudo-arcs.

As a consequence of Theorem 3.2, we have:

4.1. Theorem. *The class of continuously irreducible continua is a class of continua for which the set function $\mathcal{T}$ is continuous.*

The following theorem is due to Mohler and Oversteegen [15, Corollary 1.1 and Theorem 2.1].

4.2. Theorem. *Let X be any continuously irreducible one-dimensional continuum. Then there exist a continuously irreducible one-dimensional continuum $\widehat{X}$ such that every nondegenerate subcontinuum of $\widehat{X}$ contains an arc, and an atomic map $g : \widehat{X} \to X$.*

As a consequence of Theorems 4.2 and 3.2, we have:

4.3. Theorem. *For each one-dimensional continuously irreducible continuum X, there exist a one-dimensional continuously irreducible continuum $\widehat{X}$ such that $\mathcal{T}_{\widehat{X}}(\{\hat{x}\})$ does not contain a pseudo-arc for any $\hat{x} \in \widehat{X}$, and an atomic map $g : \widehat{X} \to X$.*

Let $\mathcal{Z}$ be the class of one-dimensional continuously irreducible continua and let $\widehat{\mathcal{Z}} = \{\widehat{X} \mid X \in \mathcal{Z}\}$, where $\widehat{X}$ is given in Theorem 4.3. Hence, we have the following:

4.4. Theorem. *The class $\widehat{\mathcal{Z}}$, defined above, is a class of one-dimensional continua $\widehat{X}$ for which the set function $\mathcal{T}_{\widehat{X}}$ is continuous such that $\mathcal{T}_{\widehat{X}}(\{\hat{x}\})$ does not contain a pseudo-arc for any $\hat{x} \in \widehat{X}$. In particular, no element of $\widehat{\mathcal{Z}}$ contains a pseudo-arc.*

5. Maps

We prove that if f is a map of a continuously irreducible continuum into itself and its image is not contained in a layer, then the image of f is a subcontinuum which is also continuously irreducible (Theorem 5.1). We also prove that if f is a map of a continuously irreducible continuum into itself with connected fibres, then f has a fixed point (Theorem 5.4).

5.1. Theorem. *Let X be a continuously irreducible continuum. If $f : X \to X$ is a map and $x \in f(X)$, then either $f(X) \subset \mathcal{T}_X(\{x\})$ or $\mathcal{T}_X(\{x\}) \subset f(X)$. Moreover, in the second case, $f(X) = \bigcup\{\mathcal{T}_X(\{x\}) \mid x \in f(X)\}$.*

Proof. Let X be a continuously irreducible continuum, let $f : X \to X$ be a map, and let $x \in f(X)$. Note that $f(X) \cap \mathcal{T}_X(\{x\}) \neq \emptyset$. Since $f(X)$ is a continuum and $\mathcal{T}_X(\{x\})$ is a terminal subcontinuum of X (Corollary 3.4), either $f(X) \subset \mathcal{T}_X(\{x\})$ or $\mathcal{T}_X(\{x\}) \subset f(X)$.

Suppose $\mathcal{T}_X(\{x\}) \subset f(X)$. If $\mathcal{T}_X(\{x\}) = f(X)$, then since $\mathcal{T}_X$ is continuous (Theorem 3.2), $\mathcal{T}_X$ is idempotent [1, Lemma 3]. Hence, $f(X) = \bigcup\{\mathcal{T}_X(\{x\}) \mid x \in f(X)\}$ [8, 3.1.53]. Next assume that $\mathcal{T}_X(\{x\}) \neq f(X)$. Then for every $z \in f(X) \setminus \mathcal{T}_X(\{x\})$, we also have that $\mathcal{T}_X(\{z\}) \subset f(X)$. Therefore,

$$f(X) = \bigcup\{\mathcal{T}_X(\{x\}) \mid x \in f(X)\}. \qquad \square$$

As a consequence of Theorem 5.1, we have:

5.2. Corollary. *Let X be a continuously irreducible continuum. If $f : X \to X$ is a map such that either $f(X) \not\subset \mathcal{T}_X(\{x\})$ or $f(X) \neq \mathcal{T}_X(\{x\})$ for any $x \in X$, then $\mathcal{T}_X(\{x\}) = \mathcal{T}_{f(X)}(\{x\})$ for each $x \in f(X)$.*

5.3. Theorem. *Let X be a continuously irreducible continuum. If $f : X \to X$ is a map, then there exists $x \in X$ such that $f(x) \in \mathcal{T}_X(\{x\})$.*

Proof. Let X be a continuously irreducible continuum, and let $q : X \to [0,1]$ be the quotient map given by the finest continuous monotone decomposition of X. Let $f : X \to X$ be a map and consider the following sets:

$$A = \{x \in X \mid q(x) < q(f(x))\};$$
$$B = \{x \in X \mid q(x) = q(f(x))\};$$
$$C = \{x \in X \mid q(x) > q(f(x))\}.$$

We show that $B \neq \emptyset$. To this end, suppose $B = \emptyset$. Note that if $x_0 \in q^{-1}(0)$ and $x_1 \in q^{-1}(1)$, then $x_0 \in A$ and $x_1 \in C$. Also observe that $A \cap C = \emptyset$ and $X = A \cup C$.

Let $x \in Cl_X(A)$. Then there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of points of A converging to x. Since q and f are both continuous, $\{q(a_n)\}_{n=1}^{\infty}$ converges to $q(x)$ and $\{q(f(a_n))\}_{n=1}^{\infty}$ converges to $q(f(x))$. Since $\{a_n\}_{n=1}^{\infty} \subset A$, $q(a_n) < q(f(a_n))$ for every positive integer n. Hence, $q(x) \leqslant q(f(x))$. Since $B = \emptyset$, $q(x) < q(f(x))$. Thus, $x \in A$. Therefore, A is closed in X. Similarly, C is closed in X too. Thus, X is not connected, a contradiction. Therefore, $B \neq \emptyset$.

Let $x \in B$. Then, by Theorem 3.2, $q^{-1}(q(x)) = \mathcal{T}_X(\{x\})$. Therefore, $f(x) \in \mathcal{T}_X(\{x\})$. □

G.R. Gordh Jr. proved that if X is a continuum of type λ and each layer of X has the fixed point property, then any monotone map from X onto itself has a fixed point [5, Theorem 3B.1]. In the following theorem we prove a similar result for continuously irreducible continua, but we do not assume that the map is surjective.

5.4. Theorem. *Let X be a continuously irreducible continuum such that $\mathcal{T}_X(\{x\})$ has the fixed point property for each $x \in X$. If $f : X \to X$ is a map such that f is monotone onto $f(X)$, then f has a fixed point.*

Proof. Let X be a continuously irreducible continuum. Suppose $\mathcal{T}_X(\{x\})$ has the fixed point property for every $x \in X$. Let $f : X \to X$ be a map such that f is monotone onto $f(X)$. If there exists $x \in X$ such that either $f(X) \subset \mathcal{T}_X(\{x\})$ or $f(X) = \mathcal{T}_X(\{x\})$, then there is nothing to prove. Assume $f(X) \not\subset \mathcal{T}_X(\{x\})$ and $f(X) \neq \mathcal{T}_X(\{x\})$ for any $x \in X$. By Theorem 5.3, there exists $x_0 \in X$ such that $f(x_0) \in \mathcal{T}_X(\{x_0\})$. Hence, $\mathcal{T}_X(\{f(x_0)\}) = \mathcal{T}_X(\{x_0\})$. Since f is monotone onto $f(X)$, $f(\mathcal{T}_X(\{x_0\})) \subset \mathcal{T}_{f(X)}(\{f(x_0)\})$ [2, Theorem 1(b), p. 5]. Since $\mathcal{T}_{f(X)}(\{f(x_0)\}) = \mathcal{T}_X(\{f(x_0)\})$ (Corollary 5.2), we have $f(\mathcal{T}_X(\{x_0\})) \subset \mathcal{T}_X(\{f(x_0)\})$. Thus, $f|_{\mathcal{T}_X(\{x_0\})} : \mathcal{T}_X(\{x_0\}) \to \mathcal{T}_X(\{x_0\})$ is well defined. Since $\mathcal{T}_X(\{x_0\})$ has the fixed point property, there exists $z \in \mathcal{T}_X(\{x_0\})$ such that $f(z) = z$. □

6. Hyperspaces

We study the hyperspace of subcontinua of a continuously irreducible continuum, we show that if X is such a continuum, then its hyperspace of subcontinua is locally a 2-cell at the top (Theorem 6.1). We also prove that if X is a continuously irreducible continuum with nondegenerate layers, then its hyperspace of subcontinua admits a Whitney map whose levels admit a continuous decomposition into subcontinua such that the quotient space is $[0,1]$ (Theorem 6.2). Finally we show that if X is continuously irreducible continuum, then $\mathcal{F}_n(X)$ is a Z-set in the hyperspaces 2^X and $\mathcal{C}_n(X)$ for any positive integer n (Theorem 6.4).

6.1. Theorem. *If X is a continuously irreducible continuum, then $\mathcal{C}(X)$ is locally a 2-cell at the top.*

Proof. Let X be a continuously irreducible continuum. Then $\mathcal{T}_X$ is continuous for X, Theorem 3.2. Let $q : X \to [0,1]$ be the quotient map given by the finest continuous monotone decomposition of X. Let $g : 2^{[0,1]} \to 2^X$ be given by $g(B) = q^{-1}(B)$. By [7, Theorem 2, p. 165], g is continuous. In fact, $2^q \circ g = 1_{2^{[0,1]}}$. In particular, $g : 2^{[0,1]} \to g(2^{[0,1]})$ is a homeomorphism. By the proof of [11, Theorem 3.3], we have that $\mathcal{T}_X(2^X) = g(2^{[0,1]})$. It is easy to see that $\mathcal{T}_X(\mathcal{C}(X)) = g(\mathcal{C}[0,1])$, $g([0,1]) = X$ and $g(\mathcal{F}_1([0,1])) = \{\mathcal{T}_X(\{x\}) \mid x \in X\}$. It is well known that $\mathcal{C}([0,1])$ is a 2-cell [16, (0.54)]. Therefore, $\mathcal{T}_X(\mathcal{C}(X))$ is a 2-cell.

Let us consider a Whitney map $\mu : \mathcal{C}(X) \to [0,1]$ [16, (0.50)]. Let $t_0 = \max\{\mu(\mathcal{T}_X(\{x\})) \mid x \in X\}$. Note that $0 < t_0 < 1$. Also observe that if $D \in \mu^{-1}([t_0,1])$, then $\mathcal{T}_X(D) = D$, Lemma 3.6. Thus, $\mu^{-1}([t_0,1]) \subset \mathcal{T}_X(\mathcal{C}(X))$. Hence, $\mathcal{T}_X(\mathcal{C}(X))$ is a neighborhood of X in $\mathcal{C}(X)$. Therefore, $\mathcal{C}(X)$ is locally a 2-cell at the top. □

6.2. Theorem. *Let X be a continuously irreducible continuum such that $\mathcal{T}_X(\{x\}) \notin \mathcal{F}_1(X)$ for any $x \in X$. Then there exists a Whitney map $\mu : \mathcal{C}(X) \to [0,1]$ such that $\mu^{-1}(t)$ admits a continuous decomposition into continua such that the quotient space is* $[0,1]$.

Proof. Let X be a continuously irreducible continuum such that $\mathcal{T}_X(\{x\}) \notin \mathcal{F}_1(X)$ for any $x \in X$, and let $t_0 \in (0,1)$. Then, by [19, Theorem 3.1], there exists a Whitney map $\mu : \mathcal{C}(X) \to [0,1]$ such that $\mu(\mathcal{T}_X(\{x\})) = t_0$ for each $x \in X$. We show that $\mathcal{T}_X(\mathcal{C}(X)) = \mu^{-1}([t_0,1])$. Let $A \in \mathcal{C}(X)$. Then for each $a \in A$, $\mathcal{T}_X(\{a\}) \subset \mathcal{T}_X(A)$. Since μ is a Whitney map, $\mu(\mathcal{T}_X(A)) \geqslant t_0$. Thus, $\mathcal{T}_X(A) \in \mu^{-1}([t_0,1])$, and $\mathcal{T}_X(\mathcal{C}(X)) \subset \mu^{-1}([t_0,1])$. Next, let $D \in \mu^{-1}([t_0,1])$. Then, by Lemma 3.6, $D \in \mathcal{T}_X(\mathcal{C}(X))$, and $\mu^{-1}([t_0,1]) \subset \mathcal{T}_X(\mathcal{C}(X))$. Thus, $\mathcal{T}_X(\mathcal{C}(X)) = \mu^{-1}([t_0,1])$.

Let $q : X \to [0,1]$ be the quotient map given by the finest continuous monotone decomposition of X. As we saw in the proof of Theorem 6.1, $\mathcal{T}_X(\mathcal{C}(X)) = g(\mathcal{C}[0,1])$, and $g|_{\mathcal{C}([0,1])}$ is a homeomorphism. Let $\omega : \mathcal{C}([0,1]) \to [0,1]$ be given by $\omega(B) = \frac{\mu(g(B)) - t_0}{1 - t_0}$. Then ω is a Whitney map for $\mathcal{C}([0,1])$. By [16, (14.6)], $\omega^{-1}(s)$ is an arc for each $s \in [0,1)$. Let $t \in [t_0,1]$.

We show that $\mu^{-1}(t) = g(\omega^{-1}(\frac{t-t_0}{1-t_0}))$. To this end, let $A \in \mu^{-1}(t)$. Then $\omega(g^{-1}(A)) = \frac{\mu(g(g^{-1}(A)))-t_0}{1-t_0} = \frac{\mu(A)-t_0}{1-t_0} = \frac{t-t_0}{1-t_0}$. Thus, $A \in g(\omega^{-1}(\frac{t-t_0}{1-t_0}))$, and $\mu^{-1}(t) \subset g(\omega^{-1}(\frac{t-t_0}{1-t_0}))$. Now, let $D \in g(\omega^{-1}(\frac{t-t_0}{1-t_0}))$. Then $\frac{t-t_0}{1-t_0} = \omega(g^{-1}(D)) = \frac{\mu(g(g^{-1}(D)))-t_0}{1-t_0} = \frac{\mu(D)-t_0}{1-t_0}$. This implies that $\mu(D) = t$. Hence, $D \in \mu^{-1}(t)$, and $\mu^{-1}(t) \subset g(\omega^{-1}(\frac{t-t_0}{1-t_0}))$. Hence, $\mu^{-1}(t) = g(\omega^{-1}(\frac{t-t_0}{1-t_0}))$. Therefore, for each $t \in [t_0, 1]$, $\mu^{-1}(t)$ is an arc.

Next, suppose $t \in (0, t_0)$. For each $x \in X$, let $\mathcal{G}_x(t) = \{A \in \mu^{-1}(t) \mid A \subset \mathcal{T}_X(\{x\})\} = \mu^{-1}(t) \cap \mathcal{C}(\mathcal{T}_X(\{x\}))$. Then $\mathcal{G}_x(t)$ is a continuum [4, (1.4)], and $\mathfrak{G}_t = \{\mathcal{G}_x(t) \mid x \in X\}$ is a monotone decomposition of $\mu^{-1}(t)$.

Define $h_t : \mu^{-1}(t) \to [0,1]$ by $h_t = r \circ \mathcal{C}(q)|_{\mu^{-1}(t)}$, where $r : \mathcal{F}_1([0,1]) \to [0,1]$ is the natural isometry defined by $r(\{s\}) = s$. Hence, h_t is continuous, monotone and surjective ($\mathcal{C}(q)$ is monotone by [17, Lemma 2.1]). In fact, for each $s \in [0,1]$, there exists $x \in X$ such that $h_t^{-1}(s) = \mathcal{G}_x(t)$. To see that h_t is open it suffices to show that $\mathcal{C}(q)|_{\mu^{-1}(t)}$ is open. To this end, note that $\mathcal{C}(q) \circ g = 1_{\mathcal{C}([0,1])}$. Since open maps have the composition factor property [13, (5.15)], $\mathcal{C}(q)$ is open. Let $\langle U_1, \dots, U_n\rangle$ be an open set in $\mathcal{C}(X)$ such that $\langle U_1, \dots, U_n\rangle \subset \mu^{-1}([0, t_0))$ and $\langle U_1, \dots, U_n\rangle \cap \mu^{-1}(t) \neq \emptyset$. We show that $\mathcal{C}(q)(\langle U_1, \dots, U_n\rangle) = \mathcal{C}(q)(\langle U_1, \dots, U_n\rangle \cap \mu^{-1}(t))$. It is clear that $\mathcal{C}(q)(\langle U_1, \dots, U_n\rangle \cap \mu^{-1}(t)) \subset \mathcal{C}(q)(\langle U_1, \dots, U_n\rangle)$. Let $A \in \langle U_1, \dots, U_n\rangle$. Since A is a subcontinuum of X and $\mu(A) < t_0$, there exists $x \in X$ such that $A \subset \mathcal{T}_X(\{x\})$. Let $B \in \mathcal{G}_x(t)$. Then $\mathcal{C}(q)(A) = \mathcal{C}(q)(B) = \mathcal{C}(q)(\mathcal{T}_X(\{x\}))$. Hence, $\mathcal{C}(q)(\langle U_1, \dots, U_n\rangle) \subset \mathcal{C}(q)(\langle U_1, \dots, U_n\rangle \cap \mu^{-1}(t))$, and $\mathcal{C}(q)(\langle U_1, \dots, U_n\rangle) = \mathcal{C}(q)(\langle U_1, \dots, U_n\rangle \cap \mu^{-1}(t))$. Therefore, $\mathcal{C}(q)|_{\mu^{-1}(t)}$ is open, and h_t is open too. Therefore, $\mathfrak{G}_t = \{\mathcal{G}_x(t) \mid x \in X\}$ is a continuous decomposition of $\mu^{-1}(t)$ such that $\mu^{-1}(t)/\mathfrak{G}_t$ is $[0,1]$. □

The following result is easy to establish:

6.3. Lemma. *Let X be a continuum, and let $\mu : 2^X \to [0,1]$ be a Whitney map. If $\varepsilon > 0$, then there exists $t \in (0,1)$ such that $\operatorname{diam}(A) < \varepsilon$ for every $A \in \mu^{-1}(t)$.*

As a consequence of Theorem 6.2 and Lemma 6.3, we have the following:

6.4. Theorem. *Let X be a continuously irreducible continuum such that $\mathcal{T}_X(\{x\}) \notin \mathcal{F}_1(X)$ for any $x \in X$. Then $\mathcal{F}_n(X)$ is a Z-set in 2^X and in $\mathcal{C}_n(X)$ for any positive integer n.*

Proof. Let X be a continuously irreducible continuum such that $\mathcal{T}_X(\{x\}) \notin \mathcal{F}_1(X)$ for any $x \in X$. Let $t_0 \in (0,1)$. Then, by [19, Theorem 3.1], there exists a Whitney map $\mu : \mathcal{C}(X) \to [0,1]$ such that $\mu(\mathcal{T}_X(\{x\})) = t_0$ for each $x \in X$. For each $t \in (0, t_0)$, define $g_t : \mathcal{F}_1(X) \to \mathcal{C}(\mathcal{C}(X))$ by $g_t(\{x\}) = \mathcal{G}_x(t)$. Since $\mathfrak{G}_t = \{\mathcal{G}_x(t) \mid x \in X\}$ is a continuous decomposition of $\mu^{-1}(t)$ (Theorem 6.2), g_t is continuous. Let $\sigma : 2^{2^X} \to 2^X$ be given by $\sigma(\mathcal{A}) = \bigcup \mathcal{A}$. Then σ is continuous [16, (1.48)].

Let $\varepsilon > 0$. By Lemma 6.3, there exists $t \in (0, t_0)$ such that $\operatorname{diam}(A) < \frac{\varepsilon}{3}$ for every $A \in \mu^{-1}(t)$. Since for each $x \in X$, $\lim_{t\to 0} \operatorname{diam}(\mathcal{G}_x(t)) = 0$ and $\mu^{-1}(t)$ is compact for every $t \in [0,1]$, we also assume that $\operatorname{diam}(\mathcal{G}_x(t)) < \frac{\varepsilon}{3}$ for every $\mathcal{G}_x(t) \in \mathfrak{G}_t$. Define $f_\varepsilon : \mathcal{F}_1(X) \to \mathcal{C}(X) \setminus \mathcal{F}_1(X)$ by

$$f_\varepsilon\big(\{x\}\big) = \sigma \circ g_t\big(\{x\}\big) = \bigcup \mathcal{G}_x(t).$$

Note that f_ε is well defined, i.e., $f_\varepsilon(\{x\}) \in \mathcal{C}(X)$ [16, (1.49)]. Let $A_x \in \mathcal{G}_x(t)$ such that $x \in A_x$. Let $y \in f_\varepsilon(\{x\})$. Then there exists $A_y \in \mathcal{G}_x(t)$ such that $y \in A_y$. Let $z_x \in A_x$ and $z_y \in A_y$ such that $d(A_x, A_y) = d(z_x, z_y)$. Observe that $d(z_x, z_y) \leqslant \mathcal{H}(A_x, A_y)$ [16, (0.4)]. Thus, $d(x, y) \leqslant d(x, z_x) + d(z_x, z_y) + d(z_y, y) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$. Hence, $\mathcal{H}(\{x\}, f_\varepsilon(\{x\})) < \varepsilon$. Therefore, $\mathcal{F}_1(X)$ is a Z-set in $\mathcal{C}(X)$ [12, 2.1].

Since $\mathcal{F}_1(X)$ is a Z-set in $\mathcal{C}(X)$, $\mathcal{F}_n(X)$ is a Z-set in 2^X and in $\mathcal{C}_n(X)$ for any positive integer n [12, 2.2]. □

Acknowledgement

The author thanks the referee for his/her careful reading.

References

[1] D.P. Bellamy, Continua for which the set function $\mathcal{T}$ is continuous, Trans. Amer. Math. Soc. 151 (1970) 581–587.
[2] D.P. Bellamy, Some topics in modern continua theory, in: R.H. Bing, W.T. Eaton, M.P. Starbird (Eds.), Continua Decompositions Manifolds, University of Texas Press, 1983, pp. 1–26.
[3] D.P. Bellamy, H.S. Davis, Continuum neighborhoods and filterbases, Proc. Amer. Math. Soc. 27 (1971) 371–374.
[4] C. Eberhart, S.B. Nadler Jr., Irreducible Whitney levels, Houston J. Math. 6 (1980) 355–363.
[5] G.R. Gordh Jr., Monotone decompositions of irreducible Hausdorff continua, PhD Dissertation, University of California, Riverside, 1971.
[6] G.R. Gordh Jr., Monotone decompositions of irreducible Hausdorff continua, Pacific J. Math. 36 (1971) 647–658.
[7] K. Kuratowski, Topology, vol. I, Academic Press and PWN, New York, N. Y., London, Warszawa, 1966.
[8] S. Macías, Topics on Continua, Pure Appl. Math. Ser., vol. 275, Chapman & Hall/CRC, Taylor & Francis Group, Boca Raton, London, New York, Singapore, 2005.
[9] S. Macías, A class of one-dimensional, nonlocally connected continua for which the set function $\mathcal{T}$ is continuous, Houston J. Math. 32 (2006) 161–165.
[10] S. Macías, Homogeneous continua for which the set function $\mathcal{T}$ is continuous, Topology Appl. 153 (2006) 3397–3401.

[11] S. Macías, A decomposition theorem for a class of continua for which the set function $\mathcal{T}$ is continuous, Colloq. Math. 109 (2007) 163–170.
[12] S. Macías, S.B. Nadler Jr., Z-sets in hyperspaces, Questions Answers Gen. Topology 19 (2001) 227–241.
[13] T. Maćkowiak, Continuous mappings on continua, Dissertationes Math. (Rozprawy Mat.) 158 (1979) 1–95.
[14] T. Maćkowiak, Singular arc-like continua, Dissertationes Math. (Rozprawy Mat.) 257 (1986) 1–35.
[15] L. Mohler, L.G. Oversteegen, On the structure of tranches in continuously irreducible continua, Colloq. Math. 54 (1987) 23–28.
[16] S.B. Nadler Jr., Hyperspaces of Sets: A Text with Research Questions, Monogr. Textbooks Pure Appl. Math., vol. 49, Marcel Dekker, New York, Basel, 1978.
[17] S.B. Nadler Jr., Induced universal maps and some hyperspaces with the fixed point property, Proc. Amer. Math. Soc. 100 (1987) 749–754.
[18] E.S. Thomas Jr., Monotone decompositions of irreducible continua, Dissertationes Math. (Rozprawy Mat.) 50 (1966) 1–74.
[19] L.E. Ward Jr., Extending Whitney maps, Pacific J. Math. 93 (1981) 465–469.