Semi-orthogonal Parseval frame wavelets and generalized multiresolution analyses

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Abstract
We study Parseval frame wavelets in $L^2(\mathbb{R}^d)$ with matrix dilations of the form $(Df)(x) = \sqrt{2}f(Ax)$, where $A$ is an arbitrary expanding $n \times n$ matrix with integer coefficients, such that $|\det A| = 2$. In our study we use generalized multiresolution analyses (GMRA) $(V_j)$ in $L^2(\mathbb{R}^d)$ with dilations $D$. We describe, in terms of the underlying multiresolution structure, all GMRA Parseval frame wavelets and, a posteriori, all semi-orthogonal Parseval frame wavelets in $L^2(\mathbb{R}^d)$. As an application, we include an explicit construction of an orthonormal wavelet on the real line whose dimension function is essentially unbounded.

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1. Introduction

We work in $\mathbb{R}^d$ with two groups of unitary operators: $T_k, D^j \in B(L^2(\mathbb{R}^d))$, $k \in \mathbb{Z}^d$, $j \in \mathbb{Z}$, where $(T_k f)(x) = f(x - k)$ and $(D^j f)(x) = \sqrt{2}f(Ax)$. Here $A \in M_d(\mathbb{Z})$ denotes a fixed, but arbitrary, expanding matrix such that $|\det A| = 2$. (By expanding, we mean that all eigenvalues of $A$ have absolute value greater than 1.) Notice that the choice $d = 1$, $A = 2$, gives us the classical dyadic dilation on the real line, as a special case.

Definition 1.1. We say that a function $\psi \in L^2(\mathbb{R}^d)$ is an orthonormal wavelet if the system

$$\{D^j T_k \psi: j \in \mathbb{Z}, \ k \in \mathbb{Z}^d\} = \{2^j \psi(A^j x - k): j \in \mathbb{Z}, \ k \in \mathbb{Z}^d\}$$

(1.1)

is an orthonormal basis for $L^2(\mathbb{R}^d)$.

A more general concept is that of a Parseval frame wavelet.

Recall that a sequence $(x_n)$ in a separable Hilbert space $(H, \langle \cdot, \cdot \rangle)$ is a Parseval frame for $H$ if each $x \in H$ satisfies $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, x_n \rangle|^2$. A well-known fact is that this is equivalent to the reconstruction property $x =...$
\( \sum_{n=1}^{\infty} \langle x, x_n \rangle x_n, \ \forall x \in H. \) Notice that the elements of a Parseval frame need not be linearly independent; in particular, they are not necessarily orthogonal to each other.

**Definition 1.2.** We say that a function \( \psi \in L^2(\mathbb{R}^d) \) is a Parseval frame wavelet if the system (1.1) is a Parseval frame for \( L^2(\mathbb{R}^d). \)

A Parseval frame wavelet \( \psi \) is said to be semi-orthogonal if \( D^{h_1}T_{k_1}\psi \perp D^{h_2}T_{k_2}\psi \) for all \( j_1 \neq j_2 \) and for all \( k_1, k_2 \) in \( \mathbb{Z}^d. \)

Obviously, the class of all semi-orthogonal Parseval frame wavelets contains all orthonormal wavelets. The converse is not true. Also, there are Parseval frame wavelets that are not semi-orthogonal. For a detailed discussion on semi-orthogonal Parseval frame wavelets we refer the reader to [2,10].

Both orthonormal and Parseval frame wavelets are studied extensively over the last two decades even for more general dilations on \( L^2(\mathbb{R}^d). \) For technical reasons, in this paper we restrict ourselves only to the case of dilations \( D \) induced by integer valued matrices \( A \) such that \( |\det A|=2. \)

In particular, here we focus our attention to a method of construction of Parseval frame wavelets: generalized multiresolution analysis.

**Definition 1.3.** [1] We say that a sequence \( (V_j), \ j \in \mathbb{Z}, \) of closed subspaces of \( L^2(\mathbb{R}^d) \) is a generalized multiresolution analysis (briefly: GMRA) if the following conditions are satisfied:

(i) \( V_j \subseteq V_{j+1}, \ \forall j \in \mathbb{Z}; \)
(ii) \( DV_j = V_{j+1}, \ \forall j \in \mathbb{Z}; \)
(iii) \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \ \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^d); \)
(iv) The core space \( V_0 \) is shift invariant; i.e. \( f \in V_0 \Rightarrow T_k f \in V_0, \ \forall k \in \mathbb{Z}^d. \)

This definition extends naturally the concept of a classical multiresolution analysis. Recall that a sequence \( (V_j) \) of closed subspaces of \( L^2(\mathbb{R}^d) \) is called a multiresolution analysis (MRA) if \( (V_j) \) satisfies (i)–(iv) from Definition 1.3 and, additionally, there exists a function \( \varphi \in V_0 \) such that the system \( \{T_k \varphi: k \in \mathbb{Z}^d\} \) is an orthonormal basis for \( V_0. \) If this is the case, the function \( \varphi \) is said to be a scaling function for \( (V_j). \)

In what follows, the system of all integer translates \( \{T_k f: k \in \mathbb{Z}^d\} \) of a function \( f \in L^2(\mathbb{R}^d) \) will be denoted by \( T(f). \)

There is a standard procedure for constructing wavelets from a given GMRA \( (V_j). \)

First, one defines \( W_j = V_{j+1} \ominus V_j \) for all \( j \in \mathbb{Z}. \) As an easy consequence of conditions (i), (ii), (iii) from Definition 1.3, one obtains \( L^2(\mathbb{R}^d) = \bigoplus_{j \in \mathbb{Z}} W_j \) and \( W_{j+1} = DW_j, \ \forall j \in \mathbb{Z}. \) Suppose now that there exists a function \( \psi \in W_0 \) such that the system \( T(\psi) \) is a Parseval frame for \( W_0. \) Then \( \{D^{j} T_k \psi: k \in \mathbb{Z}^d\} \) is a Parseval frame for \( W_j, \ \forall j \in \mathbb{Z}, \) and, consequently, \( \{D^{j} T_k \psi: j \in \mathbb{Z}, k \in \mathbb{Z}^d\} \) is a Parseval frame for \( L^2(\mathbb{R}^d). \)

**Definition 1.4.** A Parseval frame wavelet \( \psi \) is said to be associated with a GMRA \( (V_j) \) if \( T(\psi) \) is a Parseval frame for \( W_0 = V_1 \ominus V_0. \) We say that a Parseval frame wavelet is a GMRA wavelet if it is associated with some GMRA.

Notice that each GMRA wavelet is necessarily semi-orthogonal; this follows immediately from the construction explained above.

Secondly, one observes: the question of finding all Parseval frame wavelets associated with any GMRA \( (V_j) \) reduces to the identification of all those functions \( \psi \in W_0 \) such that \( T(\psi) \) is a Parseval frame for \( W_0. \) However, it is not clear how to describe all such functions. Note that an appropriate description of such functions could also lead to a concrete method of construction of Parseval frame wavelets associated with any given GMRA. Finally, a general description of GMRA wavelets may be useful in developing methods of construction of more general Parseval frame wavelets. Thus, we may pose the following problem:

**Problem 1.5.** Let \( (V_j) \) be a GMRA.

- Describe all Parseval frame wavelets associated with \( (V_j). \)
- Find a method of construction of all Parseval frame wavelets associated with \( (V_j). \)
The aim of the present paper is to give a solution of Problem 1.5. A description of Parseval frame wavelets associated with any GMRA \((V_j)\), as well as the resulting construction method, will be given in terms of the core space \(V_0\).

Before describing the main result of the paper, let us briefly explain the scope of the solution of Problem 1.5. As we already observed, the class of all GMRA wavelets is contained in the class of all semi-orthogonal Parseval frame wavelets. A natural question is to identify those semi-orthogonal Parseval frame wavelets that arise from the GMRA construction.

Recall that the dimension function \(D_\psi\) of an orthonormal wavelet \(\psi\) on the real line is defined by

\[
D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}} |\hat{\psi}(2^j(\xi + k))|^2, \tag{1.2}
\]

where \(\hat{\psi}\) denotes the Fourier transform of \(\psi\). Obviously, \(D_\psi\) is a well-defined integrable function. Also, it is known that \(D_\psi(\xi)\) is finite a.e. and for every \(\xi\) such that \(D_\psi(\xi)\) is finite, we have that \(D_\psi(\xi)\) is the dimension of certain finite-dimensional vector space; hence, \(D_\psi(\xi) \in \{0\} \cup \mathbb{N}\).

The dimension function of a Parseval frame wavelet \(\psi\) on the real line is defined in the same way, i.e. by (1.2). However, if \(\psi\) is an arbitrary Parseval frame wavelet, \(D_\Psi\) may take non-integer values (see [10]).

Similarly, if \(\psi \in L^2(\mathbb{R}^d)\) is a Parseval frame wavelet (with matrix dilation \(D = D_A\)), we define its dimension function \(D_\psi\) by

\[
D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^d} |\hat{\psi}(B^j(\xi + k))|^2, \tag{1.3}
\]

where \(B\) denotes the transpose of \(A\); \(B = A^t\).

The following theorem characterizes GMRA Parseval frame wavelets.

**Theorem 1.6.** [11, Theorem 4] The classes of all GMRA Parseval frame wavelets and all semi-orthogonal Parseval frame wavelets coincide. In particular, each orthonormal wavelet is a GMRA wavelet.

**Proof.** The last statement, restricted to the case of orthonormal wavelets on the real line, is precisely Theorem 4 from [11].

To prove the general case, we must show that each semi-orthogonal Parseval frame wavelet is a GMRA wavelet.

The proof for semi-orthogonal Parseval frame wavelets on the real line follows by a straightforward extension of the original Papadakis’ argument. The key step in the original proof is the observation that the dimension function of each orthonormal wavelet on the real line is integer valued a.e. Since by Theorem 3.1 in [10] a Parseval frame wavelet \(\psi \in L^2(\mathbb{R})\) is semi-orthogonal if and only if \(D_\psi(\xi) \in \{0\} \cup \mathbb{N}\), one can apply the rest of the original argument without changes.

Finally, by the same reasoning as in [11], one obtains the proof in \(L^2(\mathbb{R}^d)\) for all Parseval frame wavelets with integer valued dimension functions. To complete the proof, one only needs to apply Theorem 4.15 from [2]: a Parseval frame wavelet \(\psi \in L^2(\mathbb{R}^d)\) is semi-orthogonal if and only if \(D_\psi\) is integer valued a.e. \(\square\)

By Theorem 1.6, the solution of Problem 1.5 will provide us with a description of all semi-orthogonal Parseval frame wavelets in terms of the underlying GMRA structure.

Let us now explain the main result of the paper—the solution of Problem 1.5. First observe that an explicit solution is well known in the classical MRA case [8, Proposition 2.2.13]: it is given in terms of the scaling function and the high-pass filter. To obtain a similar description for a GMRA \((V_j)\), we first need some additional information about the core space \(V_0\). Besides, it turns out (see Theorem 1.8) that an arbitrary GMRA \((V_j)\) may not admit associated wavelets. Observe that this is a new phenomenon in comparison with the MRA case: it is a well-known fact that each MRA possesses associated wavelets (and all of them are necessarily orthonormal; cf. [8, Proposition 2.2.13], see also [2]).

To state all relevant results, we will make use of the canonical decomposition of an arbitrary shift invariant space.

For a function \(f \in L^2(\mathbb{R}^d)\), we denote by \(\langle f \rangle\) the minimal closed shift invariant subspace that contains \(f\). Notice that \(\langle f \rangle = \text{span} T(f)\). If the system \(T(f)\) is a Parseval frame for \(\langle f \rangle\), we say that \(\langle f \rangle\) is a principal shift invariant space.
For \( f, g \in L^2(\mathbb{R}^d) \), let \([f, g]\) be the function defined a.e. by \([f, g] (\xi) = \sum_{k \in \mathbb{Z}^d} f(\xi + k) g(\xi - k)\). Notice that \([f, g] \in L^1(\mathbb{T}^d)\), where \(L^1(\mathbb{T}^d)\) denotes the space of all integrable \(\mathbb{Z}^d\)-periodic functions on the \(d\)-dimensional torus \(\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d\). In particular, we write \(\sigma_f := [f, f]\).

Recall that, for \(\varphi \in L^2(\mathbb{R}^d)\), the system \(T(\varphi)\) is a Parseval frame for the Hilbert space \(\langle \varphi \rangle\) if and only if there exists a measurable \(\mathbb{Z}^d\)-periodic set \(\Omega \subseteq \mathbb{R}^d\) such that \(\sigma_\varphi = \chi_\Omega\) a.e., where \(\chi_\Omega\) denotes the characteristic function of the set \(\Omega\). In particular, if \(T(\varphi)\) is an orthonormal basis for \(\langle \varphi \rangle\), we have \(\Omega = \mathbb{R}^d\), up to a set of measure zero.

Now we recall the result from [5, Theorem 3.3] that provides us with the canonical decomposition of shift invariant spaces.

Suppose that \(V \subseteq L^2(\mathbb{R}^d)\) is a shift invariant space. Then there exist \(n \in \mathbb{N} \cup \{\infty\}\) and a (possibly finite) sequence of functions \((\varphi_i), i = 1, 2, \ldots\), with the following properties:

(i) \(V = \bigoplus_{i=1}^{n} \langle \varphi_i \rangle\);
(ii) \(T(\varphi_i)\) is a Parseval frame for \(\langle \varphi_i \rangle\) for all \(i = 1, 2, \ldots\);
(iii) \(\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_i \supseteq \cdots\), where \(\sigma_{\varphi_i} = \chi_{\Omega_i}\) a.e., for all \(i = 1, 2, \ldots\).

From now on, we will always assume that the core space \(V_0\) of any GMRA \((V_j)\) is decomposed as above and \((\varphi_i)\) will be called the sequence of scaling functions of \((V_j)\). In some situations it will be convenient to specify the number of scaling functions of \((V_j)\). When this is the case, we shall say that \((V_j)\) is an \(n\)-GMRA. When \((V_j)\) is an \(n\)-GMRA with \(n \in \mathbb{N}\), we can write \(\varphi_i = 0\) and \(\Omega_i = \emptyset\), for all \(i > n\). This will enable us to state some results without specifying the effective number of generators of \(V_0\).

Now we turn to the description of those GMRA’s that admit associated Parseval frame wavelets.

**Definition 1.7.** A GMRA \((V_j)\) is said to be admissible if there exist Parseval frame wavelets associated with \((V_j)\).

A description of admissible GMRA’s is known; see [1,5,7]. (We also note that a related results for 1-GMRA’s can be found in [2,4,9].)

To formulate a characterization of admissible GMRA’s, it is convenient to introduce an auxiliary function that can be defined for any GMRA.

Let

\[
\hat{h}(\xi) = \sum_{i=1}^{\infty} \chi_{\Omega_i}(\xi) + \sum_{i=1}^{\infty} \chi_{\Omega_i}(\xi + \beta) - \sum_{i=1}^{\infty} \chi_{\Omega_i}(B\xi),
\]

where \(\beta = B^{-1}\alpha\), and \(\alpha\) is an arbitrary element of \(\mathbb{Z}^d \setminus B\mathbb{Z}^d\). Throughout the paper we will fix such \(\alpha\) and \(\beta\). Observe that, since \(|\det B| = 2\), the order of the group \(\mathbb{Z}^d / B\mathbb{Z}^d\) is equal to 2. We have \(\mathbb{Z}^d = B\mathbb{Z}^d \cup (B\mathbb{Z}^d + \alpha)\) (a disjoint union) and, consequently, \(B^{-1}\mathbb{Z}^d = \mathbb{Z}^d \cup (\mathbb{Z}^d + \beta)\).

In the dyadic case on the real line one can take \(\alpha = 1\) and \(\beta = 1/2\). In the quincunx case, that is, for \(A = [\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array}]\), the simplest choice is \(\alpha = (1, 0), \beta = (\frac{1}{2}, \frac{1}{2})\).

Notice that the definition of the function \(\hat{h}\) is independent on the particular choice of \(\alpha\) and \(\beta\). The function \(\hat{h}\) is obviously integer-valued a.e. (when it is finite) and \(B^{-1}\mathbb{Z}^d\)-periodic (i.e. \(\beta\)-periodic) because \(2\beta \in \mathbb{Z}^d\) and all \(\Omega_i\)’s are \(\mathbb{Z}^d\)-periodic sets.

We can now summarize Theorem 1.3 from [1], Theorem 2.9 from [7] and Example from [5, pp. 297, 298]. The following theorem is known to the experts in the field and can also be deduced from other literature. In the first assertion we use the symbol \(|S|\) to denote the Lebesgue measure of a set \(S\).

**Theorem 1.8.** A GMRA \((V_j)\) is admissible if and only if the following two conditions are satisfied:

(i) \(\bigcap_{i=1}^{\infty} \Omega_i = \emptyset\).

(ii) There exists a measurable \(\mathbb{Z}^d\)-periodic set \(\Omega \subseteq \mathbb{R}^d\) such that

\[
h(\xi) = \chi_{B^{-1}\Omega}(\xi) \quad \text{a.e.}
\]
If \((V_j)\) is an admissible GMRA then the dimension function of each associated Parseval frame wavelet \(\psi\) satisfies

\[
D_\psi(\xi) = \sum_{j=1}^{\infty} \chi_{\Omega_i}(\xi) \quad \text{a.e.}
\]  

(1.7)

Let us note a few comments on the above theorem.

First, observe that (1.5) is fulfilled in a trivial way for all \(n\)-GMRA’s such that \(n < \infty\). Hence, for such GMRA’s, a necessary and sufficient condition for the existence of associated wavelets reduces to (1.6).

Second, in the case \(n = \infty\), the above condition (1.5) ensures that the function \(h\) is finite a.e. Indeed, (1.5) implies that for a.e. \(\xi\) there exists an integer \(n(\xi) \geq 0\) such that \(\xi \notin \Omega_i, \forall i > n(\xi)\).

Notice that (1.6) simply means that \(h\) is equal a.e. to the characteristic function of some measurable set, say \(\Sigma\). Since \(h\) is \(\beta\)-periodic, \(\Sigma\) is a \(\beta\)-periodic set; thus \(\Omega := B\Sigma\) is \(\mathbb{Z}^d\)-periodic.

In the course of our description of GMRA wavelets we will give another, constructive proof that (1.5) and (1.6) are sufficient for the existence of Parseval frame wavelets associated with a GMRA. In particular, we refer the reader to Remark 2.6 in Section 2 for a comment on the necessity of the condition (1.6).

It is known that condition (1.6) may not be satisfied in the case of one generator (see [2,4,9]). As an example of a non-admissible \(n\)-GMRA for \(n \in \mathbb{N}, n \geq 2\), one can take \((V_j)\) such that \(V_0 = \langle \varphi_1 \rangle \oplus \cdots \oplus \langle \varphi_n \rangle\) and \(\sigma_{\varphi_i}(\xi) = 1\) a.e. \(\forall i = 1, \ldots, n\). Then, obviously, \(h(\xi) = n\) a.e. It will be seen that we always have \(h(\xi) \geq 0\) a.e. It can be proved that those GMRA’s for which \(h(\xi) \geq 2\) is satisfied on a set of positive measure, always admit associated multi-wavelets. The minimal number of generators is then equal to the highest value that \(h\) takes on a set of positive measure. Since in the present paper we are interested only in singly generated wavelets, a study of GMRA multi-wavelets will appear elsewhere.

Let \(\psi\) be an arbitrary semi-orthogonal Parseval frame wavelet. Then, by Theorem 1.6, \(\psi\) is a GMRA wavelet. We know that \(D_\psi(\xi)\) is finite and integer valued a.e. Let us denote by \(n\) the maximum value that the function \(D_\psi\) attains on a set of positive measure. Notice that we have \(n \in \mathbb{N} \cup \{\infty\}\). Now, the last assertion of Theorem 1.6 shows that \(\psi\) must be associated with some \(n\)-GMRA \((V_j)\). In particular, if \(D_\psi\) is essentially unbounded, then \(\psi\) is an \(\infty\)-GMRA wavelet.

Let us now turn to the solution of Problem 1.5.

**Theorem 1.9.** Let \((V_j)\) be an admissible \(n\)-GMRA, \(n \in \mathbb{N} \cup \{\infty\}\). Denote by \(\Omega\) the set with the property \(h(\xi) = \chi_{B^{-1}\Omega}(\xi)\) a.e. For each \(\xi\) there exists a vector \((v_1(\xi), v_2(\xi), \ldots)\) in \(\mathbb{C}^n\) (where \(\mathbb{C}^n\) stands for \(L^2\) in the case \(n = \infty\)) with the following properties:

(i) the map \(\xi \mapsto v(\xi) = ((v_1(\xi), v_2(\xi), \ldots), (v_1(\xi + \beta), v_2(\xi + \beta), \ldots))\) is a measurable \(\mathbb{Z}^d\)-periodic function;

(ii) \(u_1(\xi) = 0, \forall \xi \notin \Omega_i, \forall i;\)

(iii) \(\|v(\xi)\| = \chi_{B^{-1}\Omega}(\xi)\) a.e.;

(iv) a function \(\psi \in L^2(\mathbb{R}^d)\) belongs to \(W_0 = V_1 \ominus V_0\) if and only if there exists a measurable \(\mathbb{Z}^d\)-periodic function \(s\) such that \(\hat{\psi}(B\xi) = s(B\xi)\sum_{i=1}^{n} v_i(\xi)\hat{\varphi}_i(\xi),\)

(1.8)

where, in the case \(n = \infty\), the series on the right-hand side converges in norm of \(L^2(\mathbb{R}^d)\).

In particular, \(\psi\) is a Parseval frame wavelet associated with \((V_j)\) if and only if \(\psi\) satisfies (1.8) with a unimodular function \(s\).

Each Parseval frame wavelet \(\psi\) associated with \((V_j)\) satisfies

\[
\sigma_\psi(\xi) = \chi_{\Omega}(\xi) \quad \text{a.e.}
\]  

(1.9)

Let us point out that, by assertion (iv), we have the same vector valued function \(\xi \mapsto (v_1(\xi), v_2(\xi), \ldots)\) that serves as a universal filter, up to the unimodular factor \(s(B\xi)\), for all \(\psi \in W_0\). Thus, the function \(\hat{\xi} \mapsto \psi(\xi) =\)
((v_1(\xi), v_2(\xi), \ldots), (v_1(\xi + \beta), v_2(\xi + \beta), \ldots)) should be considered as an object that is attached to (V_j) and may be called the characteristic function of (V_j).

Theorems 1.8 and 1.9 provide us with a classification of all GMRA’s. Each GMRA belongs to one of the following three classes:

(N) Non-admissible GMRA’s.
(O) Admissible GMRA’s such that \( h(\xi) = 1 \) a.e. There exist associated wavelets and all of them are orthonormal.
(P) Admissible GMRA’s such that \(|\{\xi: h(\xi) = 0\}| > 0\). There exist associated Parseval frame wavelets, but none of them is orthonormal.

Theorem 1.9 will be proved in Sections 2 and 3. In the course of the proof the characteristic function \( v(\xi) \) will be explicitly determined. It is important to note that the proof of Theorem 1.9 provides us with an algorithm for obtaining \( v(\xi) \). This will enable us to obtain explicit formulae for all wavelets associated with any given admissible GMRA.

The paper is organized in the following way: in Sections 2 and 3 we prove Theorem 1.9. First, in Section 2, we give a proof for admissible GMRA’s \((V_j)\) with finitely many generators of the core space \( V_0 \). In Section 3 the argument is then extended to the infinite case.

Section 4 is devoted to examples. This is enabled by Proposition 4.1 that provides a construction of GMRA’s from certain sequences of scaling functions. Main result of the section is the explicit construction of an orthonormal wavelet \( \psi \in L^2(\mathbb{R}) \) whose dimension function \( D_\psi \) is essentially unbounded.

Starting from a suitable dimension function, by applying Theorem 1.9, we obtain an MSF wavelet with this property. It is proved in [7, Example 5.6] that such wavelets exist. Also, Example 5.9 in [7] provides a family of multi-wavelets with unbounded dimension functions. However, the wavelet constructed in Section 4 seems to be the first concrete example of a wavelet whose dimension function is essentially unbounded.

We end this introductory section by establishing the rest of our notation and by recalling some useful technical facts.

The standard unit cube \( [-\frac{1}{2}, \frac{1}{2}]^d \) in \( \mathbb{R}^d \) is denoted by \( C \). We denote by \( \{C_0, C_1\} \) the partition of \( C \) from Lemma 2.3 in [2] with the property \( C_0 \sim B^{-1}C, C_0 + \beta \sim C_1 \), where \( X \sim Y \) means that the sets \( X \) and \( Y \) are \( \mathbb{Z}^d \)-translations equivalent. In the dyadic case on the real line we have \( C_0 = [-\frac{1}{4}, \frac{1}{4}], C_1 = [-\frac{1}{2}, -\frac{1}{4}) \cup [\frac{1}{4}, \frac{1}{2}] \). We refer the reader to Section 2 in [2] for this and other facts concerning expanding matrices \( A \) with integer coefficients such that \(|\det A| = 2 \).

\( L^p(\mathbb{T}^d) \) denotes the space of all \( \mathbb{Z}^d \)-periodic functions \( f \) (meaning that \( f \) is 1-periodic in each variable) such that \( \int_{\mathbb{T}^d} |f(\xi)|^p \, d\xi < \infty \). A function \( f \) is said to be unimodular if \(|f(\xi)| = 1 \) a.e.

We use the Fourier transform in the form \( \hat{f}(\xi) = \int_{\mathbb{T}^d} f(x) e^{-2\pi i (x, \xi)} \, dx \) where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product in \( \mathbb{R}^d \) (and in \( \mathbb{C}^d \)). It is useful to note the following formulae for any \( f \in L^2(\mathbb{R}^d) \):

\[
\widehat{T_k f}(\xi) = e^{-2\pi i (k, \xi)} \hat{f}(\xi), \quad \forall k \in \mathbb{Z}^d, \tag{1.10}
\]

\[
\widehat{D^j f}(\xi) = 2^{-\frac{j}{2}} \hat{f}(B^{-j} \xi), \quad \forall j \in \mathbb{Z}, \tag{1.11}
\]

and, in particular,

\[
\widehat{Df}(\xi) = \frac{1}{\sqrt{2}} \hat{f}(B^{-1} \xi). \tag{1.12}
\]

2. The finite case

In this section we prove Theorem 1.9 for all admissible \( n \)-GMRA’s such that \( n \in \mathbb{N} \).

We start with two useful facts concerned with principal shift invariant spaces. Since both statements in the following remark are well known, we omit their proofs.

**Remark 2.1.** (a) Let \( T(\varphi) \) be a Parseval frame for \( \langle \varphi \rangle, \varphi \in L^2(\mathbb{R}^d) \). Then

\[
\langle \varphi \rangle = \{ f \in L^2(\mathbb{R}^d) : \hat{f}(\xi) = m(\xi) \hat{\varphi}(\xi), \ m \in L^2(\mathbb{Z}^d) \}. \tag{2.1}
\]
Moreover, for each $f \in \langle \varphi \rangle$ there exists a unique function $m_f \in L^2(\mathbb{T}^d)$ such that $\hat{f}(\xi) = m_f(\xi)\varphi(\xi)$ and $m_f(\xi) = 0$, $\forall \xi \notin \Omega$, where $\sigma_f(\xi) = \chi_\Omega(\xi)$ a.e. It is known that $m_f = [\hat{f}, \varphi]$.

(b) Let $f, g \in L^2(\mathbb{R}^d)$. Then $\langle f \rangle \perp \langle g \rangle$ if and only if $[\hat{f}, \hat{g}](\xi) = 0$ a.e.

**Lemma 2.2.** Let $(V_j)$ be an $n$-GMRA with $n \in \mathbb{N}$. Then

$$V_0 = \left\{ g \in L^2(\mathbb{R}^d) : \hat{g} = \sum_{i=1}^n [\hat{g}, \varphi_i] \varphi_i \right\}.$$  

(2.2)

**Proof.** Suppose first that $h \in \langle \varphi_1 \rangle$ for some $i \leq n$. Then, clearly, $\langle h \rangle \subseteq \langle \varphi_i \rangle$. This implies $\langle h \rangle \perp \langle \varphi_j \rangle$, $\forall j \neq i$; hence, by Remark 2.1(b), $[\hat{h}, \varphi_j] = 0$, $\forall j \neq i$.

Let us now take an arbitrary $g \in V_0 = \bigoplus_{i=1}^n \varphi_i).$ Then we have $g = \sum_{i=1}^n g_i$, $g_i \in \langle \varphi_i \rangle$, which implies $\hat{g} = \sum_{i=1}^n \hat{g}_i$. Using Remark 2.1(a), we can write $\hat{g}_i = [\hat{g}_i, \varphi_i] \varphi_i$, $\forall i = 1, \ldots, n$. Since $[\cdot, \cdot]$ is linear in the first, and antilinear in the second argument, by the observation from the beginning of the proof, we can write $[\hat{g}_i, \varphi_i] = \sum_{j=1}^n [\hat{g}_j, \varphi_j] = [\hat{g}_i, \varphi_i]$. This gives one inclusion in (2.2). The opposite inclusion is obvious. $\square$

**Lemma 2.3.** Let $(V_j)$ be an $n$-GMRA with $n \in \mathbb{N}$. Then

$$V_j = \left\{ f \in L^2(\mathbb{R}^d) : \hat{f}(B^j \xi) = \sum_{i=1}^n t_i(\xi)\varphi_i(\xi), \ t_i \in L^2(\mathbb{T}^d), \ \forall i \in \mathbb{Z} \right\}.$$  

(2.3)

Moreover, for every $j \in \mathbb{Z}$, and for each $f \in V_j$, there exist uniquely determined functions $t_i \in L^2(\mathbb{T}^d)$ such that $\hat{f}(B^j \xi) = \sum_{i=1}^n t_i(\xi)\varphi_i(\xi)$ and $t_i(\xi) = 0$, $\forall \xi \notin \Omega, \forall i = 1, \ldots, n$.

**Proof.** Let $f \in V_j$. By definition, we have $D^{-j} f \in V_0$ and, in particular, $f(A^{-j} \cdot) \in V_0$. Since by (1.11) we have $f(A^{-j} \cdot)(\xi) = 2^j D^{-j} f(\xi) = 2^j \hat{f}(B^j \xi)$, (2.2) gives us $\hat{f}(B^j \xi) = \sum_{i=1}^n t_i(\xi)\varphi_i(\xi)$, for some $t_i \in L^2(\mathbb{T}^d)$, $i = 1, \ldots, n$. The converse is proved in the same way. The last assertion is deduced from Remark 2.1(a). $\square$

By the preceding lemma, for each $f \in V_1$, there exists a unique $n$-tuple $t = (t_1, \ldots, t_n)$, $t_i \in L^2(\mathbb{T}^d)$, such that

$$\hat{f}(B^1 \xi) = \sum_{i=1}^n t_i(\xi)\varphi_i(\xi), \ t_i(\xi) = 0, \ \forall \xi \notin \Omega, \forall i = 1, \ldots, n.$$  

(2.4)

In the sequel, we shall write $\mu(f) = t$ and $\mu(f)$ will be called the minimal vector-filter for $f \in V_1$.

Since $V_0 \subseteq V_1$, this also applies to $\varphi_1, \ldots, \varphi_n$. For each $i \leq n$, we will write $\mu(\varphi_i) = m_i = (m_{i1}, \ldots, m_{in})$.

**Lemma 2.4.** Let $(V_j)$ be an $n$-GMRA with $n \in \mathbb{N}$. Suppose that $f, g \in V_1$, $\mu(f) = t = (t_1, \ldots, t_n)$, $\mu(g) = s = (s_1, \ldots, s_n)$. Then

$$[\hat{f}, \hat{g}](B^j \xi) = \sum_{j=1}^n t_j(\xi) s_j(\xi) + \sum_{j=1}^n t_j(\xi + \beta)s_j(\xi + \beta) \quad a.e.$$  

(2.5)

**Proof.**

$$[\hat{f}, \hat{g}](B^j \xi) = \sum_{k \in \mathbb{Z}^d} \hat{f}(B^j \xi + k) \varphi(B^j \xi + k) = \sum_{k \in \mathbb{Z}^d} \hat{f}(B(\xi + B^{-1}k)) \varphi(B(\xi + B^{-1}k))$$

(using Lemma 2.2 from [2])

$$= \sum_{k \in \mathbb{Z}^d} \left( \sum_{i=1}^n t_i(\xi + k)\varphi_i(\xi + k) \right) \left( \sum_{j=1}^n s_j(\xi + k)\varphi_j(\xi + k) \right)$$

$$+ \sum_{k \in \mathbb{Z}^d} \left( \sum_{i=1}^n t_i(\xi + \beta + k)\varphi_i(\xi + \beta + k) \right) \left( \sum_{j=1}^n s_j(\xi + \beta + k)\varphi_j(\xi + \beta + k) \right)$$
(using the $\mathbb{Z}^d$-periodicity of $t_i$’s and $s_j$’s)

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} t_i(\xi) s_j(\xi)[\hat{\varphi}_i, \hat{\varphi}_j](\xi) + \sum_{i=1}^{n} \sum_{j=1}^{n} t_i(\xi + \beta) s_j(\xi + \beta)[\hat{\varphi}_i, \hat{\varphi}_j](\xi + \beta)$$

(by Remark 2.1(b))

$$= \sum_{j=1}^{n} t_j(\xi) s_j(\xi) \chi_{\Omega_j}(\xi) + \sum_{j=1}^{n} t_j(\xi + \beta) s_j(\xi + \beta) \chi_{\Omega_j}(\xi + \beta).$$

It remains to observe that $t_j(\xi) s_j(\xi) \chi_{\Omega_j}(\xi) = t_j(\xi) s_j(\xi)$ for all $j = 1, \ldots, n$. \hfill \Box

**Corollary 2.5.** Let $(V_j)$ be an $n$-GMRA, $n \in \mathbb{N}$, with scaling functions $\varphi_1, \ldots, \varphi_n$; put $\mu(\varphi_i) = (m_{i1}, \ldots, m_{in})$, $i = 1, \ldots, n$. Then

$$\sum_{j=1}^{n} |m_{ij}(\xi)|^2 + \sum_{j=1}^{n} |m_{ij}(\xi + \beta)|^2 = \chi_{\Omega_j}(B\xi), \quad \text{a.e.} \forall i = 1, \ldots, n, \quad (2.6)$$

$$\sum_{j=1}^{n} m_{ij}(\xi)m_{ij}(\xi) + \sum_{j=1}^{n} m_{ij}(\xi + \beta)m_{ij}(\xi + \beta) = 0, \quad \text{a.e.} \forall i \neq l. \quad (2.7)$$

If $\psi$ is any function in $W_0$ with $\mu(\psi) = (t_1, \ldots, t_n)$, then

$$\sum_{j=1}^{n} m_{ij}(\xi)t_j(\xi) + \sum_{j=1}^{n} m_{ij}(\xi + \beta)t_j(\xi + \beta) = 0, \quad \text{a.e.} \forall i = 1, \ldots, n. \quad (2.8)$$

In particular, if $\psi$ is a Parseval frame wavelet associated with $(V_j)$, there exists a measurable $\mathbb{Z}^d$-periodic set $\Omega_\psi \subseteq \mathbb{R}^d$ such that

$$\sigma_\psi(\xi) = \chi_{\Omega_\psi}(\xi), \quad \text{a.e.} \quad (2.9)$$

and

$$\sum_{j=1}^{n} |t_j(\xi)|^2 + \sum_{j=1}^{n} |t_j(\xi + \beta)|^2 = \chi_{\Omega_\psi}(B\xi), \quad \text{a.e.} \quad (2.10)$$

**Remark 2.6.** Let $(V_j)$ be an $n$-GMRA, $n \in \mathbb{N}$, with scaling functions $\varphi_1, \ldots, \varphi_n$. Suppose that $\psi$ is a Parseval frame wavelet associated with $(V_j)$. If we put $\mu(\varphi_i) = (m_{i1}, \ldots, m_{in})$, $i = 1, \ldots, n$, and $\mu(\psi) = (t_1, \ldots, t_n)$, then

$$\sum_{i=1}^{n} |m_{ij}(\xi)|^2 + |t_j(\xi)|^2 = \chi_{\Omega_j}(\xi), \quad \text{a.e.} \forall j = 1, \ldots, n, \quad (2.11)$$

$$\sum_{i=1}^{n} m_{ij}(\xi)m_{ij}(\xi + \beta) + t_j(\xi)t_j(\xi + \beta) = 0, \quad \text{a.e.} \forall j = 1, \ldots, n, \quad (2.12)$$

$$\sum_{i=1}^{n} m_{ij}(\xi)m_{il}(\xi + \beta) + t_j(\xi)t_l(\xi + \beta) = 0, \quad \text{a.e.} \forall j \neq l. \quad (2.13)$$

$$\sum_{i=1}^{n} m_{ij}(\xi)m_{il}(\xi + \beta) + t_j(\xi)t_l(\xi + \beta) = 0, \quad \text{a.e.} \forall j \neq l. \quad (2.14)$$

The proof of all these assertions is essentially the same as the proof of Lemma 3.11 from [2]. Since we do not need these formulae in the sequel, we omit the details.
Let us just observe: if we add equalities (2.11) for all \( f = 1, \ldots, n \), written with \( \xi \), and then with \( \xi + \beta \) instead of \( \xi \), and take into account (2.6) and (2.10), we find \( h(\xi) = \chi_{B^{-1} \Omega(\xi)}(\xi) \) a.e. In this way, one obtains an alternative proof of the necessity of condition (1.6) for the existence of associated Parseval frame wavelets.

Now we can begin with the proof of Theorem 1.9. Let us fix an admissible \( n \)-GMRA \((V_j), n \in \mathbb{N} \), and denote by \( \Omega \) the set with the property \( h(\xi) = \chi_{B^{-1} \Omega(\xi)}(\xi) \) a.e. If \( f \) is an arbitrary function in \( W_0 \) and \( \mu(f) = t = (t_1, \ldots, t_n) \), we write, for any \( \xi, t(\xi) = (t_1(\xi), \ldots, t_n(\xi)) \in \mathbb{C}^n \). As before, we denote by \( m_i = (m_{i1}, \ldots, m_{ini}) \) the minimal vector-filter \( \mu(\phi_i) \) for \( \phi_i, i = 1, \ldots, n \).

Notice that, by Corollary 2.5, the set \[ \{(m_1(\xi), m_1(\xi + \beta)), \ldots, (m_n(\xi), m_n(\xi + \beta)), (t(\xi), t(\xi + \beta))\} \] is an orthogonal system in \( \mathbb{C}^n \otimes \mathbb{C}^n \) consisting of \( n + 1 \) vectors, for every \( f \in W_0 \) and for a.e. \( \xi \).

Note a difficulty that arises in the case \( n > 1 \). In the case \( n = 1 \) the system (2.15) consists of two mutually orthogonal vectors in \( \mathbb{C}^2 \); thus \( (t(\xi), t(\xi + \beta)) \) is determined by \( (m(\xi), m(\xi + \beta)) \), up to a scalar. When \( n > 1 \) the only immediate conclusion one can deduce from the orthogonality of the system (2.15) is that \( (t(\xi), t(\xi + \beta)) \) belongs to a subspace of \( \mathbb{C}^2n \) whose dimension is at least \( n \).

However, the following proposition tells us much more.

**Proposition 2.7.** Let \((V_j)\) be an admissible \( n \)-GMRA, \( n \in \mathbb{N} \). Then

(i) \( (t(\xi), t(\xi + \beta)) = 0 \) for a.e. \( \xi \notin B^{-1} \Omega \) and \( \forall f \in W_0, t = \mu(f) \).

(ii) For a.e. \( \xi \in B^{-1} \Omega \) there exist a subspace \( L(\xi) \subset \mathbb{C}^n \otimes \mathbb{C}^n \) such that \( \dim L(\xi) = 1 \) and \( (t(\xi), t(\xi + \beta)) \in L(\xi), \forall f \in W_0, t = \mu(f) \).

**Proof.** Let \( h_1(\xi) = \chi_{\Omega_1}(\xi) + \cdots + \chi_{\Omega_n}(\xi) + \chi_{\Omega_1}(\xi + \beta) + \cdots + \chi_{\Omega_n}(\xi + \beta) \) and \( h_2(\xi) = \chi_{B^{-1} \Omega_1}(\xi) + \cdots + \chi_{B^{-1} \Omega_n}(\xi) \). Then 1.6 can be rewritten as

\[ h_1(\xi) = h_2(\xi) + \chi_{B^{-1} \Omega}(\xi) \tag{2.16} \]

Consider first the case \( \xi \notin B^{-1} \Omega_1 \).

For such \( \xi \) we have \( h_2(\xi) = 0 \). Now the possibilities are \( \xi \in B^{-1} \Omega \) and \( \xi \notin B^{-1} \Omega_1 \).

If \( \xi \notin B^{-1} \Omega_1 \), (2.16) implies one and only one of the following two conclusions:

\[(I_a) \quad \xi \notin \Omega_1 \setminus \Omega_2, \xi + \beta \notin \Omega_1, \forall i = 1, \ldots, n. \]

\[(I_b) \quad \xi \notin \Omega_1, \forall i = 1, \ldots, n, \xi + \beta \in \Omega_1 \setminus \Omega_2. \]

In the subcase \((I_a)\) we have \( t_i(\xi) = 0, \forall i > 1 \) and \( t_i(\xi + \beta) = 0, \forall i \). Thus, \( (t(\xi), t(\xi + \beta)) \) is co-linear with \((0, \ldots, 0), (0, \ldots, 0)\). Similarly, in the subcase \((I_b)\), \( L(\xi) \) is spanned by \((0, \ldots, 0), (0, \ldots, 0)\).

It remains to analyze the possibility \( \xi \notin B^{-1} \Omega_1 \). In this case (2.16) implies \( h_1(\xi) = 0 \). This means that \( \xi \notin \Omega_1, \forall i = 1, \ldots, n \) and \( \xi + \beta \notin \Omega_1, \forall i = 1, \ldots, n \). Clearly, this gives \( (t(\xi), t(\xi + \beta)) = 0 \).

Consider now the case \( h_2(\xi) = r \geq 1, \) i.e. \( \xi \in B^{-1} \Omega_r \setminus B^{-1} \Omega_{r+1} \) with the convention \( \Omega_{n+1} = \emptyset \). Again, the possibilities are \( \xi \in B^{-1} \Omega_1 \) and \( \xi \notin B^{-1} \Omega_r \).

If \( \xi \in B^{-1} \Omega_r \) is the case, (2.16) implies \( h_1(\xi) = r + 1 \). Thus, there exist integers \( p, q \geq 0 \) such that \( 2p + q = r + 1 \) and, precisely one of the following two subcases must occur:

\[(II_a) \quad \xi \in \Omega_{p+q} \setminus \Omega_{p+q+1}, \xi + \beta \in \Omega_p \setminus \Omega_{p+1}. \]

\[(II_b) \quad \xi \in \Omega_p \setminus \Omega_{p+1}, \xi + \beta \in \Omega_{p+q} \setminus \Omega_{p+q+1}. \]

Consider the subcase \((II_a)\). Since, by assumption, \( \xi \in B^{-1} \Omega_r \setminus B^{-1} \Omega_{r+1}, (2.6) \) implies that the set \( \{m_1(\xi), m_1(\xi + \beta), \ldots, m_r(\xi), m_r(\xi + \beta)\} \) is orthonormal, while the vectors \( (m_1(\xi), m_r(\xi + \beta)) \) are trivial for \( r + 1 \leq i \leq n \).

In particular, if we take into account the position of \( \xi \) and \( \xi + \beta \) given by \((II_a)\) and the fact that each \( m_{ij} \) vanishes outside \( \Omega_j \), we conclude that the vectors
Proposition 2.9. Let \( C \)

Let us retain the notation from the proof of the preceding proposition. Since all functions involved are

Proof. (because here we have \( 2p + q = r + 1 \))

Additionally, (2.8) tells us that \( (t(\xi), t(\xi + \beta)) \in \mathbb{C}^{r+1} \) must be orthogonal to each of the vectors in (2.17). It is now clear that \( (t(\xi), t(\xi + \beta)) \) must belong to a 1-dimensional subspace \( L(\xi) \) of \( \mathbb{C}^n \oplus \mathbb{C}^n \) that is uniquely determined by (2.17).

Remark 2.8. The preceding proposition is concerned with admissible GMRA’s. However, the proof shows that for an arbitrary \( n \)-GMRA \( (V_j) \), \( n \in \mathbb{N} \), we have \( h(\xi) \geq 0 \) a.e.

Indeed, suppose that \( h_1(\xi) < h_2(\xi) \) on a set of positive measure. Then, following the preceding proof (see, for example, the subcase (II\( a_n \)), we would end up with an orthonormal set (2.17) in \( \mathbb{C}^{h(\xi)} \) that consists of \( h_2(\xi) \) vectors—a contradiction.

In the following proposition we will fix a suitable unit vector in \( L(\xi) \) for each \( \xi \in B^{-1}\Omega \). We shall need the operator \( J \) on \( \mathbb{C}^n \oplus \mathbb{C}^n \) given by \( J = \left[ \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right] \), where \( I \) is the identity operator on \( \mathbb{C}^n \). Notice that \( J((\xi_1, \xi_2, \ldots), (\eta_1, \eta_2, \ldots)) = ((\eta_1, \eta_2, \ldots), (\xi_1, \xi_2, \ldots)) \).

Proposition 2.9. Let \( (V_j) \) be an admissible GMRA, \( n \in \mathbb{N} \). For each \( \xi \in B^{-1}\Omega \) there exists a unit vector \( v(\xi) = (v_1(\xi), \ldots, v_n(\xi)), (w_1(\xi), \ldots, w_n(\xi)) \in L(\xi) \) with the following properties:

(i) \( \|v(\xi)\| = 1 \);
(ii) the map \( \xi \mapsto v(\xi) \) is a measurable \( \mathbb{Z}^d \)-periodic function;
(iii) \( v_1(\xi) = 0, \forall \xi \notin \Omega_i, \forall i = 1, \ldots, n \);  
(iv) \( v(\xi + \beta) = J(v(\xi)), \forall \xi \in B^{-1}\Omega \).

Proof. Let us retain the notation from the proof of the preceding proposition. Since all functions involved are \( \mathbb{Z}^d \)-periodic, it is enough to find \( v(\xi) \) with the above properties for all \( \xi \) in \( C \cap B^{-1}\Omega \).

We will make use of the partition \([C_0, C_1]\) of \( C \) from Section 1.

For each \( \xi \in C_0 \cap B^{-1}\Omega \) we can find a unit vector \( v(\xi) \in L(\xi) \). If we, additionally, require that the first non-trivial component of \( v(\xi) \) is positive, \( v(\xi) \) will be uniquely determined.

Let us now take an arbitrary \( \xi \in C_1 \cap B^{-1}\Omega \). Suppose that \( \xi \) falls into the case (II\( a_n \)). Then \( v(\xi) = (v_1(\xi), \ldots, v_{p+q}(\xi), 0, \ldots, 0, w_1(\xi), \ldots, w_p(\xi), 0, \ldots, 0) \) and, secondly, \( v(\xi) \) must be orthogonal to the set (2.17).

Since \( \xi + \beta \in C_0 \cap B^{-1}\Omega \), \( v(\xi + \beta) \) is already defined (provided that \( v \) is extended to \( C_0 + \mathbb{Z}^d \cap B^{-1}\Omega \) by \( \mathbb{Z}^d \)-periodicity). Notice that \( \xi + \beta \) falls into the case (II\( b_n \)) because the cases (II\( a_n \)) and (II\( b_n \)) are dual to each other with respect to the change \( \xi \leftrightarrow \xi + \beta \). Hence, by the \( \mathbb{Z}^d \)-periodicity of \( m_{ij} \)'s, \( v(\xi + \beta) \) is of the form \( v(\xi + \beta) = (v_1(\xi + \beta), \ldots, v_{p+q}(\xi + \beta), 0, \ldots, 0, w_1(\xi + \beta), \ldots, w_{p+q}(\xi + \beta), 0, \ldots, 0) \), where \( (v_1(\xi + \beta), \ldots, v_p(\xi + \beta), v_1(\xi + \beta), \ldots, w_{p+q}(\xi + \beta)) \in \mathbb{C}^{r+1} \) is orthogonal to the set

\[
\begin{align*}
(m_{11}(\xi), & m_{11}(\xi + \beta), m_{1p}(\xi + \beta)), \\
(m_{21}(\xi), & m_{21}(\xi + \beta), m_{2p}(\xi + \beta)), \\
& \vdots \\
(m_{r1}(\xi), & m_{r1}(\xi + \beta), m_{rp}(\xi + \beta))
\end{align*}
\]  

(2.17) make up an orthonormal set in \( \mathbb{C}^{r+1} \) (because \( 2p + q = r + 1 \)) consisting of \( r \) elements.

On the other hand, we know that, for each \( f \in W_0 \), \( (t(\xi), t(\xi + \beta)) \) must be of the form

\[
(t(\xi), t(\xi + \beta)) = (t_1(\xi), \ldots, t_{p+q}(\xi), 0, \ldots, 0, t_1(\xi + \beta), \ldots, t_p(\xi + \beta), 0, \ldots, 0).
\]  

(2.18)
This proves (iv) from Theorem 1.9.

Proof of Theorem 1.9. Let us now define
\[ \lambda(\xi) \]
Comparing (2.20) with (2.22), by the uniqueness of \( J \), we claim that
\[ s(\xi) \]
Indeed, for \( \xi \in B^{-1}\Omega \), we can write \( s(\xi) = \lambda(B^{-1}\xi) \) and then, for any \( k \in \mathbb{Z}^d \), we have either \( B^{-1}k = k' \) or \( B^{-1}k = k'' + \beta \) with \( k', k'' \in \mathbb{Z}^d \). Now, using (2.23), in both cases one easily verifies the equality \( s(\xi + k) = s(\xi) \).

Comparing (2.20) with (2.22), by the uniqueness of \( \lambda(\xi) \), we find
\[ \lambda(\xi) = \lambda(\xi + \beta). \] (2.23)

Let us now define
\[ s(B\xi) = \begin{cases} \lambda(\xi), & \xi \in B^{-1}\Omega, \\ 1, & \xi \notin B^{-1}\Omega. \end{cases} \] (2.24)

We claim that \( s \) is a \( \mathbb{Z}^d \)-periodic function.

Indeed, for \( \xi \in B^{-1}\Omega \), we can write \( s(\xi) = \lambda(B^{-1}\xi) \) and then, for any \( k \in \mathbb{Z}^d \), we have either \( B^{-1}k = k' \) or \( B^{-1}k = k'' + \beta \) with \( k', k'' \in \mathbb{Z}^d \). Now, using (2.23), in both cases one easily verifies the equality \( s(\xi + k) = s(\xi) \).

Finally, we conclude from (2.20)
\[ t_j(\xi) = s(B\xi)v_j(\xi) \text{ a.e., } \forall j = 1, \ldots, n; \] (2.25)

hence,
\[ \hat{f}(B\xi) = s(B\xi)\sum_{j=1}^{n}v_j(\xi)\hat{\phi}_j(\xi) \text{ a.e.} \] (2.26)

This proves (iv) from Theorem 1.9.

Suppose now that \( \psi \) is a function in \( W_0 \). By (2.26), we have \( \hat{\psi}(B\xi) = s(B\xi)\sum_{j=1}^{n}v_j(\xi)\hat{\phi}_j(\xi) \) for some \( \mathbb{Z}^d \)-periodic function \( s \). Since \( v_j(\xi) = 0, \forall \xi \notin \Omega_j, \forall j = 1, \ldots, n \), we conclude that \( \mu(\psi)(\xi) = s(B\xi)(v_1(\xi), \ldots, v_n(\xi)) \). Recall from Proposition 2.9(iv) (and from the extension of the function \( v(\xi) \)) to the set \( B^{-1}\Omega \) from the beginning of this proof) that we have \( \|v(\xi)\|^2 = \sum_{j=1}^{n}|v_j(\xi)|^2 + \sum_{i=1}^{n}|v_i(\xi + \beta)|^2 = \chi_{B^{-1}\Omega}(\xi) \). An application of Lemma 2.4 gives us
\[ \sigma_\psi(B\xi) = \|s(B\xi)\|^2\left(\sum_{j=1}^{n}|v_j(\xi)|^2 + \sum_{j=1}^{n}|v_j(\xi + \beta)|^2\right) = \|s(B\xi)\|^2\|v(\xi)\|^2. \] (2.27)

From this we conclude: if \( \psi \) is a Parseval frame wavelet associated with \( (V_j) \), \( s \) must be unimodular. Conversely, if \( s \) is unimodular, then \( \sigma_\psi(B\xi) = \|v(\xi)\|^2 = \chi_{B^{-1}\Omega}(\xi) = \chi_\Omega(B\xi) \). This implies that the system \((T_k\psi)_k\) is a Parseval frame for \( \langle \psi \rangle \).
Thus, it remains to prove: if \( \hat{\psi}(B\xi) = s(B\xi) \sum_{j=1}^{n} v_j(\xi) \hat{\psi}_j(\xi) \) with a unimodular function \( s \), then \( W_0 = \langle \psi \rangle \).

It is obvious \( W_0 \subseteq \langle \psi \rangle \). Let us prove the reverse inclusion. Take an arbitrary function \( f \in W_0 \). By (2.26), we have \( \hat{f}(B\xi) = s_1(B\xi) \sum_{j=1}^{n} v_j(B^{-1}\xi) \hat{\psi}_j(B^{-1}\xi) \) for some measurable \( \mathbb{Z}^d \)-periodic function \( s_1 \). From this we find \( \hat{f}(\xi) = s_1(\xi) \sum_{j=1}^{n} v_j(B^{-1}\xi) \hat{\psi}_j(B^{-1}\xi) = \frac{s_1(\xi)}{s(\xi)} \hat{\psi}(\xi) \). Since \( \frac{s_1(\xi)}{s(\xi)} \in L^2(T^d) \), we conclude, by Remark 2.1(a), that \( f \in \langle \psi \rangle \). \( \Box \)

We end the section by observing how formula (1.7) from Theorem 1.8 for the dimension function of a Parseval frame wavelet associated with an admissible GMRA can be deduced from the preceding results.

**Remark 2.10.** Let \( \psi \) be a Parseval frame wavelet associated with an \( n \)-GMRA \((V_j)\), \( n \in \mathbb{N} \); put \( \mu(\psi) = t \). First recall that from (2.4) we have \( \hat{\psi}(B\xi) = \sum_{j=1}^{n} t_j(\xi) \hat{\psi}_j(\xi) \) and \( \hat{\psi}_i(B\xi) = \sum_{j=1}^{n} m_{ij}(\xi) \hat{\psi}_j(\xi), \forall i = 1, \ldots, n \). This gives us

\[
|\hat{\psi}(B\xi)|^2 = \sum_{j=1}^{n} |t_j(\xi)|^2 |\hat{\psi}_j(\xi)|^2 + \sum_{j=1}^{n} \sum_{l \neq j} t_j(\xi) \overline{t_l(\xi)} \hat{\psi}_j(\xi) \overline{\hat{\psi}_l(\xi)}. \tag{2.28}
\]

and, for each \( i = 1, \ldots, n \),

\[
|\hat{\psi}_i(B\xi)|^2 = \sum_{j=1}^{n} |m_{ij}(\xi)|^2 |\hat{\psi}_j(\xi)|^2 + \sum_{j=1}^{n} \sum_{l \neq j} m_{ij}(\xi) m_{il}(\xi) \hat{\psi}_j(\xi) \overline{\hat{\psi}_l(\xi)}. \tag{2.29}
\]

Adding (2.28) and (2.29) and using (2.11) and (2.12), we obtain

\[
|\hat{\psi}(B\xi)|^2 + \sum_{i=1}^{n} |\hat{\psi}_i(B\xi)|^2 = \sum_{j=1}^{n} \chi_{\Omega_j}(\xi) |\hat{\psi}_j(\xi)|^2 \quad \text{a.e.,}
\]

which, clearly, can be written as

\[
|\hat{\psi}(B\xi)|^2 + \sum_{i=1}^{n} |\hat{\psi}_i(B\xi)|^2 = \sum_{i=1}^{n} |\hat{\psi}_i(\xi)|^2 \quad \text{a.e.} \tag{2.30}
\]

By a simple induction, (2.30) gives us

\[
\sum_{j=1}^{p} |\hat{\psi}(B^j\xi)|^2 + \sum_{i=1}^{n} |\hat{\psi}_i(B^p\xi)|^2 = \sum_{i=1}^{n} |\hat{\psi}_i(\xi)|^2 \quad \text{a.e.,} \quad \forall p \in \mathbb{N}. \tag{2.31}
\]

Now, we can use Lemma 3.23 from [2] to conclude \( \lim_{p \to \infty} |\hat{\psi}_i(B^p\xi)|^2 = 0, \forall i = 1, \ldots, n \). Therefore, if we let \( p \to \infty \) in (2.31), we find

\[
\sum_{j=1}^{\infty} |\hat{\psi}(B^j\xi)|^2 = \sum_{i=1}^{n} |\hat{\psi}_i(\xi)|^2 \quad \text{a.e.} \tag{2.32}
\]

Finally, from this we have

\[
D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k \in \mathbb{Z}^d} |\hat{\psi}(B^j(\xi + k))|^2 = \sum_{k \in \mathbb{Z}^d} \sum_{i=1}^{n} |\hat{\psi}_i(\xi + k)|^2 = \sum_{i=1}^{n} \chi_{\Omega_i}(\xi) \quad \text{a.e.}
\]

**Corollary 2.11.** Each Parseval frame wavelet \( \psi \) associated with an \( n \)-GMRA \((V_j)\), \( n \in \mathbb{N} \), satisfies \( \|\psi\|^2 = \sum_{i=1}^{n} \|\psi_i\|^2 \).

**Proof.** Immediate from (2.32). \( \Box \)
3. The infinite case

The aim of this short section is to extend the argument from Section 2 to the infinite case. First we prove an analogue of Lemma 2.2.

**Lemma 3.1.** Let \((V_j)\) be an \(\infty\)-GMRA. Then

\[
V_0 = \left\{ g \in L^2(\mathbb{R}^d) : \hat{g} = \sum_{i=1}^{\infty} [\hat{g}, \hat{\phi}_i] \hat{\phi}_i \right\}
\]

with the convergence of the series in (3.1) in norm in \(L^2(\mathbb{R}^d)\).

**Proof.** Let us denote by \(P_t\) the orthogonal projection defined on the Hilbert space \(V_0\) with the range space \(\langle \phi_i \rangle, i \in \mathbb{N}\). The sequence \(\left(\sum_{i=1}^{n} P_i\right)_{n \in \mathbb{N}}\) converges to the identity operator on \(V_0\) in the strong operator topology. Hence, if we denote, for an arbitrary \(g \in V_0\), \(P_t g = g_i \in \langle \phi_i \rangle, i \in \mathbb{N}\), we have \(g = \lim_{n \to \infty} \sum_{i=1}^{n} g_i\). Since the Fourier transform \(g \mapsto \hat{g}\) is a unitary operator, we also have \(\hat{g} = \lim_{n \to \infty} \sum_{i=1}^{n} \hat{g}_i\). Now, as in the proof of Lemma 2.2, we conclude \(\hat{g} = \lim_{n \to \infty} \sum_{i=1}^{n} \hat{g}_i\). Hence, (3.2) can be written as (3.1).

By the continuity of \([\cdot, \cdot]\) in each variable with respect to the norm in \(L^2(\mathbb{R}^d)\), we conclude \(\sum_{j=1}^{\infty} [\hat{g}_j, \hat{\phi}_i] = [\hat{g}, \hat{\phi}_i]\); hence, (3.2) can be written as (3.1). \(\square\)

Now, we can proceed as in the proof of Lemma 2.3: if \((V_j)\) is an \(\infty\)-GMRA, we have

\[
V_j = \left\{ f \in L^2(\mathbb{R}^d) : \hat{f}(B^j \xi) = \sum_{i=1}^{\infty} t_i(\xi) \hat{\phi}_i(\xi), \; t_i \in L^2(\mathbb{T}^d), \; \forall i \right\}, \quad \forall j \in \mathbb{Z}.
\]

(3.3)

Moreover, for every \(j \in \mathbb{Z}\), and for each \(f \in V_j\), there exist uniquely determined functions \(t_i \in L^2(\mathbb{T}^d)\) such that

\[
\hat{f}(B^j \xi) = \sum_{i=1}^{\infty} t_i(\xi) \hat{\phi}_i(\xi) \text{ and } t_i(\xi) = 0, \; \forall \xi \notin \Omega_i, \; \forall i \in \mathbb{N}.
\]

In particular, for each \(f \in V_1\), there exists a unique sequence \(t = (t_j)_{j \in \mathbb{N}}, t_j \in L^2(\mathbb{T}^d)\), such that

\[
\hat{f}(B^j \xi) = \sum_{j=1}^{\infty} t_j(\xi) \hat{\phi}_j(\xi), \; t_j(\xi) = 0, \; \forall \xi \notin \Omega_j, \; \forall j \in \mathbb{N}.
\]

(3.4)

Notice that, by Lemma 3.1, we have norm-convergence of the series that appears in (3.3) and in (3.4).

Again, if \(t\) is the sequence as in (3.4), we write \(\mu(f) = t\) and \(t\) is called the minimal vector-filter for \(f\).

In particular, for each \(i \in \mathbb{N}\), we denote \(\mu(\phi_i) = m_i = (m_{ij})_{j \in \mathbb{N}}\).

Basically, these are the only differences in comparison with Section 2. Observe that all computations and the results obtained in Section 2, starting from Lemma 2.4, are pointwise in nature. Here, in the infinite case, we should operate with vectors of the form \(t(\xi) = (t_j(\xi))_{j \in \mathbb{N}}\), where \(t = \mu(f)\) and \(f\) is an arbitrary function in \(V_1\). Such vectors are infinite sequences and, in fact, elements of \(I^2 \oplus I^2\). We will also need double sequences of the form \((t(\xi), t(\xi + \beta)) \in I^2 \oplus I^2\).

However, if \((V_j)\) is an admissible \(\infty\)-GMRA, for a.e. \(\xi\), using (1.5), we can find the minimal non-negative integer \(n = n(\xi)\) such \(\xi \notin \Omega_i, \; \forall i > n(\xi)\), and \(\xi + \beta \notin \Omega_i, \; \forall i > n(\xi)\). Recall that, for every \(f \in V_1\) with \(\mu(f) = t\), we have, by (3.4), \(t_j(\xi) = 0\), \(\forall \xi \notin \Omega_j\). This implies that, for a.e. \(\xi\), our double sequences of the form \((t(\xi), t(\xi + \beta)) \in I^2 \oplus I^2\), \(f \in V_1, \; t = \mu(f)\), can be regarded as elements in \(C^n(\xi) \oplus C^n(\xi)\). This will make our computations (locally) finite dimensional. The only difference is that now, instead of working in \(C^n\) with fixed \(n \in \mathbb{N}\), we work in \(C^n(\xi), n(\xi) \in \mathbb{N}\), where \(n(\xi)\) depends on \(\xi\).

To make the exposition complete, we will briefly repeat the arguments from Section 2. For simplicity, we shall operate with infinite sums (like, e.g., \(\sum_{j=1}^{\infty} t_j(\xi) \hat{\phi}_j(\xi)\)), having in mind that, for each particular \(\xi\), the summation effectively runs only up to the \(n(\xi)\)th term.
Lemma 3.2. Let \((V_j)\) be an \(\infty\)-GMRA. Suppose that \(f, g \in V_1, \mu(f) = t = (t_j), \mu(g) = s = (s_j)\). Then
\[
\hat{f} \hat{g}(B\xi) = \sum_{j=1}^{\infty} t_j(\xi) s_j(\xi) + \sum_{j=1}^{\infty} t_j(\xi + \beta) s_j(\xi + \beta) \quad \text{a.e.} \tag{3.5}
\]

Proof. As in the proof of Lemma 2.4, we find
\[
\hat{f} \hat{g}(B\xi) = \sum_{k \in \mathbb{Z}^d} \left( \sum_{i=1}^{\infty} t_i(\xi + k) \phi_i(\xi + k) \right) \left( \sum_{j=1}^{\infty} s_j(\xi + k) \phi_j(\xi + k) \right)
\]
\[+ \sum_{k \in \mathbb{Z}^d} \left( \sum_{i=1}^{\infty} t_i(\xi + \beta + k) \phi_i(\xi + \beta + k) \right) \left( \sum_{j=1}^{\infty} s_j(\xi + \beta + k) \phi_j(\xi + \beta + k) \right).
\]

Now observe that, for a.e. \(\xi\), each of the above sums over \(i\) and over \(j\) has at most \(n(\xi)\) non-trivial terms. The rest of the computation is the same as in the proof of Lemma 2.4. \(\Box\)

Corollary 3.3. Let \((V_j)\) be an \(\infty\)-GMRA. Then
\[
\sum_{j=1}^{\infty} |m_{ij}(\xi)|^2 + \sum_{j=1}^{\infty} |m_{ij}(\xi + \beta)|^2 = \chi_{\Omega_i}(B\xi), \quad \text{a.e., } \forall i \in \mathbb{N}, \tag{3.6}
\]
\[
\sum_{j=1}^{\infty} m_{ij}(\xi \bar{m}_{ij}(\xi + \beta) + \sum_{j=1}^{\infty} m_{ij}(\xi + \beta) \bar{m}_{ij}(\xi + \beta) = 0, \quad \text{a.e., } \forall i \neq l. \tag{3.7}
\]

If \(\psi\) is any function in \(W_0\) with \(\mu(\psi) = (t_j)\), then
\[
\sum_{j=1}^{\infty} m_{ij}(\xi) \bar{t}_j(\xi) + \sum_{j=1}^{\infty} m_{ij}(\xi + \beta) \bar{t}_j(\xi + \beta) = 0, \quad \text{a.e., } \forall i \in \mathbb{N}. \tag{3.8}
\]

In particular, if \(\psi\) is a Parseval frame wavelet associated with \((V_j)\), there exists a measurable \(\mathbb{Z}^d\)-periodic set \(\Omega_\psi \subseteq \mathbb{R}^d\) such that
\[
\sigma_\psi(\xi) = \chi_{\Omega_\psi}(\xi) \quad \text{a.e.} \tag{3.9}
\]
and
\[
\sum_{j=1}^{\infty} |t_j(\xi)|^2 + \sum_{j=1}^{\infty} |t_j(\xi + \beta)|^2 = \chi_{\Omega_\psi}(B\xi) \quad \text{a.e.} \tag{3.10}
\]

Remark 3.4. One can prove for, any \(\infty\)-GMRA \((V_j)\), the formulae that are analogous to those from Remark 2.6:

Suppose that \(\psi\) is a Parseval frame wavelet associated with \((V_j)\). If we put \(\mu(\psi) = (t_j)\), then
\[
\sum_{i=1}^{\infty} |m_{ij}(\xi)|^2 + |t_j(\xi)|^2 = \chi_{\Omega_j}(\xi), \quad \text{a.e., } \forall j \in \mathbb{N}, \tag{3.11}
\]
\[
\sum_{i=1}^{\infty} m_{ij}(\xi) \bar{m}_{ij}(\xi + \beta) + t_j(\xi) \bar{t}_j(\xi + \beta) = 0, \quad \text{a.e., } \forall j \in \mathbb{N}, \tag{3.12}
\]
\[
\sum_{i=1}^{\infty} m_{ij}(\xi) \bar{m}_{i}(\xi + \beta) + t_j(\xi) \bar{t}_j(\xi + \beta) = 0, \quad \text{a.e., } \forall j \neq l, \tag{3.13}
\]
\[
\sum_{i=1}^{\infty} m_{ij}(\xi) \bar{m}_{il}(\xi + \beta) + t_j(\xi) \bar{t}_l(\xi + \beta) = 0, \quad \text{a.e., } \forall j \neq l. \tag{3.14}
\]

Again, the details are omitted.
It remains to provide the proof of Theorem 1.9 in the infinite case. This will be enabled by proving analogues of Propositions 2.7 and 2.9. Having these results, we can prove Theorem 1.9 in the infinite case exactly in the same way as in the finite case.

Since one can easily verify that all the arguments from the above mentioned proofs from Section 2 apply without changes, we omit the details.

4. Examples

In this section we use Theorem 1.9 to compute wavelets associated with various concrete GMRA’s. We open the section with a proposition that serves as a tool for constructing GMRA’s. In general, each method of construction of GMRA’s starts with a shift invariant space $V_0$ with the property $V_0 \subseteq D V_0$. However, it is a non-trivial question whether the sequence $(V_j)$, $V_j = D^j V_0$, $j \in \mathbb{Z}$, satisfies condition (iii) from Definition 1.3 (see [6]). Here we provide a sufficient condition in the special case when $V_0$ is generated in the Fourier domain by a sequence of characteristic functions. For simplicity, we restrict ourselves to the dyadic case on the real line.

Proposition 4.1. Let $(S_i)$, $i \in \mathbb{N}$, be a sequence of measurable sets with the following properties:

1. $|S_i \cap S_j| = 0$, \forall $i \neq j$;
2. $|S_i \cap (S_i + k)| = 0$, \forall $k \in \mathbb{Z} \setminus \{0\}$, \forall $i \in \mathbb{N}$;
3. $\Omega_1 \supseteq \Omega_2 \supseteq \cdots \supseteq \Omega_i \supseteq \cdots$, where $\Omega_i = S_i + \mathbb{Z}$, $i \in \mathbb{N}$;
4. for each $i$ in $\mathbb{N}$ there exist a sequence $(m_{ij})$ of functions in $L^2\left([\frac{1}{2}, 1)\right)$ such that $\chi_{S_i}(2\xi) = \sum_{j=1}^{\infty} m_{ij}(\xi) \chi_{S_j}(\xi)$ in $L^2$-norm;
5. $\sum_{i=1}^{\infty} |S_i| < \infty$;
6. $\lim_{m \to \infty} \sum_{i=1}^{\infty} |S_i| \chi_{S_i}(\frac{1}{2^n} \xi)|^2 = 1$ a.e.

Let $\hat{\phi} = \chi_{S_i}$, $i \in \mathbb{N}$. Then $\langle \phi_i \rangle \perp \langle \phi_j \rangle$, $i \neq j$, and the sequence of subspaces $(V_j)$ defined by $V_j = D^j (\bigoplus_{i=1}^{\infty} \langle \phi_i \rangle)$, $j \in \mathbb{Z}$, is a GMRA in $L^2(\mathbb{R})$.

Proof. First observe that (2) implies $\sigma_{\phi_i} = \chi_{\Omega_i}$, \forall $i \in \mathbb{N}$; hence each $\langle \phi_i \rangle$ is a principal shift-invariant space. Using (1) and applying Remark 2.1(b), we conclude $\langle \phi_i \rangle \perp \langle \phi_j \rangle$, $i \neq j$. Thus, $V_0 := \bigoplus_{i=1}^{\infty} \langle \phi_i \rangle$ is a shift invariant space. Moreover, since by the assumption (3) the sequence $(\Omega_i)$ decreases, $\bigoplus_{i=1}^{\infty} \langle \phi_i \rangle$ is the canonical decomposition of $V_0$. Put $V_j = D^j V_0$, $j \in \mathbb{Z}$. As in Lemmas 2.3 and 3.1, we conclude that (4) implies $V_0 \subseteq V_1$. It remains to show that the sequence $(V_j)$ satisfies condition (iii) from Definition 1.3.

First note that $V_0 = \{ f \in L^2(\mathbb{R}) : \hat{f} = \sum_{i=1}^{\infty} |\hat{f}, \phi_i| \hat{\phi}_i \}$. This implies $\| f \|^2 = \| \hat{f} \|^2 = \sum_{j=1}^{\infty} \| |\hat{f}, \phi_i| \hat{\phi}_i \|^2, \forall f \in V_0$. Now we use (2) to obtain $\| \hat{f}, \phi_i \| \hat{\phi}_i \|^2 = \int_{S_i} |\hat{f}(\xi)|^2 d\xi$. This implies $\hat{f}(\xi) \hat{\phi}_i(\xi) = \int_{S_i} \hat{f}(\xi) \hat{\phi}_i(\xi) \hat{\phi}_i(\xi) d\xi$ for all $f \in V_0$; thus, $\hat{f}(\xi) \hat{\phi}_i(\xi) = 0$ for a.e. $\xi \notin \bigcup_{i=1}^{\infty} S_i$.

Suppose now $f \in \bigcap_{j \in \mathbb{Z}} V_j$. Since $f \in V_{-n}$, \forall $n \in \mathbb{N}$, a computation similar to that in the preceding paragraph gives $\hat{f}(\xi) = 0$ for a.e. $\xi \notin \bigcup_{j=1}^{\infty} S_j$ for each $n \in \mathbb{N}$. From this we conclude $\{ \sup f \leq 2^{-n} \sum_{i=1}^{\infty} |S_i|, \forall n \in \mathbb{N} \}$; thus, by (5), $\{ \sup f \leq 0$ and $f = 0$.

To prove $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})$ we follow the argument from the proof of Proposition 7.5.2 in [8]. Denote by $P_j$ the orthogonal projection to $V_j$. It suffices to show that $\| P_n f \| \to \| f \|$ as $n \to \infty$ for each $f \in L^2(\mathbb{R})$ such that $\hat{f}$ is compactly supported.

Since $\{D^k T_{ik} \phi_k, k \in \mathbb{Z}, i \in \mathbb{N} \}$ is a Parseval frame for $V_n$ for each $n$ in $\mathbb{N}$, we first have $\| P_n f \|^2 = \sum_{i=1}^{\infty} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} 2^{n/2} \hat{f}(2^n \xi) e^{2\pi i k \xi} \overline{\phi_i(\xi)} d\xi|^2$ for each $f \in L^2(\mathbb{R})$.

If $\sup \hat{f}$ is a compact set, we can find a natural number $n_0$ such that $\sup \hat{f} \subseteq 2^n [-\frac{1}{2}, \frac{1}{2}]$, $\forall n \geq n_0$. Then the preceding formula can be rewritten in the form $\| P_n f \|^2 = 2^n \sum_{i=1}^{\infty} \sum_{k \in \mathbb{Z}} \int_{[-\frac{1}{2}, \frac{1}{2}]} \hat{f}(2^n \xi) e^{2\pi i k \xi} \overline{\phi_i(\xi)} d\xi|^2, \forall n \geq n_0$. From this we obtain
We will find two different 2-GMRA’s with scaling functions (where the last equality is obtained by applying the monotone convergence theorem). Finally, since $S_i$’s are essentially disjoint, we have
\[
\sum_{i=1}^{\infty} \| \tilde{\phi}_i (2^{-n} \xi) \|^2 = \sum_{i=1}^{\infty} \| \tilde{\phi}_i (2^{-n} \xi) \|^2 \, d\xi = \sum_{i=1}^{\infty} \| \tilde{\phi}_i (2^{-n} \xi) \|^2 \, d\xi
\]
(where the last equality is obtained by applying the monotone convergence theorem). Finally, since $S_i$’s are essentially disjoint, we have $\sum_{i=1}^{\infty} \| \tilde{\phi}_i (2^{-n} \xi) \|^2 = \sum_{i=1}^{\infty} |\chi_{S_i}(2^{-n} \xi)|^2 \leq 1$ a.e. Applying the dominated convergence theorem and using the remaining assumption (6), we conclude $\lim_{n \to \infty} \| P_n f \|^2 = \| f \|^2$. □

With Proposition 4.1 in hand, we can construct GMRA’s and apply Theorem 1.9 to concrete examples. Let us begin with the simplest non-MRA example: the Journé wavelet.

**Example 4.2.** Let $A_1 = \left[ -\frac{1}{2}, -\frac{3}{4} \right] \cup \left[ -\frac{3}{4}, \frac{1}{4} \right] \cup \left[ \frac{1}{4}, \frac{3}{4} \right]$, $A_2 = \left[ -\frac{1}{4}, \frac{1}{4} \right]$, and $D(\xi) = \sum_{i=1}^{\infty} \chi_{O_i}(\xi)$. It is well known that the function $D$ defined in this way is the dimension function of the Journé wavelet. We will find two different 2-GMRA’s with scaling functions $\phi_i$, $i = 1, 2$, such that $\sigma_i(\xi) = \chi_{O_i}(\xi)$ a.e., $i = 1, 2$. Basically, the idea is to put $\tilde{\phi}_i = \chi_{S_i}$, $i = 1, 2$, where $S_1 = A_1$ and $S_2$ is some integer translation of the set $A_2$. Such a translation is needed in order to meet condition (1) from Proposition 4.1.

Notice that the dimension function of any orthonormal wavelet on the real line satisfies $D(\xi) + D(\xi + \frac{1}{2}) = D(2\xi) = 1$ a.e. This implies: if $(V_j)$ is a GMRA whose scaling functions $\varphi_1, \varphi_2$, satisfy $\sigma_{\varphi_j} = \chi_{O_j}$, $i = 1, 2$, and $[\tilde{\phi}_1, \tilde{\phi}_2] = 0$ a.e., then $(V_j)$ is admissible.

(a) Let $\tilde{\phi}_1 = \chi_{A_1}$ and $\tilde{\phi}_2 = \chi_{\left[ -\frac{1}{2}, -\frac{3}{4} \right] \cup \left( \frac{1}{4}, \frac{3}{4} \right]}$. We shall apply Proposition 4.1 to conclude that these two functions generate a GMRA. We only need to find the corresponding filters since all other conditions from Proposition 4.1 are clearly fulfilled. One easily obtains $m_{11} = \chi_{\frac{1}{2}A_2}$, $m_{12} = 0$, $m_{21} = \chi_{\left[ -\frac{1}{4}, -\frac{3}{4} \right] \cup \left( \frac{1}{4}, \frac{3}{4} \right]}$, $m_{22} = 0$ and then extends these functions by $\mathbb{Z}$-periodicity.

Now one easily finds the characteristic function $v(\xi)$ for the GMRA $(V_j)$ constructed in this way. It turns out
\[
v(\xi) = \begin{cases} 
(1, 0), (0, 0), & \xi \in \left[ 0, \frac{3}{4} \right] \cup \left[ \frac{3}{4}, \frac{1}{2} \right], \\
(0, 0), (1, 0), & \xi \in \left[ \frac{1}{2}, \frac{3}{4} \right] \cup \left[ \frac{3}{4}, 1 \right]. 
\end{cases}
\]

Notice that, by Proposition 2.9, we have $v(\xi + \frac{1}{2}) = J(v(\xi))$, so from the above formula one easily finds the values of $v$ on the interval $[-\frac{1}{2}, 0]$. Finally, we extend $v(\xi)$ by $\mathbb{Z}$-periodicity.

Now we can apply the last two assertions of Theorem 1.9. If we choose $s \equiv 1$, the resulting orthonormal wavelet is given by
\[
\hat{\psi} = \chi_{\left[ -\frac{1}{2}, -\frac{3}{4} \right] \cup \left[ -\frac{1}{4}, -\frac{3}{4} \right] \cup \left( \frac{1}{4}, \frac{1}{2} \right) \cup \left( \frac{1}{2}, \frac{3}{4} \right) \cup \left[ \frac{3}{4}, \frac{1}{2} \right] \cup \left[ \frac{1}{2}, 2 \right]}.
\] (4.1)

(b) Another possibility is to define $\tilde{\phi}_1 = \chi_{\left[ -\frac{1}{2}, -\frac{3}{4} \right] \cup \left( -\frac{1}{4}, \frac{1}{2} \right) \cup \left( \frac{1}{2}, \frac{3}{4} \right) \cup \left[ \frac{3}{4}, \frac{1}{2} \right] \cup \left[ \frac{1}{2}, 2 \right]}$ and $\tilde{\phi}_2 = \chi_{\left[ -\frac{1}{4}, -\frac{3}{4} \right] \cup \left( \frac{1}{4}, \frac{3}{4} \right] \cup \left[ \frac{3}{4}, \frac{1}{2} \right] \cup \left[ \frac{1}{2}, 2 \right]}$. This two functions define a different GMRA. By an easy computation one finds the corresponding filters $m_{ij}(\xi)$, $i = 1, 2$, and the characteristic function $v(\xi)$. It turns out that the choice $s \equiv 1$ this time gives the Journé wavelet
\[
\hat{\psi} = \chi_{\left[ -\frac{1}{2}, -\frac{3}{4} \right] \cup \left[ -\frac{1}{4}, -\frac{3}{4} \right] \cup \left( \frac{1}{4}, \frac{1}{2} \right) \cup \left( \frac{1}{2}, \frac{3}{4} \right) \cup \left[ \frac{3}{4}, \frac{1}{2} \right] \cup \left[ \frac{1}{2}, 2 \right]}.
\] (4.2)

**Example 4.3.** For a fixed $n \in \mathbb{N}$, consider the sets
\[
A_1 = \left[ \frac{1}{2}, -\frac{1}{2} + \frac{1}{2(2^n + 1)}, 1 \right] \cup \left[ -\frac{2^n}{2(2^n + 1) - 1}, \frac{2^n}{2(2^n + 1) - 1} \right] \cup \left[ \frac{1}{2} - \frac{1}{2}, \frac{1}{2} \right] \cup \left[ \frac{1}{2}, \frac{1}{2} \right]
\]
\[
A_2 = \left[ \frac{1}{2}, -\frac{2^n}{2(2^n + 1) - 1}, \frac{2^n}{2(2^n + 1) - 1} \right]
\]
\[
\vdots
\]
\[ A_i = \left[ \begin{array}{c} -\frac{2n-i+1}{2(2n+1-1)}, \frac{2n-i+1}{2(2n+1-1)} \end{array} \right] \]

\[ A_n = \left[ \begin{array}{c} -\frac{2}{2(2n+1-1)}, \frac{2}{2(2n+1-1)} \end{array} \right]. \]

Let \( \Omega_i = A_i + \mathbb{Z}, i = 1, \ldots, n \), and \( D_n(\xi) = \sum_{i=1}^n \chi_{\Omega_i}(\xi) \). It is known that, for each \( n \geq 2 \), the function \( D_n \) is the dimension function of an orthonormal wavelet on the real line. These dimension functions are introduced in [3]; see also [7]. Notice that, for \( n = 1 \), we have \( D_1(\xi) = 1 \) a.e., which is the dimension function of every orthonormal wavelet, while \( D_2 \) coincides with the dimension function of the Journé wavelet from Example 4.2.

For each \( n \geq 2 \) we can find an \( n \)-GMRA \( (V_j) \) such that \( V_0 \) is generated by \( n \) functions \( \varphi_i, i = 1, \ldots, n \), with the properties \( \sigma_{\varphi_i}(\xi) = \chi_{\Omega_i}(\xi), \forall i \), and \( \{\hat{\varphi}_i, \hat{\varphi}_j\}(\xi) = 0, \forall i \neq j \) a.e. This can be done using Proposition 4.1 and following the pattern from Example 4.2(a). Again, we shall work with the original set \( S_1 = A_1 \) and some integer translations \( S_i \) of \( A_i, i = 2, \ldots, n; \) as before, this will ensure condition (1) from Proposition 4.1.

Consider, first the case \( n = 3 \). Let us define \( \hat{\varphi}_1 = \chi_{A_1}, \hat{\varphi}_2 = \chi_{[-2, -\frac{4}{15}] \cup \{\frac{2}{15}, 2\}} \) and \( \hat{\varphi}_3 = \chi_{[-1, -\frac{4}{15}) \cup \{\frac{4}{15}, 1\}} \). After finding the filters and the characteristic function \( v \), one obtains, for \( s = 1 \), the orthonormal wavelet \( \psi_3 \) given by

\[ \hat{\psi}_3 = \chi_{[-4, -4 + \frac{4}{15}) \cup \{\frac{4}{15}, 2\}} \cup \{\frac{2}{15}, 4\}. \] (4.3)

For an arbitrary \( n \geq 2 \), we define \( \hat{\psi}_1 = \chi_{A_1}, \hat{\psi}_n = \chi_{[0, \frac{1}{2n+1-1}] - 2^n \hat{\psi}_1} \) and \( \hat{\varphi}_i = \chi_{[\frac{i}{2n+1-1}, \frac{i+1}{2n+1-1}) - 2^n \hat{\psi}_1} \). Again, by a simple calculation, one finds the filters and the characteristic function, and obtains (applying Theorem 1.9 with \( s = 1 \)) the orthonormal wavelet \( \psi_n \) given by

\[ \hat{\psi}_n = \chi_{W_n}, \]

\[ W_n = \left[ -2^n-1, -2^n-1 + \frac{2^n}{2n+1-1} \right] \cup \left[ -\frac{2^n}{2n+1-1}, -\frac{1}{2} \right] \cup \left[ -\frac{1}{2}, -\frac{2^n-1}{2n+1-1} \right] \]

\[ \cup \left[ -\frac{2^n-1}{2n+1-1}, -\frac{2^n}{2n+1-1} \right] \cup \left[ -1, -\frac{2^n-1}{2n+1-1} \right] \cup \left[ -\frac{2^n-1}{2n+1-1}, 2^n-1 \right]. \] (4.4)

We omit the computational details, since they are very similar to those in the following example.

Notice that formula (4.4) for \( n = 1 \) gives the Shannon wavelet, for \( n = 2 \), \( \psi_2 \) coincides with the wavelet obtained in Example 4.2(a) defined by (4.1), and, for \( n = 3 \), (4.4) coincides with (4.3).

We now turn to the construction of an orthonormal wavelet whose dimension function is essentially unbounded. It is proved in [7] (see also [1]) that such wavelets do exist. Example 5.9 in [7] provides a family of multi-wavelets with this property, but there are no examples in the existing literature of (singly generated) orthonormal wavelets with unbounded dimension function. Also, no concrete examples of such dimension functions are known.

As the starting point of our construction, we shall provide an essentially unbounded integer valued measurable function that is the dimension function of an orthonormal wavelet. This will give rise to an admissible GMRA with infinitely many scaling functions. Finally, an application of Theorem 1.9 will give as a wavelet with the desired property.

Recall from [7] that a non-negative, periodic, measurable integer valued function \( D \) is the dimension function of some orthonormal wavelet on the real line if and only if \( D \) satisfies the following three conditions:

\[ \liminf_{m \to \infty} D\left(\frac{1}{2m} \xi\right) \geq 1 \quad \text{a.e.,} \] (4.5)

\[ D(\xi) + D\left(\xi + \frac{1}{2}\right) - D(2\xi) = 1 \quad \text{a.e.,} \] (4.6)

\[ D(\xi) \leq \sum_{k \in \mathbb{Z}} \chi_D(\xi + k) \quad \text{a.e.,} \] (4.7)

where \( \Delta = \{\xi : D\left(\frac{1}{2m} \xi\right) \geq 0, \forall m \in \mathbb{N} \cup \{0\}\} \).
Example 4.4. We define an integer valued function on the unit interval \([-\frac{1}{2}, \frac{1}{2}\]) by defining its values separately on the intervals \([-\frac{1}{2}, -\frac{1}{4})\) and \([-\frac{1}{4}, -\frac{1}{8})\), then on the series on intervals \(\frac{1}{27}[-\frac{1}{4}, -\frac{1}{8})\), \(j \in \mathbb{N}\), and, finally, on \([0, \frac{1}{2})\).

\[
D(\xi) = \begin{cases} 
1, & \xi \in \left[-\frac{1}{2}, -\frac{3}{8}\right), \\
0, & \xi \in \left[-\frac{3}{8}, -\frac{5}{16}\right), \\
n, & \xi \in \left[-\frac{2}{7} + \frac{1}{2\cdot 896}, -\frac{2}{7} + \frac{1}{2\cdot 896}\right), n \in \mathbb{N}, \\
1, & \xi \in \left[-\frac{2}{7} + \frac{1}{2\cdot 896}, -\frac{1}{4}\right). 
\end{cases}
\]

\[
D(\xi) = \begin{cases} 
2, & \xi \in \left[-\frac{5}{32}, \frac{3}{16}\right], \\
1, & \xi \in \left[-\frac{3}{16}, -\frac{5}{32}\right), \\
n + 1, & \xi \in \left[-\frac{1}{7} + \frac{1}{4\cdot 896}, -\frac{1}{7} + \frac{1}{4\cdot 896}\right), n \in \mathbb{N}, \\
2, & \xi \in \left[-\frac{1}{7} + \frac{1}{4\cdot 896}, -\frac{1}{8}\right). 
\end{cases}
\]

\[
D(\xi) = \begin{cases} 
2, & \xi \in \left[\frac{1}{27}[-\frac{1}{4}, -\frac{3}{16}\right), \\
1, & \xi \in \left[\frac{1}{27}[-\frac{3}{16}, -\frac{5}{32}\right), \\
n, & \xi \in \left[\frac{1}{27}[-\frac{5}{32}, -\frac{1}{7}\right), j \in \mathbb{N}, \\
1, & \xi \in \left[\frac{1}{27}[-\frac{1}{7}, -\frac{1}{7} + \frac{1}{4\cdot 896}\right), \\
2, & \xi \in \left[\frac{1}{27}[-\frac{1}{7} + \frac{1}{4\cdot 896}, -\frac{1}{8}\right). 
\end{cases}
\]

\[
D(\xi) = \begin{cases} 
1, & \xi \in \left[0, \frac{1}{4}\right), \\
0, & \xi \in \left[\frac{1}{4}, \frac{3}{8}\right), \\
n, & \xi \in \left[\frac{3}{8}, \frac{3}{7}\right), n \in \mathbb{N}, \\
1, & \xi \in \left[\frac{7}{16}, \frac{7}{12}\right), 
\end{cases}
\]

Finally, we extend \(D\) by \(\mathbb{Z}\)-periodicity.

We claim that \(D\) is a dimension function. As \(D\) is measurable, integer valued and periodic, it suffices to verify conditions (4.5), (4.6) and (4.7). Obviously, \(D\) satisfies (4.5).

Claim 1. The function \(D\) satisfies (4.6).

Proof. We will explain how the function \(D\) was constructed as well as the role of the numbers that appear in the definition. It will be clear from the construction that \(D\) satisfies the consistency condition (4.6).

Consider the function \(D\) defined on \([0, \frac{1}{2})\) by

\[
D(\xi) = \begin{cases} 
1, & \xi \in [0, \frac{1}{4}), \\
0, & \xi \in \left[\frac{1}{4}, \frac{3}{8}\right), \\
\delta(\xi), & \xi \in \left[\frac{3}{8}, \frac{7}{12}\right), \\
1, & \xi \in \left[\frac{7}{16}, \frac{7}{12}\right), 
\end{cases}
\]

(4.8)

where \(\delta\) is an integer valued measurable function, temporarily unspecified.

Now we use (4.6) to define \(D\) on the interval \([-\frac{1}{2}, -\frac{1}{4})\) by

\[
D\left(\xi - \frac{1}{2}\right) = 1 - D(\xi) + D(2\xi), \quad \xi \in \left[0, \frac{1}{4}\right). 
\]

(4.9)
i.e.
\[
D(\xi) = \begin{cases} 
1, & \xi \in \left[ -\frac{1}{2}, -\frac{3}{8} \right), \\
0, & \xi \in \left[ -\frac{3}{8}, -\frac{5}{16} \right), \\
\delta(2\xi + 1), & \xi \in \left[ -\frac{5}{16}, -\frac{9}{32} \right), \\
1, & \xi \in \left[ -\frac{9}{32}, -\frac{1}{4} \right].
\end{cases}
\] (4.10)

Using (4.6) again, we now define \( D \) on the interval \( \left[ -\frac{1}{4}, -\frac{1}{8} \right) \) by
\[
D(\xi) = 1 - D\left(\xi + \frac{1}{2}\right) + D(2\xi), \quad \xi \in \left[ -\frac{1}{4}, -\frac{1}{8} \right).
\] (4.11)

Since \( D(\xi) = 0 \) for all \( \xi \in \left[ \frac{1}{4}, \frac{3}{8} \right) \), we get
\[
D(\xi) = \begin{cases} 
2, & \xi \in \left[ -\frac{1}{4}, -\frac{3}{16} \right), \\
1, & \xi \in \left[ -\frac{3}{16}, -\frac{5}{32} \right), \\
1 + \delta(4\xi + 1), & \xi \in \left[ -\frac{5}{32}, -\frac{9}{64} \right), \\
2, & \xi \in \left[ -\frac{9}{64}, -\frac{1}{8} \right).
\end{cases}
\] (4.12)

The idea is to proceed inductively in the same fashion. Notice that the next step is the critical one, since \( \left[ -\frac{1}{8}, -\frac{1}{16} \right) + \frac{1}{2} = \left[ \frac{3}{8}, \frac{7}{16} \right) \). Using the formula from (4.11), this time for \( \xi \in \left[ -\frac{1}{8}, -\frac{1}{16} \right) \), we have
\[
D(\xi) = \begin{cases} 
3 - \delta(\xi + \frac{1}{2}), & \xi \in \left[ -\frac{1}{8}, -\frac{3}{16} \right), \\
2 - \delta(\xi + \frac{1}{2}), & \xi \in \left[ -\frac{3}{16}, -\frac{5}{32} \right), \\
2 + \delta(8\xi + 1) - \delta(\xi + \frac{1}{2}), & \xi \in \left[ -\frac{5}{32}, -\frac{9}{64} \right), \\
3 - \delta(\xi + \frac{1}{2}), & \xi \in \left[ -\frac{9}{64}, -\frac{1}{16} \right).
\end{cases}
\] (4.13)

We are now in position to choose \( \delta \). Our goal is to find an unbounded integer valued function \( \delta \) that makes the above formula as simple as possible. First of all, we can make the first two lines and the last line in (4.13) constant, if we choose \( \delta \) as a constant function on the intervals \( \left[ \frac{3}{8}, \frac{7}{16} \right) \) and \( \left[ \frac{55}{128}, \frac{7}{16} \right) \). Furthermore, one observes that \( 8\xi + 1 = \xi + \frac{1}{2} \) has the only solution \( \xi = -\frac{1}{16} \). Since \( -\frac{1}{14} + \frac{1}{2} = \frac{3}{7} \), it is also convenient to have \( \delta \) constant on the interval \( \left( \frac{2}{7}, \frac{3}{7} \right) \). The remaining interval where we are going to allow \( \delta \) to be unbounded is \( \left[ \frac{3}{7}, \frac{55}{128} \right) \). Notice that \( \frac{55}{128} = \frac{3}{7} + \frac{1}{896} = \frac{3}{7} + \frac{1}{727} \).

Further, if we choose an unbounded function \( \delta \) on the interval \( \left[ \frac{3}{7}, \frac{3}{7} + \frac{1}{896} \right) \), the preceding part of the construction shows that the function \( D \) will have unbounded peaks at the points \( \frac{3}{7}, -\frac{2}{7} \) and \( -\frac{1}{7} \). Recall that we must continue our inductive definition of the function \( D \) by determining its values on the intervals \( \left[ \frac{1}{7}, -\frac{1}{16} \right) \). Notice that the function \( \xi \to 8\xi + 1 \) runs eight times faster than \( \xi \to \xi + \frac{1}{2} \). So, if we want to prevent our inductive procedure reproducing further unbounded peaks at the points \( -\frac{1}{14}, \frac{1}{14}, \frac{1}{896} \), the third line of (4.13) suggests to take constant values of \( \delta \) on the intervals \( \left[ \frac{3}{7} + \frac{1}{896}, \frac{3}{7} + \frac{1}{896} \right) \).

Taking into account all of these desired properties of \( \delta \), we now define \( D(\xi) \) on the interval \( \left[ 0, \frac{1}{2} \right) \) by
\[
D(\xi) = \begin{cases} 
1, & \xi \in \left[ 0, \frac{1}{4} \right), \\
0, & \xi \in \left[ \frac{1}{4}, \frac{3}{8} \right), \\
1, & \xi \in \left[ \frac{3}{8}, \frac{3}{7} \right), \\
li(w \in \mathbb{N}, \xi \in \left[ \frac{3}{7} + \frac{1}{896} + \frac{1}{896} \right), \\
1, & \xi \in \left[ \frac{3}{7} + \frac{1}{896} + \frac{1}{2} \right).
\end{cases}
\]

Starting with \( D(\xi) \) defined in this way on the interval \( \left[ 0, \frac{1}{2} \right) \) we can apply the inductive procedure described above to define \( D \) on the interval \( \left[ -\frac{1}{2}, 0 \right) \). Clearly, this yields the function introduced in Example 4.4. \( \Box \)
To prove that the function $D$ given in Example 4.4 is a dimension function, it only remains to see that $D$ satisfies (4.7). Instead of verifying (4.7) directly, we can do that in the following way: Put

$$\Omega_n = \left\{ \xi : D(\xi) \geq n \right\}, \quad n \in \mathbb{N}. \quad (4.14)$$

Obviously, $(\Omega_n)$ is a decreasing sequence of measurable $\mathbb{Z}$-periodic sets and

$$D(\xi) = \sum_{n=1}^{\infty} \chi_{\Omega_n}(\xi). \quad (4.15)$$

We will construct a GMRA $(V_j)$ such that the sequence of scaling functions $(\varphi_n)$ satisfies $\sigma_{\varphi_n}(\xi) = \chi_{\Omega_n}(\xi), \forall n \in \mathbb{N}$. Notice that each GMRA with this property must be admissible. Indeed, condition (1.5) is ensured by (4.14) and the definition of $D$, while (1.6) follows from (4.15) and the fact that $D$ satisfies (4.6).

Let $S_n = \Omega_n \cap \left[ -\frac{1}{2}, \frac{1}{2} \right], n \in \mathbb{N}$. Notice that $\Omega_n = S_n + \mathbb{Z}, \forall n \in \mathbb{N}$, and

$$S_1 = \left[ -\frac{1}{2}, -\frac{3}{8} \right] \cup \left[ -\frac{5}{16}, \frac{1}{4} \right] \cup \left[ \frac{3}{8}, \frac{1}{2} \right],$$

$$S_2 = \left[ -\frac{2}{7}, -\frac{2}{7} + \frac{1}{2} \frac{8}{896} \right] \cup \left[ \frac{1}{2j}, -\frac{1}{4}, -\frac{3}{16} \right] \cup \left[ \frac{5}{32}, -\frac{1}{7} \right] \cup \left[ -\frac{1}{4}, -\frac{3}{16} \right] \cup \left[ \frac{5}{32}, -\frac{1}{7} \right],$$

$$S_n = \left[ -\frac{2}{7}, -\frac{2}{7} + \frac{1}{2} \frac{8n-1}{896} \right] \cup \left[ -\frac{1}{7}, -\frac{1}{7} + \frac{1}{4} \frac{8n-2}{896} \right] \cup \left[ \frac{3}{7}, \frac{1}{8} \right], \quad \forall n \geq 3. \quad (4.16)$$

We shall apply Proposition 4.1 to the sequence $(S'_n)$, where $S'_1 = S_1$ and each $S'_n, n \geq 2$, is an integer translation of $S_n$ (again, this will ensure condition (1) from Proposition 4.1). However, the sequence $(S'_n)$ should also satisfy (4) (among other conditions from Proposition 4.1). This will be achieved by finding an appropriate partition of each $S_n, n \geq 2$, and by translating the sets $S_n, n \geq 2$, by parts. Denote, for typographical reasons,

$$A = \bigcup_{j=0}^{\infty} \frac{1}{2j} \left[ -\frac{1}{4}, -\frac{3}{16} \right], \quad B = \bigcup_{j=0}^{\infty} \frac{1}{2j} \left[ -\frac{5}{32}, -\frac{1}{7} \right], \quad C = \bigcup_{j=0}^{\infty} \frac{1}{2j} \left[ -\frac{1}{7} + \frac{1}{4} \frac{8n-2}{896}, -\frac{1}{8} \right],$$

$$D_n = \left[ -\frac{2}{7}, -\frac{2}{7} + \frac{1}{2} \frac{8n-1}{896} \right], \quad n \geq 2, \quad E_n = \left[ -\frac{1}{7}, -\frac{1}{7} + \frac{1}{4} \frac{8n-2}{896} \right], \quad n \geq 2,$$

$$F_n = \left[ \frac{3}{7}, \frac{3}{7} + \frac{1}{8n-1} \frac{896}{896} \right], \quad n \geq 2. \quad (4.17)$$

Let

$$\hat{\varphi}_1 = \chi_{S_1} = \chi_{\left[ -\frac{1}{2}, -\frac{3}{8} \right]} + \chi_{\left[ -\frac{5}{16}, \frac{1}{4} \right]} + \chi_{\left[ \frac{3}{8}, \frac{1}{2} \right]},$$

$$\hat{\varphi}_2 = \chi_{A+1} + \chi_{B+1} + \chi_{C+1} + \chi_{D_2+2} + \chi_{E_2+1} + \chi_{F_2+3},$$

$$\hat{\varphi}_n = \chi_{D_n+\frac{12}{896}n-1} + \chi_{F_n+\frac{6}{896}n-1} + \chi_{F_n+\frac{3}{896}n-1}, \quad \forall n \geq 3. \quad (4.18)$$

Claim 2. The sequence $(\varphi_n)$ defined by (4.18) is a sequence of scaling functions for an admissible $\infty$-GMRA $(V_j)$ and satisfies $\sigma_{\varphi_n} = \chi_{\Omega_n}, \forall n \in \mathbb{N}$.

Proof. We use Proposition 4.1. Obviously, we only need to find the corresponding filters $m_{ij}$, since the other properties from Proposition 4.1 are clearly satisfied. Also, as already observed, the GMRA $(V_j)$ generated by the sequence $(\varphi_n)$ must be admissible because $D$ satisfies (4.6) and (4.15). Notice that our claim also implies that $D$ is the dimension function of an orthonormal wavelet.
To find filters for $\varphi_n$’s, observe: $\frac{1}{2} S_1 \subseteq S_1, \frac{1}{2} A + \frac{1}{2} \subseteq S_1, \frac{1}{2} B + \frac{1}{2} \subseteq S_1, \frac{1}{2} C + \frac{1}{2} \subseteq S_1, \frac{1}{2} E_2 + \frac{1}{2} \subseteq S_1, \frac{1}{2} (D_2 + 2) \subseteq E_2 + 1, \frac{1}{2} (F_2 + 3) = D_2 + 2$ and, for all $n \geq 3$, $\frac{1}{2} (E_n + \frac{68^{n-2}+1}{7}) \subseteq F_{n-1} + \frac{38^{n-2}-3}{7}, \frac{1}{2} (D_n + \frac{128^{n-2}+2}{7}) \subseteq E_n + \frac{68^{n-2}+1}{7}$ and $\frac{1}{4} (F_n + \frac{38^{n-1}-3}{7}) = D_n + \frac{128^{n-2}+2}{7}$.

From these relations we conclude:

$$
\begin{align*}
    m_{11} &= X_{[- \frac{1}{2}, \frac{1}{2}]} + X_{[- \frac{5}{16}, \frac{1}{16}]} + X_{[\frac{1}{16}, \frac{1}{16}]} , \\
    m_{ij} &= 0, \quad \forall j \geq 2, \\
    m_{21} &= X_{[\frac{1}{2} A + \frac{1}{2}]} + X_{[\frac{1}{2} B + \frac{1}{2}]} + X_{[\frac{1}{2} C + \frac{1}{2}]} + X_{\frac{1}{2} E_2 + \frac{1}{2} } , \\
    m_{22} &= X_{\frac{1}{2} D_2 + 1} + X_{D_2 + 2} , \\
    m_{2j} &= 0, \quad \forall j \geq 3 , \\
    m_{n,n-1} &= X_{\frac{1}{2} (E_n + \frac{68^{n-2}+1}{7})} , \quad \forall n \geq 3 , \\
    m_{nn} &= X_{\frac{1}{2} (D_n + \frac{128^{n-2}+2}{7})} + X_{\frac{1}{2} D_n + \frac{128^{n-2}+2}{7} } , \quad \forall n \geq 3 , \\
    m_{n,j} &= 0, \quad \forall j \geq n + 1, \quad \forall n \geq 3 .
\end{align*}
$$

(4.19)

After extending the obtained functions $m_{ij}$ by $Z$-periodicity, we conclude that $m_{ij} \in L^2([-\frac{1}{2}, \frac{1}{2}])$ and $\hat{\varphi}_i (2\xi) = \sum_{j=1}^{\infty} m_{ij}(\xi) \hat{\varphi}_j (\xi)$, $\forall i \in \mathbb{N}$. □

We now know that $(V_j)$ is an admissible GMRA. To obtain associated wavelets, we must find the characteristic function $v(\xi)$ for $(V_j)$. Our computation will follow the proof of Theorem 1.9. Recall that we only need to determine the values of $v$ on the interval $[-\frac{1}{2}, 0)$, since from the form of $v(\xi)$ (see assertion (i) in Theorem 1.9) we have $v(\xi + \beta) = J(v(\xi)), \forall \xi$. This will give us the values of $v$ on the interval $[0, \frac{1}{2})$. Finally, $v(\xi)$ will be extended by $Z$-periodicity.

To compute $v(\xi)$, we first need to determine the sets $[-\frac{1}{2}, 0) \setminus \frac{1}{2} \Omega_1$ and $\left( \frac{1}{2} \Omega_n \setminus \frac{1}{2} \Omega_{n+1} \right) \cap [-\frac{1}{2}, 0)$, $\forall n \in \mathbb{N}$. Since $\Omega_n = S_n + Z$, using (4.16), we find the following set differences in the interval $[-\frac{1}{2}, 0)$:

$$
\begin{align*}
    \left[ -\frac{1}{2}, 0 \right) \setminus \frac{1}{2} \Omega_1 &= \left[ -\frac{3}{8}, -\frac{5}{16} \right] \cup \left[ -\frac{3}{16}, -\frac{32}{16} \right] , \\
    \left( \frac{1}{2} \Omega_1 \setminus \frac{1}{2} \Omega_2 \right) &= \left[ -\frac{1}{2}, -\frac{3}{8} \right] \cup \left[ -\frac{5}{16}, -\frac{2}{16} \right] \cup \left[ -\frac{2}{7}, \frac{1}{2} \frac{1}{8} \frac{896}{896}, \frac{3}{16} \right] \cup \left[ -\frac{5}{32}, -\frac{7}{32} \right] , \\
    &\quad \cup \left[ -\frac{1}{7}, \frac{1}{2} \frac{1}{8} \frac{896}{896}, -\frac{1}{8} \right] \cup \bigcup_{j=1}^{\infty} \left[ -\frac{3}{16}, -\frac{5}{32} \right] \cup \bigcup_{j=2}^{\infty} \left[ -\frac{7}{16}, -\frac{7}{32} + \frac{1}{8} \frac{896}{896} \right] , \\
    \left( \frac{1}{2} \Omega_2 \setminus \frac{1}{2} \Omega_3 \right) &= \bigcup_{j=1}^{\infty} \left[ -\frac{1}{4}, -\frac{3}{16} \right] \cup \bigcup_{j=1}^{\infty} \left[ -\frac{5}{32}, -\frac{7}{16} \right] \cup \bigcup_{j=1}^{\infty} \left[ -\frac{7}{16}, -\frac{7}{32} + \frac{1}{8} \frac{896}{896} \right] , \\
    &\quad \cup \left[ -\frac{1}{7}, \frac{1}{2} \frac{1}{8} \frac{896}{896}, -\frac{1}{8} \right] \cup \left[ -\frac{2}{7}, \frac{1}{8} \frac{896}{896}, -\frac{2}{7} + \frac{1}{2} \frac{1}{8} \frac{896}{896} \right] \cup \left[ -\frac{1}{7} + \frac{1}{2} \frac{1}{8} \frac{896}{896}, -\frac{7}{16} + \frac{1}{8} \frac{896}{896} \right] , \\
    \left( \frac{1}{2} \Omega_3 \setminus \frac{1}{2} \Omega_4 \right) &= \left[ -\frac{1}{14}, -\frac{1}{8} \frac{896}{896}, -\frac{1}{14} + \frac{1}{8} \frac{896}{896} \right] , \\
    \left( \frac{1}{2} \Omega_n \setminus \frac{1}{2} \Omega_{n+1} \right) &= \left[ -\frac{2}{7}, \frac{1}{2} \frac{1}{8} \frac{896}{896}, -\frac{2}{7} + \frac{1}{2} \frac{1}{8} \frac{896}{896} \right] \cup \left[ -\frac{1}{7}, \frac{1}{2} \frac{1}{8} \frac{896}{896}, -\frac{1}{7} + \frac{1}{8} \frac{896}{896} \right] , \quad \forall n \geq 3 .
\end{align*}
$$

(4.20)

Now we can determine the characteristic function $v(\xi)$ on each of the above sets. Recall that $v(\xi) = (v_1(\xi), v_2(\xi), \ldots, v_1(\xi + \frac{1}{2}), v_2(\xi + \frac{1}{2}), \ldots))$. As before, we write $h_1(\xi) = \sum_{i=1}^{\infty} \chi_{\frac{1}{2} \Omega_i}(\xi) + \sum_{i=1}^{\infty} \chi_{\frac{1}{2} \Omega_i}(\xi + \frac{1}{2})$. Notice that, using (4.6) (that is, $\sum_{i=1}^{\infty} \chi_{\frac{1}{2} \Omega_i}(\xi) + \sum_{i=1}^{\infty} \chi_{\frac{1}{2} \Omega_i}(\xi + \frac{1}{2}) = 1 + \sum_{i=1}^{\infty} \chi_{\frac{1}{2} \Omega_i}(\xi)$ a.e.), we have, for a.e. $\xi$, $\xi \in \frac{1}{2} \Omega_n \setminus \frac{1}{2} \Omega_{n+1} \Rightarrow h_1(\xi) = n + 1$.
The computation of \( v(\xi) \) will be divided into a number of cases that correspond to the sets listed in (4.20). The only exception is \( \left[ -\frac{2}{7} + \frac{1}{8}, \frac{1}{8} \right] \cup \left[ -\frac{3}{10}, \frac{1}{10} \right] \subseteq \frac{1}{2} \Omega_1 \setminus \frac{1}{2} \Omega_2 \) that requires two separate subcases (cf. (2c) and (2d) below).

(1) \( \xi \in \left[ -\frac{1}{2}, 0 \right) \setminus \frac{1}{2} \Omega_1 \), \( h_1(\xi) = 1 \).

(a) \( \xi \in \left[ -\frac{3}{8}, -\frac{5}{16} \right) \). Since \( \xi \in \Omega_1 + \frac{1}{2} \), we have \( v(\xi) = ((0, 0, \ldots), (1, 0, \ldots)) \).

(b) \( \xi \in \left[ -\frac{3}{16}, -\frac{5}{32} \right) \). Since \( \xi \in \Omega_1 \), here we have \( v(\xi) = ((1, 0, \ldots), (0, 0, \ldots)) \).

(2) \( \xi \in \frac{1}{2} \Omega_2 \setminus \frac{1}{2} \Omega_2 \), \( h_1(\xi) = 2 \).

(a) \( \xi \in \left[ -\frac{1}{2}, -\frac{3}{8} \right) \). These points belong to \( \Omega_1 \cap \left( \Omega_1 + \frac{1}{2} \right) \). Thus, we have

\[
m_1(\xi) = ((0, 0, \ldots), (1, 0, \ldots)) \quad \text{and} \quad v(\xi) = ((1, 0, \ldots), (0, 0, \ldots)).
\]

(b) \( \xi \in \left[ -\frac{5}{16}, -\frac{3}{8} \right) \). These points belong to \( \Omega_1 \cap \left( \Omega_1 + \frac{1}{2} \right) \). Again, we have

\[
m_1(\xi) = ((0, 0, \ldots), (1, 0, \ldots)) \quad \text{and} \quad v(\xi) = ((1, 0, \ldots), (0, 0, \ldots)).
\]

(c) \( \xi \in \left[ -\frac{5}{16}, -\frac{3}{8} \right] \). Again, \( \xi \in \Omega_1 \cap \left( \Omega_1 + \frac{1}{2} \right) \). As above,

\[
m_1(\xi) = ((0, 0, \ldots), (1, 0, \ldots)) \quad \text{and} \quad v(\xi) = ((1, 0, \ldots), (0, 0, \ldots)).
\]

(d) \( \xi \in \left[ -\frac{1}{4}, -\frac{3}{16} \right) \). Here we have \( \xi \in \Omega_1 \cap \Omega_2 \). It follows

\[
m_1(\xi) = ((1, 0, \ldots), (0, 0, \ldots)) \quad \text{and} \quad v(\xi) = ((0, 1, 0, \ldots), (0, 0, \ldots)).
\]

(e) \( \xi \in \left[ -\frac{5}{32}, -\frac{1}{4} \right) \). As above, \( \xi \in \Omega_1 \cap \Omega_2 \). It follows

\[
m_1(\xi) = ((1, 0, \ldots), (0, 0, \ldots)) \quad \text{and} \quad v(\xi) = ((0, 1, 0, \ldots), (0, 0, \ldots)).
\]

(f) \( \xi \in \left[ -\frac{1}{4}, -\frac{5}{32} \right] \). Again, \( \xi \in \Omega_1 \cap \Omega_2 \). It follows

\[
m_1(\xi) = ((1, 0, \ldots), (0, 0, \ldots)) \quad \text{and} \quad v(\xi) = ((0, 1, 0, \ldots), (0, 0, \ldots)).
\]

(g) \( \xi \in \bigcup_{j=1}^{\infty} \left[ \frac{1}{2} \right] \left[ -\frac{3}{16}, -\frac{5}{32} \right] \). These points are in \( \Omega_1 \cap \left( \Omega_1 + \frac{1}{2} \right) \). One finds

\[
m_1(\xi) = ((1, 0, \ldots), (0, 0, \ldots)) \quad \text{and} \quad v(\xi) = ((0, 0, \ldots), (1, 0, \ldots)).
\]

(h) \( \xi \in \bigcup_{j=1}^{\infty} \left[ \frac{1}{2} \right] \left[ -\frac{1}{4}, -\frac{1}{4} + \frac{1}{4} - \frac{1}{8} \right] \). Again, \( \xi \in \Omega_1 \cap \left( \Omega_1 + \frac{1}{2} \right) \). One finds

\[
m_1(\xi) = ((1, 0, \ldots), (0, 0, \ldots)) \quad \text{and} \quad v(\xi) = ((0, 0, \ldots), (1, 0, \ldots)).
\]

(3) \( \xi \in \frac{1}{2} \Omega_2 \setminus \frac{1}{2} \Omega_3 \), \( h_1(\xi) = 3 \).

(a) \( \xi \in \bigcup_{j=1}^{\infty} \left[ \frac{1}{2} \right] \left[ -\frac{1}{4}, -\frac{3}{16} \right] \). Observe \( \xi \in \Omega_1 \cap \Omega_2 \cap \left( \Omega_1 + \frac{1}{2} \right) \). Here we find

\[
m_1(\xi) = ((1, 0, \ldots), (0, 0, \ldots)), \quad m_2(\xi) = ((0, 0, \ldots), (1, 0, \ldots)); \quad \text{hence,}

v(\xi) = ((0, 1, 0, \ldots), (0, 0, \ldots)).
\]

(b) \( \xi \in \bigcup_{j=1}^{\infty} \left[ \frac{1}{2} \right] \left[ -\frac{5}{32}, -\frac{1}{4} \right] \). Again, \( \xi \in \Omega_1 \cap \Omega_2 \cap \left( \Omega_1 + \frac{1}{2} \right) \); we find

\[
m_1(\xi) = ((1, 0, \ldots), (0, 0, \ldots)), \quad m_2(\xi) = ((0, 0, \ldots), (1, 0, \ldots)) \quad \text{and}

v(\xi) = ((0, 1, 0, \ldots), (0, 0, \ldots)).
\]

(c) \( \xi \in \bigcup_{j=1}^{\infty} \left[ \frac{1}{2} \right] \left[ -\frac{1}{4} + \frac{1}{4} - \frac{1}{8} \right] \). Again, \( \xi \in \Omega_1 \cap \Omega_2 \cap \left( \Omega_1 + \frac{1}{2} \right) \); we find

\[
m_1(\xi) = ((1, 0, \ldots), (0, 0, \ldots)), \quad m_2(\xi) = ((0, 0, \ldots), (1, 0, \ldots)) \quad \text{and}

v(\xi) = ((0, 1, 0, \ldots), (0, 0, \ldots)).
\]

(d) \( \xi \in \left[ -\frac{2}{7} + \frac{1}{8}, \frac{1}{8} \right] \cup \left[ -\frac{2}{7} + \frac{1}{8} \right] \). These points also belong to the set \( \Omega_1 \cap \Omega_2 \cap \left( \Omega_1 + \frac{1}{2} \right) \), but here we have

\[
m_1(\xi) = ((0, 0, \ldots), (1, 0, \ldots)), \quad m_2(\xi) = ((0, 1, 0, \ldots), (0, 0, \ldots)) \quad \text{and}

v(\xi) = ((1, 0, \ldots), (0, 0, \ldots)).
\]
(c) \( \xi \in \left[ -\frac{1}{7} + \frac{1}{4} \frac{1}{896}, -\frac{1}{7} + \frac{1}{4} \frac{1}{896} \right) \). In this case we have \( \xi \in \Omega_1 \cap \Omega_2 \cap \Omega_3 \). It follows
\[
m_1(\xi) = ((1,0,\ldots),(0,0,\ldots)), \quad m_2(\xi) = ((0,1,0,\ldots),(0,0,\ldots)) \quad \text{and}
\]
\[
v(\xi) = ((0,0,\ldots),(0,1,0,\ldots)).
\]

(f) \( \xi \in \left[ -\frac{1}{14} + \frac{1}{2} \frac{1}{896}, -\frac{1}{14} + \frac{1}{2} \frac{1}{896} \right) \). Here we have \( \xi \in \Omega_1 \cap (\Omega_1 + \frac{1}{2}) \cap (\Omega_2 + \frac{1}{2}) \). It turns out
\[
m_1(\xi) = ((1,0,\ldots),(0,0,\ldots)), \quad m_2(\xi) = ((0,0,\ldots),(1,0,\ldots)) \quad \text{and}
\]
\[
v(\xi) = ((0,0,\ldots),(0,1,0,\ldots)).
\]

(4) \( \xi \in \frac{1}{2} \Omega_n \setminus \frac{1}{2} \Omega_{n+1}, n \geq 3, \ h_1(\xi) = n + 1 \).

Observe that the subcases (a), (b), (c) listed below are analogous to the preceding subcases (3)(d), (3)(e), (3)(f).

(a) \( \xi \in \left[ -\frac{1}{7} + \frac{1}{2} \frac{1}{896}, -\frac{1}{7} + \frac{1}{2} \frac{1}{896} \right) \). These points belong to the sets \( \Omega_i, i = 1,\ldots,n \), and \( (\Omega_1 + \frac{1}{2}) \). One finds
\[
m_1(\xi) = ((0,0,\ldots),(1,0,\ldots)), \quad m_2(\xi) = ((0,1,0,\ldots),(0,0,\ldots))
\]
and, for \( 2 \leq i \leq n \), \( m_i(\xi) = ((0,\ldots,0,1,0,\ldots),(0,0,\ldots)) \) with 1 on the \( i \)th position. Hence, \( v(\xi) = ((1,0,\ldots),(0,\ldots,0)) \).

(b) \( \xi \in \left[ -\frac{1}{7} + \frac{1}{2} \frac{1}{896}, -\frac{1}{7} + \frac{1}{2} \frac{1}{896} \right) \). In this case we have \( \xi \in \Omega_i, i = 1,\ldots,n+1 \). One finds \( m_i(\xi) = ((0,\ldots,0,1,0,\ldots),(0,0,\ldots)) \) with 1 on the \( i \)th position. Hence, \( v(\xi) = ((0,\ldots,0,1,0,\ldots),(0,0,\ldots)) \).

(c) \( \xi \in \left[ -\frac{1}{14} + \frac{1}{2} \frac{1}{896}, -\frac{1}{14} + \frac{1}{2} \frac{1}{896} \right) \). Here we have \( \xi \in \Omega_1 \) and \( \xi \in (\Omega_1 + \frac{1}{2}), i = 1,\ldots,n+1 \). It turns out
\[
m_1(\xi) = ((1,0,\ldots),(0,0,\ldots)), m_2(\xi) = ((0,0,\ldots),(1,0,\ldots))
\]
and, for \( 2 \leq i \leq n \), \( m_i(\xi) = ((0,\ldots,0,1,0,\ldots),(0,0,\ldots)) \) with 1 on the position \( n+1 \). This implies \( v(\xi) = ((0,\ldots,0,1,0,\ldots),(0,\ldots,0)) \) with 1 on the \( n \)th coordinate.

It remains to apply the equality \( v(\xi + \frac{1}{2}) = J(v(\xi)) \) to obtain the values of \( v \) on the interval \( \left[ 0, \frac{1}{2} \right) \), and, after that, to extend \( v \) by \( \mathbb{Z} \)-periodicity. As before, in applying Theorem 1.9 we will choose \( s \equiv 1 \). Then the formula \( \psi(2^k) = \sum_{n=1}^{\infty} v_n(\xi) \hat{\psi}_n(\xi) \) gives us the orthonormal wavelet \( \psi \) such that \( \hat{\psi}(2^k) = \chi_V(\xi) \) where the set \( V \) is the following union:
\[
\begin{align*}
\left[ \frac{1}{8}, \frac{3}{16} \right] \cup \left[ -\frac{3}{16}, -\frac{5}{32} \right] \cup \left[ -\frac{1}{2}, -\frac{3}{8} \right] \cup \left[ -\frac{5}{16}, -\frac{2}{7} \right] \cup \left[ -\frac{2}{7} + \frac{1}{2} \frac{1}{896}, -\frac{1}{4} \right] \\
\cup \left[ -\frac{1}{4}, -\frac{3}{16} \right] + 1 \cup \left[ -\frac{5}{32}, -\frac{1}{7} \right] + 1 \cup \left[ -\frac{1}{7} + \frac{1}{2} \frac{1}{896}, -\frac{1}{8} \right] + 1 \\
\cup \left[ \frac{1}{2} \right] \cup \left[ -\frac{3}{16}, -\frac{5}{32} \right] + \frac{1}{2} \cup \left[ -\frac{1}{7}, -\frac{1}{7} + \frac{1}{4} \frac{1}{896} \right] + \frac{1}{2} \\
\cup \left[ \frac{1}{2} \right] \cup \left[ -\frac{1}{4}, -\frac{3}{16} \right] + 1 \cup \left[ -\frac{5}{32}, -\frac{1}{7} \right] + 1 \cup \left[ -\frac{1}{7} + \frac{1}{4} \frac{1}{896}, -\frac{1}{8} \right] + 1 \\
\cup \left[ -\frac{2}{7} + \frac{1}{2} \frac{1}{896}, -\frac{1}{7} + \frac{1}{2} \frac{1}{896} \right] \cup \left[ -\frac{1}{7} + \frac{1}{4} \frac{1}{896}, -\frac{1}{7} + \frac{1}{4} \frac{1}{896} \right] + 7 \\
\cup \left[ \frac{3}{7} + \frac{1}{8} \frac{1}{896}, \frac{3}{7} + \frac{1}{8} \frac{1}{896} \right] + 3 \cup \left[ -\frac{2}{7} + \frac{1}{2} \frac{1}{896}, -\frac{1}{7} + \frac{1}{2} \frac{1}{896} \right] \\
\cup \left[ \frac{1}{7} + \frac{1}{4} \frac{1}{896}, -\frac{1}{7} + \frac{1}{4} \frac{1}{896} \right] + \frac{6 \cdot 8^{n-1} + 1}{7} \\
\cup \left[ \frac{3}{7} + \frac{1}{8} \frac{1}{896}, \frac{3}{7} + \frac{1}{8} \frac{1}{896} \right] + \frac{3 \cdot 8^{n-1} - 3}{7} \quad (4.21)
\end{align*}
\]

Finally, if we put \( W = 2V \), we can write \( \hat{\psi} = \chi_W \). Explicitly, the set \( W \) is equal to the following union:

\[
\left[-1, -\frac{3}{4}\right) \cup \left[-\frac{5}{8}, -\frac{1}{2}\right) \cup \left[-\frac{3}{8}, -\frac{5}{16}\right) \cup \left[\frac{1}{4}, \frac{3}{8}\right)
\]

\[
\bigcup_{j=1}^{\infty} \left( \frac{1}{2^j} \left[ -\frac{3}{8j}, -\frac{5}{16j} \right) + 1 \right) \bigcup_{j=2}^{\infty} \left( \frac{1}{2^j} \left[ -\frac{2}{7j}, -\frac{2}{7j} + \frac{1}{2j} \right) + 1 \right)
\]

\[
\bigcup_{j=0}^{\infty} \left( \frac{1}{2^j} \left[ -\frac{1}{2j} - \frac{3}{8}, -\frac{1}{2j} \right) + 2 \right) \bigcup_{j=0}^{\infty} \left( \frac{1}{2^j} \left[ -\frac{5}{16j}, -\frac{2}{7j} \right) + 2 \right) \bigcup_{j=0}^{\infty} \left( \frac{1}{2^j} \left[ -\frac{2}{7j} + \frac{1}{2j}, -\frac{1}{4j} \right) + 2 \right)
\]

\[
\bigcup_{n=2}^{\infty} \left( \frac{1}{7} \left[ -\frac{2}{7} + \frac{1}{2j} \frac{1}{8^n}, -\frac{2}{7} + \frac{1}{2j} \frac{1}{8^{n-1}j} \right) \right) + \frac{12 \cdot 8^{n-1} + 2}{7}
\]

\[
\bigcup_{n=2}^{\infty} \left( \frac{1}{7} \left[ \frac{6}{7} + \frac{1}{2j} \frac{1}{8^{n-1}j}, \frac{6}{7} + \frac{1}{2j} \frac{1}{8^{n-2}j} \right) \right) + \frac{6 \cdot 8^{n-1} - 6}{7}
\]

(4.22)

By Theorem 1.9, \( \hat{\psi} = \chi_W \), with \( W \) given by (4.22), is an orthonormal wavelet. Using Theorem 1.8, we conclude that \( D_{\psi} = D \), where \( D \) is the function introduced in Example 4.4.

**Remark 4.5.** At the end, we note that one can get by a similar construction a family of wavelets \( \psi_n, n \geq 2 \), such that the maximal value of \( D_{\psi_n} \) is \( n, \forall n \geq 2 \).

One can also modify the preceding construction to obtain a non-orthogonal Parseval frame wavelet whose dimension function is essentially unbounded.

The details will appear elsewhere.

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**References**


