Solving singularly perturbed boundary-value problems by spline in tension

K. SURLA and M. STOJANOVIC
Institute of Mathematics, University of Novi Sad, Novi Sad, Yugoslavia

Received 20 January 1988
Revised 25 April 1988

Abstract: The difference scheme via spline in tension for the problem: \(- \epsilon y'' + p(x) y = f(x), p(x) > 0, y(0) = a_0, y(1) = a_1\), is derived. The error of the form \(O(h \min(h, \sqrt{\epsilon})\) is obtained. When \(p(x) = p = \text{const.}\), the corresponding spline in tension achieves the second order of the global uniform convergence.

Keywords: Spline in tension, singular perturbation, uniform convergence, spline difference scheme.

1. Introduction

The application of exponential splines for numerical solution of the singularly perturbed boundary problems has been described in many papers ([3], [4], [5], [8] etc.). Because the problems analyzed in these papers were more complex than ours, in none of them a uniform convergence was achieved. We consider the self-adjoint singularly perturbed problem

\[ L y = -\epsilon y'' + p(x) y = f(x), \quad p(x) > p > 0, \quad 0 < x < 1, \]
\[ y(0) = a_0, \quad y(1) = a_1, \quad a_0, a_1 \in \mathbb{R}, \]  

and for its solution we use the technique from [8] which is related to a non-self-adjoint problem. The approximate solution of the problem (1) when \(p(x) = p = \text{const.}\) we seek in the form of function \(S(x)\) which on each interval \([x_{j-1}, x_j]\) is a solution (denoted by \(S_j(x)\)) of the differential equation

\[-\epsilon S_j''(x) + p S_j(x) = \frac{x - x_j}{h} \left( -\epsilon M_j + p u_j \right) + \frac{x_j - x}{h} \left( -\epsilon M_{j-1} + p u_{j-1} \right)\]

where

\[ S_j''(x_j) = M_j, \quad S_j(x_j) = u_j, \]
\[ x \in [x_{j-1}, x_j], \quad x_j = jh, \quad j = 0(1)n, \quad h = 1/n. \]

Solving this equation we obtain

\[ S_j(x) = u_j t + (1 - t) u_{j-1} + M_j \frac{h^2}{\rho^2} \left( \frac{\text{sh} \rho t}{\text{sh} \rho} - t \right) + M_{j-1} \frac{h^2}{\rho^2} \left( \frac{\text{sh} \rho (1 - t)}{\text{sh} \rho} - (1 - t) \right), \]
\[ t = \frac{x - x_{j-1}}{h}, \quad \tilde{\rho} = \left( \frac{p}{\epsilon} \right)^{1/2}, \quad \rho = \tilde{\rho} h, \]
which is a tension spline [6], and $\tilde{\rho}$ is a tension parameter. In the rest of the paper we will use $S(x, \rho)$ instead of $S(x)$ in order to stress the dependence of $\rho$. When $\tilde{\rho} = 0$, $S(x, \rho)$ is a cubic spline and when $\tilde{\rho} \to \infty$, $S(x, \rho)$ tends to a linear spline. For some other properties of the tension spline see [6].

The corresponding difference scheme (4) is derived in Section 2. The exponentially fitted cubic spline difference scheme for the problem (1) is derived in [7] and the second order of accuracy is proved. This property has also the classical difference scheme [2].

The three mentioned schemes have approximately the same numerical results at the grid points but the estimate for the scheme (4) is optimal, in the sense of [2]. The advantage of the spline in tension over the cubic spline is a uniform convergence between the grid points, but obtained only in the case $p(x) = \rho = \text{const.}$

2. Derivation of the scheme

Because of $S(x, \rho) \in C^2[0, 1]$, we have

$$S'_j(x_j, \rho) = S'_{j+1}(x_{j+1}, \rho).$$

If we replace (2) in (3) and after that if we put $\epsilon M_j = p_j u_j - f_j$, we obtain the difference scheme

$$Ru_j = Qf_j, \quad j = 1(1)n - 1,$$

where

$$Ru_j = r_j^- u_{j-1} + r_j^c u_j + r_j^+ u_{j+1},$$

$$Qf_j = q_j^- f_{j-1} + q_j^c f_j + q_j^+ f_{j+1},$$

$$u_0 = \alpha_0, \quad u_n = \alpha_1$$

$$r_j^c = -\frac{\rho}{\sh \rho}, \quad r_j^+ = -2 \rho \cosh \rho, \quad q_j^c = -\frac{1}{p} \left(1 - \frac{\rho}{\sh \rho}\right),$$

$$q_j^+ = -\frac{2}{p} \left(-1 + \rho \cosh \rho\right).$$

When it will be clear from the context the subscripts in $r_j^\pm$ and $q_j^\pm$ will be omitted.

In the case $p(x) = \text{const.}$ we define

$$r_j^c = -\frac{p_j \pm 1}{\sh p_j}, \quad r_j^- = -2 p_j \cosh p_j, \quad q_j^c = -\frac{1}{p_j \pm 1} \left(1 - \frac{p_j \pm 1}{\sh p_j}\right),$$

$$q_j^+ = -\frac{2}{p_j} \left(-1 + p_j \cosh p_j\right), \quad p_j = p(x_j), \quad f_j = f(x_j).$$

The local truncation error $\tau_j(\phi)$ of scheme (4), for an arbitrary sufficiently smooth, function $\phi(x)$, is defined by

$$\tau_j(\phi) = R\phi_j - Q(L\phi)_j.$$
The scheme (4), (5) can be written in the matrix form:

\[ Au = F \]

where \( A \) is a matrix of the system (4), \( u \) and \( F \) are corresponding vectors. Because of \( R(y_j - u_j) = \tau_f(y) \) we have

\[ \max_j |y_j - u_j| \leq \| A^{-1} \| \max_j |\tau_f(y)|. \]  \hfill (6)

In the order to estimate values \( |y_j - u_j| \), in the next section we will estimate the truncation error and the norm of matrix \( A^{-1} \).

3. Proof of the uniform convergence at the grid points

The following lemma gives the properties of the exact solution which are important for the proof.

\textbf{Lemma 1} \[2\]. Let \( y(x) \in C^4[0, 1] \). Let \( p'(0) = p'(1) = 0 \). Then the solution of the problem (1) has the form

\[ y(x) = v(x) + w(x) + g(x) \]

where

\[ v(x) = q_0 \exp(-x(\rho(0)/\epsilon)^{1/2}), \]

\[ w(x) = q_1 \exp(-(1-x)(\rho(1)/\epsilon)^{1/2}), \]

\( q_0 \) and \( q_1 \) are bounded functions of \( \epsilon \) independent of \( x \) and

\[ |g^{(i)}(x)| \leq N(1 + \epsilon^{1-i/2}), \quad i = 0(1)4, \]

\( N \) is a constant independent of \( \epsilon \).

From Lemma 1 we have

\[ \tau_f(y) = \tau_f(v) + \tau_f(w) + \tau_f(g) \]  \hfill (8)

and we will estimate separately the parts of \( \tau_f(y) \).

We start with \( v(x) \).

Let \( h^2 \ll \epsilon \). Introduce the notation:

\[ r_j^- = r^-(_{\rho_j-1}), \quad r_j^+ = r^+(_{\rho_j+1}), \quad r_j^c = r^c(_{\rho_j}), \]

\[ Rv_j = v_j \left[ r^-(_{\rho_j}) \exp(\rho_0) + r^c(_{\rho_j}) + r^+(_{\rho_j}) \exp(-\rho_0) \right] \]

\[ + v_j \left[ (r^-(_{\rho_{j+1}}) - r^-(_{\rho_j})) \exp(\rho_0) + (r^+(_{\rho_{j+1}}) - r^+(_{\rho_j})) \exp(-\rho_0) \right] \]

\[ = 2v_j \frac{\rho_j}{\rho_j} (\text{ch } \rho_0 - \text{ch } \rho_j) + O(h^4/\epsilon) = v_j \frac{h^2}{\epsilon} (\rho_0 - \rho_j) + O(h^4/\epsilon). \]  \hfill (9)

Throughout the paper \( M \) denotes positive constants that may take different values in different
formulas, but that are always independent of ε and h. All constants in the asymptotic equalities are independent of ε and h.

\[ Q(Lv)_j = 2v_j(p_0 - p_j) \frac{1}{p_j} \left[ \left( 1 - \frac{\rho_j}{\text{sh} \rho_j} \right) c \rho_0 - 1 + \rho_j \text{cth} \rho_j \right] + \mathcal{O}(h^4/\varepsilon) \]

\[ = v_j \left( \frac{h^2}{\varepsilon} (p_0 - p_j) + \mathcal{O}(h^4/\varepsilon) \right). \quad (10) \]

From (8) and (9) we have

\[ |\tau_j(v)| = |Rv_j - Q(Lv)_j| \leq M h^4/\varepsilon. \quad (11) \]

In the similar way we obtain

\[ |\tau_j(w)| \leq M h^4/\varepsilon. \quad (12) \]

Further,

\[ \tau_j(g) = T_{2j} g_j'' + T_{3j} g_j''' + r^- h^4 \frac{h^4}{4!} y''''(\xi_1) + r^+ h^4 \frac{h^4}{4!} y''''(\xi_2) + \varepsilon q^- h^2/2 y''''(\xi_3) \]

\[ - p_{j-1} h^4 \frac{h^4}{4!} y''''(\xi_4) q^- + \varepsilon q^+ h^2/2 y''''(\xi_5) - p_{j+1} h^4 \frac{h^4}{4!} q^+ y''''(\xi_6), \]

\[ x_{j-1} \leq \xi_i \leq x_{j+1}, \quad i = 0(1)6 \]

\[ T_{2j} = (r^- + r^+) \frac{h^2}{2} + \varepsilon (q^- + q^+ + q^c) - (q^+ r_{j-1} + q^+ r_{j+1}) \frac{h^2}{2}, \]

\[ T_{3j} = (r^+ - r^-) \frac{h^3}{3!} + h \varepsilon (q^+ - q^-) + (p_{j-1} q^+ - p_{j+1} q^+ - q^-) \frac{h^3}{3!}. \quad (13) \]

Since \(|r^+ + c| \leq M, |q^+ + c| \leq Mh^2/\varepsilon, \ |q^- - q^+| \leq Mh^4/\varepsilon^2, \ |r^+ - r^-| \leq Mh^3/\varepsilon, \) from (7) we have \(|T_{2j} g_j''| \leq M h^4/\varepsilon. \)

After some Taylor developments using \(x/\text{sh} x = 1 - x^2/6 + \mathcal{O}(x^4), \)

\[ T_{2j} = (r^- + r^+) \frac{h^2}{2} + \varepsilon (q^- + q^+ + q^c) - (q^+ r_{j-1} + q^+ r_{j+1}) \frac{h^2}{2}, \]

\[ T_{3j} = (r^+ - r^-) \frac{h^3}{3!} + h \varepsilon (q^+ - q^-) + (p_{j-1} q^+ - p_{j+1} q^+ - q^-) \frac{h^3}{3!}. \quad (13) \]

Since, absolute values of the remainder terms in (13) are bounded by \(M h^4/\varepsilon, \) we have

\[ |\tau_j(g)| \leq M h^4/\varepsilon \] and ((8), (11), (12))

\[ |\tau_j(v)| \leq M h^4/\varepsilon \quad \text{when} \quad h^2 \leq \varepsilon. \quad (14) \]

Let \(\varepsilon \leq h^2, \)

\[ \tau_j(v) = Rv_j - Q(Lv)_j. \]

Putting \(p_j = p_{j-1} = p_0 = p(0) \) in \(Rv_j \) we obtain that \(Rv_j = 0. \) This expression we will denote by \(\tilde{R}v_j. \) Thus

\[ Rv_j = Rv_j - \tilde{R}v_j = (r^- (\rho_0) - r^-) v_{j-1} + (r^c (\rho_0) - r^c) v_j + (r^+ (\rho_0) - r^+) v_{j+1} \]
Since \( |r^+(\rho_0) - r^+| \leq Mx_j^2 \), \( |r^-(\rho_0) - r^-| \leq Mx_j^2 \), \( |r^c(\rho_0) - r^c| \leq Mx_j^2 \) we have \( |Rv_j| \leq Me \). Because of
\[
|q^\pm| \leq M, \quad |q^c| \leq \frac{Mh^2}{\sqrt{\epsilon} (\sqrt{\epsilon} + h)} \tag{15}
\]
we obtain \( |Q(Lv)_j| \leq Me \) and
\[
|\tau_j(v)| \leq Me. \tag{16}
\]
The same estimate holds for \( \tau_j(w) \),
\[
|\tau_j(w)| \leq Me. \tag{17}
\]
For \( \tau_j(g) \) we use the form:
\[
\tau_j(g) = r^{-} \frac{h^2}{2} g''(b_1) + r^{+} \frac{h^2}{2} g''(b_2) + q^- p_{j-1} \frac{h^2}{2} g''(b_1)
- q^+ p_{j+1} \frac{h^2}{2} g''(b_2) - \epsilon ( q^- g_{j-1}'' + q^+ g_{j+1}'') \tag{18}
\]
where \( x_{j-1} \leq b_1 \leq x_j \leq b_2 \leq x_{j+1} \).
From Lemma 1, (15) and \( |r^{\pm e}| \leq M \) we have
\[
|\tau_j(g)| \leq Mh \left( h + \sqrt{\epsilon} \right) \tag{19}
\]
Finally, from (16), (17), (18) we conclude that the estimate
\[
|\tau_j(y)| \leq Mh^2 \quad \text{for } \epsilon \leq h^2
\]
is valid.

The rest of the proof is related to the estimate of \( \|A^{-1}\| \) (see (6)).
Since \( r_j^c < 0 \), \( r_j^{\pm e} > 0 \), we have
\[
\|A^{-1}\| \leq \max_j |r_j^- + r_j^c + r_j^+|^{-1},
\]
\[
|r_j^- + r_j^c + r_j^+| = \left| \frac{\rho_j}{sh \rho_j} (-2ch \rho_j + 2) + \frac{\rho_{j+1}}{sh \rho_{j+1}} \frac{2 \rho_j}{sh \rho_j} + \frac{\rho_{j-1}}{sh \rho_{j-1}} \right|
\geq N_1 \frac{\rho_j^3}{sh \rho_j} \geq N_2 \frac{h^2}{\epsilon}
\]
when \( h^2 \leq \epsilon \).
In the opposite case we obtain
\[
|r_j^- + r_j^c + r_j^+| \geq N_4 \rho_j, \quad \text{cth } \rho_j \geq N_5 \rho_j.
\]
\( N_i, i = 1(1)5 \) are constants independent of \( \epsilon \) and \( h \).
Thus, there is a constant \( M \) such that:
\[
\|A^{-1}\| \leq \begin{cases} 
M\epsilon/h^2, & h^2 \leq \epsilon, \\
M\sqrt{\epsilon}/h, & \epsilon \leq h^2.
\end{cases} \tag{20}
\]
Replacing (20), (19) and (14) in (6) we get the following theorem.
Theorem 1. Let \( p(x), f(x) \in C^2[0, 1] \) and \( p(x) \geq p > 0, \quad p'(0) = p'(1) = 0. \) Let \( u_j, \ j = 0(1)n, \) be the approximation to the solution of (1) obtained using (4), (5). Then, there is a constant \( M \) independent of \( \epsilon \) and \( h, \) such that

\[
|y(x_j) - u_j| \leq Mh \min(h, \sqrt{\epsilon}), \quad j = 0(1)n.
\]

4. The convergence between the grid points

Let \( p(x) = p = \text{const}. \) Then spline \( S(x, \rho) \) has the basis: 1, \( x, \exp(-\rho x), \exp(\rho x). \) So the truncation errors of the scheme for the functions \( v \) and \( w \) are equal to zero.

Thus, the corresponding spline is an interpolation tension spline for the boundary layer functions \( v \) and \( w. \) Because of linearity, we have

\[
S(x, \rho, y) = S(x, \rho, u) + S(x, \rho, w) + S(x, \rho, g),
\]

\[
|S(x, \rho, y) - y| \leq |S(x, \rho, v) - v| + |S(x, \rho, w) - w| + |S(x, \rho, g) - g|,
\]

\[
S(x, \rho) = S(x, \rho, u). \tag{21}
\]

Taking into account the basis of spline \( S(x, \rho) \) and the analytical form of the functions \( v \) and \( w \) (Lemma 1) we can conclude that the first two terms in (21) are equal to zero.

The last one will be estimated. According to Theorem 1 and (2) we have

\[
y_j = u_j + O(h^2), \quad \epsilon y''(x_i) = \epsilon M_i + O(h^2)
\]

and

\[
S(x, \rho, y) = S(x, \rho, u) + O(h^2). \tag{22}
\]

The difference from the interpolation tension spline \( S(x, \rho, y) \) and collocation tension spline \( S(x, \rho, u) \) is \( O(h^2). \)

For \( S(x, \rho, g) \) we can form an interpolation cubic spline \( S_k(x, \rho, g) \), with boundary conditions

\[
S_k''(0, \rho, g) = g''(0) \quad \text{and} \quad S_k''(1, \rho, g) = g''(1).
\]

According to [6]

\[
|S(x, \rho, g) - S_k(x, \rho, g)| \leq M\rho^3h^4 \max_j |S_k''(x_j, \rho, g)| \leq Mh^2 \tag{23}
\]

for \( h^2 \leq \epsilon. \)

Namely, \( |g''(x_j)| \leq M \) and from the form of the cubic spline [1] we can see that \( |S''(x_j, \rho, g)| \leq M. \)

Let \( l_i(x) = g_{i-1}(x_i - x)/h + g_i(x - x_{i-1})/h \) be linear interpolant for \( g(x) \) on the interval \([x_{i-1}, x_i]. \) Then the inequality

\[
|S(x, \rho, g) - l_i(x)| \leq 8\rho^{-1} \max_j |F_j|
\]

where

\[
F_j = \frac{(g_j - g_{j-1})}{h} - \frac{(g_{j-1} - g_{j-2})}{h} = h g''(\xi), \quad x_{j-2} \leq \xi \leq x_j,
\]
holds for \( \bar{\rho} \) sufficiently large, i.e. \( \bar{\rho} \geq N_0 \), \( N_0 \) is a fixed constant independent of \( h \) and \( \epsilon \) [6]. In our case it means

\[
|S(x, \rho, g) - l_i(x)| \leq M_1 \epsilon h \quad \text{for } h \geq N_0 \sqrt{\epsilon}.
\]  

(24)

Let \( h \leq N_0 \sqrt{\epsilon} \). Then,

\[
|S(x, \rho, g) - g| \leq |S(x, \rho, g) - S_k(x, \rho, g)| + |S_k(x, \rho, g) - g|.
\]

Since,

\[
|S_k(x, \rho, g) - g| \leq Mh^4(1 + \epsilon^{-1}) \quad \text{for } g \in C^4[0, 1],
\]

from (23) we have

\[
|S(x, \rho, g) - g| \leq Mh^4(\bar{\rho}^2 + \epsilon^{-1}) \leq M \frac{h^4}{\epsilon}, \quad h \leq N_0 \sqrt{\epsilon}.
\]  

(25)

Also, from (24) we have

\[
|S(x, \rho, g) - g| \leq |S(x, \rho, g) - l_i(x)| + |l_i(x) - g| \leq Mh^2
\]  

(26)

for \( N_0 \sqrt{\epsilon} \leq h \).

Thus, the following theorem holds.

**Theorem 2.** Let in (1) \( p(x) = p > 0 \), \( f(x) \in C^2[0, 1] \). Then, the estimate

\[
|S(x, \rho, u) - y(x)| \leq M h^2,
\]

where \( M \) is a constant independent of \( \epsilon \) and \( h \), is valid.

**Proof.** From (25), (26), (22), (21) and above investigation related to functions \( w \) and \( v \), we have

\[
|S(x, \rho, u) - y(x)| \leq |S(x, \rho, u) - S(x, \rho, y)| + |S(x, \rho, y) - y(x)|
\]

\[
\leq M h^2.
\]

When \( p(x) \neq 0 \) we introduce the variable tension parameter \( \bar{\rho}_j = (\frac{p_j}{\epsilon})^{1/2} \) and we define tension spline \( S(x, \rho_j, u) \) in the following way

\[
S(x, \rho_j, u) = u_{j+1} + (1 - t)u_{j-1} + M_j \frac{h^2}{\rho_j^2} \left( \frac{\text{sh} \rho_j (1 - t)}{\rho_j} - t \right)
\]

\[
+ M_{j-1} \frac{h^2}{\rho_j^2} \left( \frac{\text{sh} \rho_{j-1} (1 - t)}{\rho_{j-1}} - (1 - t) \right),
\]

\[
t = \frac{x - x_{j-1}}{h}, \quad \rho_j = h \left( \frac{p_j}{\epsilon} \right)^{1/2}.
\]

This spline satisfies the differential equation (1) at the grid points. \( \square \)

5. Numerical experiments

The numerical results which support the predicted theory are based on the test of uniform convergence which is described in [2] on uniform grid.
Table 1

<table>
<thead>
<tr>
<th>[k]\</th>
<th>(\frac{\pi}{2})</th>
<th>(\frac{\pi}{4})</th>
<th>(\frac{\pi}{8})</th>
<th>(\frac{\pi}{16})</th>
<th>(\frac{\pi}{32})</th>
<th>(\frac{\pi}{64})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\epsilon = 1)</td>
<td>(0.199 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
</tr>
<tr>
<td>(\epsilon = 1/2)</td>
<td>(0.199 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
</tr>
<tr>
<td>(\epsilon = 1/4)</td>
<td>(0.199 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
</tr>
<tr>
<td>(\epsilon = 1/8)</td>
<td>(0.199 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
</tr>
<tr>
<td>(\epsilon = 1/16)</td>
<td>(0.199 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
</tr>
<tr>
<td>(\epsilon = 1/32)</td>
<td>(0.196 \times 10^1)</td>
<td>(0.199 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
</tr>
<tr>
<td>(\epsilon = 1/64)</td>
<td>(0.193 \times 10^1)</td>
<td>(0.198 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.198 \times 10^1)</td>
</tr>
<tr>
<td>(\epsilon = 1/128)</td>
<td>(0.198 \times 10^1)</td>
<td>(0.199 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.199 \times 10^1)</td>
</tr>
<tr>
<td>(\epsilon = 1/256)</td>
<td>(0.197 \times 10^1)</td>
<td>(0.199 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.199 \times 10^1)</td>
</tr>
<tr>
<td>(\epsilon = 1/512)</td>
<td>(0.194 \times 10^1)</td>
<td>(0.199 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.200 \times 10^1)</td>
<td>(0.199 \times 10^1)</td>
</tr>
</tbody>
</table>

Table 2

<table>
<thead>
<tr>
<th>(\epsilon)</th>
<th>(n = 32)</th>
<th>(64)</th>
<th>(128)</th>
<th>(256)</th>
<th>(512)</th>
<th>(1024)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\epsilon = 1/64)</td>
<td>(0.159 \times 10^{-2})</td>
<td>(0.401 \times 10^{-3})</td>
<td>(0.100 \times 10^{-3})</td>
<td>(0.251 \times 10^{-4})</td>
<td>(0.627 \times 10^{-5})</td>
<td>(0.157 \times 10^{-5})</td>
</tr>
<tr>
<td>(\epsilon = 1/512)</td>
<td>(0.159 \times 10^{-2})</td>
<td>(0.401 \times 10^{-3})</td>
<td>(0.100 \times 10^{-3})</td>
<td>(0.251 \times 10^{-4})</td>
<td>(0.627 \times 10^{-5})</td>
<td>(0.157 \times 10^{-5})</td>
</tr>
<tr>
<td>(\epsilon = 1/10000)</td>
<td>(0.775 \times 10^{-3})</td>
<td>(0.297 \times 10^{-3})</td>
<td>(0.918 \times 10^{-4})</td>
<td>(0.245 \times 10^{-4})</td>
<td>(0.624 \times 10^{-5})</td>
<td>(0.157 \times 10^{-5})</td>
</tr>
<tr>
<td>(\epsilon = 1/100000)</td>
<td>(0.288 \times 10^{-3})</td>
<td>(0.135 \times 10^{-3})</td>
<td>(0.574 \times 10^{-4})</td>
<td>(0.204 \times 10^{-4})</td>
<td>(0.519 \times 10^{-5})</td>
<td>(0.155 \times 10^{-5})</td>
</tr>
</tbody>
</table>

The exponential scheme (4) is applied to the test equation [2]:

\[-\epsilon y'' + y = -\cos^2\pi x - 2\epsilon \pi^2 \cos 2\pi x, \quad u(0) = u(1) = 0.\]

Its exact solution is:

\[y(x) = \frac{(\exp(-1/(\epsilon^2)) + \exp(-x/\sqrt{\epsilon}))/\left(1 + \exp\left(-1/\sqrt{\epsilon}\right)\right)}{\left(1 + \exp\left(-1/\sqrt{\epsilon}\right)\right)} - \cos^2\pi x.\]

Table 1 contains the numerical rate of convergence which is determined as in [2]:

\[\text{rate} = \frac{(\ln z_{k,c} - \ln z_{k+1,c})}{\ln 2},\]

where

\[z_{k,c} = \max_j |u_j^{h/c} - u_j^{h/2c}|, \quad c = 2^k, \quad k = 1(1)5,\]

and \(u_j^{h/c}\) denotes the value of \(u_j\) for the mesh length \(h/c\), \(h = 1/k\), \(p_v\) is the average of rates corresponding to one value of \(\epsilon\). Table 1 shows that the rate 2 when \(\epsilon > h^2\). The numerical rate agrees with a theoretical one.

Table 2 contains the maximum error at all mesh points:

\[\max_j |y(x_j) - u_j| \quad \text{for different} \ n \ \text{and} \ \epsilon, \ n = 1/h.\]

It indicates the convergence in \(\epsilon\).

References


