ELSEVIER Journal of Pure and Applied Algebra 165 (2001) 169–182
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Unique decompositions into ideals for Noetherian domains

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Received 7 September 1999; received in revised form 16 August 2000
Communicated by A. Blass

Abstract

A commutative ring R has the *unique decompositions into ideals* (*UDI*) property if, for any module L that decomposes into a finite direct sum of ideals, the decomposition of L into ideals is unique apart from the order of the ideals. We characterize the UDI Noetherian integral domains.

MSC: Primary 13C05; secondary 20K15

1. Introduction

A commutative ring R is said to have the *unique decompositions into ideals* (*UDI*) property, if, for any module L that decomposes into a finite direct sum of ideals, the decomposition of L into ideals is unique apart from the order of the ideals. That is, for any ideals $I_1,\dots,I_n,J_1,\dots,J_m$ of R, if $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_m$, then $n = m$ and after reindexing, $I_j \cong J_j$ for each j. If R is an integral domain, then it is easy to see that if $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_m$, then $n = m$. Thus, for integral domains, the force of UDI is the assertion that there is a reindexing of indices such that $I_j \cong J_j$ for all j.

In this article, we study UDI for Noetherian integral domains. *Throughout the paper, R always represents a Noetherian integral domain with quotient field Q, and $\bar{R}$ denotes the integral closure of R in Q.*

In Section 2 we show that the issue of characterizing Noetherian UDI domains breaks down nicely into the problem of determining local UDI domains. Specifically, R has

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0022-4049/01/$ - see front matter
PII: S0022-4049(00)00182-1

UDI if and only if R is a PID or R has exactly one nonprincipal maximal ideal M and R_M has UDI. Section 3 contains an explicit description of when a local domain R has UDI, given in terms of the splitting of its maximal ideal in certain finitely generated overrings of R inside the integral closure $\bar{R}$ of R. Our results afford a characterization of the Noetherian domains R such that every finitely generated torsion-free module decomposes uniquely into the direct sum of ideals of R, as rings such that every ideal is 2-generated and all but at most one maximal ideal is principal.

Recall that an R-submodule of Q is said to have *rank one* as a module. We also show that for Noetherian domains of Krull dimension 1, the Krull–Schmidt property for ideals implies the Krull–Schmidt property for rank one modules, thus relating UDI to a theorem of R. Baer on rank one modules over a PID. More precisely, a one-dimensional Noetherian domain has UDI if and only if whenever $X_1 \oplus \cdots \oplus X_n \cong Y_1 \oplus \cdots \oplus Y_m$ for rank one modules X_i, Y_j, then $n = m$ and after reindexing, $X_i \cong Y_i$ for all i. That is, decompositions into rank one modules are unique when decompositions into ideals are unique.

In the recent pair of articles [3,4], Levy and Odenthal analyze the Krull–Schmidt property for finitely generated modules of semiprime, module-finite algebras over commutative Noetherian rings having Krull dimension one. In particular, they characterize when such an algebra possesses the torsion-free Krull–Schmidt property (TFKS): if $A_1, \ldots, A_n, B_1, \ldots, B_m$ are indecomposable, finitely generated torsion-free Λ-modules such that $A_1 \oplus \cdots \oplus A_n \cong B_1 \oplus \cdots \oplus B_m$, then $n = m$ and after rearrangment, $A_i \cong B_i$ for all i. Our schema is somewhat different in that while we restrict to (commutative) integral domains, we allow $\Lambda = R$ to be any (not necessarily one-dimensional) Noetherian domain. And although our approach to UDI is quite different from the approach of Levy and Odenthal to the deeper property of TFKS, there is commonality among the results obtained in the one-dimensional case. In fact, comparing Theorem 1.3 of [3] and Theorem 3.2 of the present paper, one sees that the class of one-dimensional Noetherian domains satisfying TFKS is tightly contained in the class of UDI one-dimensional Noetherian domains. This containment, however, is proper: Example 4.4 guarantees the existence of a one-dimensional local domain which has UDI and whose integral closure has precisely 3 maximal ideals. Such a ring cannot satisfy TFKS [3, Theorem 3.1].

All of the modules mentioned below are torsion-free and have finite rank, where the rank of A is defined as the dimension of the Q-vector space $Q \otimes_R A$. The R-module endomorphism ring of a module A will be denoted by $E(A)$.

2. Reduction to the local case

Lemma 2.1. *If R has UDI, then all but at most one maximal ideal of R is principal.*

Proof. If R is not local and M_1 and M_2 are distinct maximal ideals of R, $M_1 \oplus M_2 \cong R \oplus M_1 \cap M_2$. Because R has UDI, one of M_1 or M_2 is principal. Thus there cannot be 2 nonprincipal maximal ideals of R. □

Recall that a domain R is called *h-local*, if (i) every nonzero element of R is contained in only finitely many maximal ideals of R, and (ii) each nonzero prime ideal of R is contained in a unique maximal ideal of R. Equivalent to the definition of h-local is the condition, for each maximal ideal M of R, $R_{[M]} \cdot R_M = Q$, where $R_{[M]} = \bigcap\{R_N \mid N$ is a maximal ideal different from $M\}$ [6]. In case R is Noetherian, (ii) implies (i); this can be seen by considering primary decompositions of an ideal.

Lemma 2.2. *If M is a maximal ideal of R such that every maximal ideal other than M is principal, then R is h-local, and $R_{[M]}$ is a PID that is also a flat R-module.*

Proof. If P be a nonzero prime ideal contained in a principal maximal ideal N, then $P = N$ by the Krull Principal Ideal Theorem. Otherwise, P is contained in M exclusively, and it follows that R is h-local. Also, $R_{[M]}$ is a flat R-module because the localizations of $R_{[M]}$ at maximal ideals of R are either $R_{[M]}R_M = Q$, or R_N where $N \neq M$. Furthermore, the maximal ideals of $R_{[M]}$ are $N \cdot R_{[M]}$, where N is a principal maximal ideal of R, so $R_{[M]}$ is a PID. □

A fundamental idea in the study of torsion-free abelian groups can be adapted to our setting and will play an important role in the proof below. Two torsion-free modules G and H are called *nearly isomorphic*, if for each ideal $I \neq 0$, there is an embedding $f : G \to H$ such that $J = \text{ann Coker } f$ is a nonzero ideal of R comaximal with I (i.e., $I + J = R$). Clearly, if G and H are isomorphic torsion-free modules, then G and H are nearly isomorphic. Also, it is not hard to check that near isomorphism implies genus isomorphism, i.e. if G is nearly isomorphic to H, then G_M is isomorphic to H_M for all maximal ideals M of R.

A consequence of Proposition 2.4, which shows the strength of the UDI hypothesis, is that isomorphism, near isomorphism and genus isomorphism all coincide for direct sums of ideals over UDI Noetherian domains. This property of UDI domains is special since for example any ideal in a Dedekind domain is nearly isomorphic to a principal ideal, yet not every ideal of a Dedekind domain need be principal.

Lemma 2.3. *Suppose R has UDI and that G and H are finitely generated torsion-free modules. If $G_M \cong H_M$ for all maximal ideals M of R, then G is nearly isomorphic to H.*

Proof. From Theorem 7.11 in [8], due to the fact that G, H are finitely generated, $Hom(G,H)_N = Hom(G_N, H_N)$ for each maximal ideal N. Let M be a fixed maximal ideal of R such that every other maximal ideal is principal (Lemma 2.1). In light of Lemma 2.2, $R_{[M]}$ is a flat R-module, so from Theorem 7.11 again, $Hom(G,H)_{[M]} = Hom(G_{[M]}, H_{[M]})$.

There is a monomorphism $f : G \to H$ such that $f_M : G_M \to H_M$ is an isomorphism. Since G and H are finitely generated and torsion-free, and for any other maximal ideal N, R_N is a DVR, $G_N \cong H_N$, and consequently there exists a monomorphism $h : G \to H$ such that h_N is an isomorphism.

Given a nonzero ideal I, let $N_0=M, N_1,\dots,N_k$ be distinct maximal ideals containing I along with M (Lemma 2.2). For each $j\geq 0$ there exists $f_j: G\to H$, such that $(f_j)_{N_j}$ is an isomorphism. With $L=Hom(G,H)$, the submodules $N_0\cdot L,\dots,N_k\cdot L$ are comaximal in L, and so are $N_j\cdot L$ and $(\Pi_{i\neq j}N_i)\cdot L$ for each j. Mimicking the proof of the Chinese Remainder Theorem in [2], we can find a map $g: G\to H$ such that g_{N_j} is an isomorphism for each j. Since $H/g(G)$ is a finitely generated torsion module that is locally zero at each N_j, $ann_R\ H/g(G)$ is an ideal of R relatively prime to I. □

Proposition 2.4. *Assume that R has UDI and that R has exactly one nonprincipal maximal ideal M. Suppose G is a finitely generated torsion-free module and H is completely decomposable. Then the following statements are equivalent*:

(1) $G_M\cong H_M$.
(2) *G and H are nearly isomorphic.*
(3) *G and H are isomorphic.*

Proof. That (1) ⇒ (2) is given by Lemma 2.3. Claim (3) ⇒ (1) is immediate, so it remains to show (2) ⇒ (3). We prove this in the rank one case first. Suppose I and J are two ideals of R such that $I_M\cong J_M$. By Lemma 2.3, I and J are nearly isomorphic. Let $f_1: I\to J$ be such that $(f_1)_M$ is an isomorphism, and let $f_2: I\to J$ be such that the ideals $A_j=ann\,Coker\,f_j$ are relatively prime. There exists $a_j\in A_j$ such that $a_1+a_2=1$. Define $f: I\oplus I\to J$ by $f(x,y)=f_1(x)+f_2(y)$. Then f is split by $g: J\to I\oplus I$ where $g(x)=(g_1(x),g_2(x))$ and $g_j=f_j^{-1}\cdot\mu_{a_j}$ for $\mu_{a_j}=$ multiplication on J by a_j. Since R has UDI, $I\cong J$, proving the claim for the rank one case.

Write $H=J_1\oplus\cdots\oplus J_n$ for ideals $J_1,\dots,J_n$ of R, and set $H_1=J_1\oplus\cdots\oplus J_{n-1}$. Since G and H are nearly isomorphic, $G_M\cong H_M$ and there is a map $f: G\to H$ such that f_M is an isomorphism. With $\pi: H\to H_1$ taken to be the coordinate projection, by Lemma 2.3, the module $X=\pi\cdot f(G)$ is nearly isomorphic to H_1 because $X_M=(H_1)_M$, and any two torsion-free modules of the same rank over a PID are isomorphic. We will show that $G\cong X\oplus J_n$, establishing the theorem by induction. Call $f'=\pi\cdot f$.

There exists $g_1\in Hom(X_M,G_M)$ such that $(f'\cdot g_1)_M=1_{X_M}$. Set $K=Ker\,f'$. If $0\neq r\in R$ is such that $rg_1: X\to G$, then the torsion module $(g_1(X)+K)/K$ is bounded by r, and consequently this torsion module has a nontrivial annihilator A_1. Additionally, $A_1\nsubseteq M$ because $(f'\cdot g_1)_M=1_{X_M}$. On the other hand, $f'_{[M]}: G_{[M]}\to X_{[M]}$ splits with splitting map $g_2\in Hom(X_{[M]},G_{[M]})$. As noted above, $Hom(X_{[M]},G_{[M]})=Hom(G,H)_{[M]}$, so the annihilator, A_2, of the torsion module $(g_2(X)+K)/K$ is nonzero, and is not contained in any maximal ideal $N\neq M$. It follows that $A_1+A_2=R$ and so $a_1+a_2=1$ for some $a_j\in A_j$.

Observe, $a_jg_j: X\to G$. Furthermore, the map $\phi: X\to G$ given by $\phi(x)=a_1g_1(x)+a_2g_2(x)$ satisfies $f'\cdot\phi(x)=a_1f'\cdot g_1(x)+a_2f'\cdot g_2(x)=a_1x+a_2x=x$, and $g(X)\cong X$ is a summand of G. It follows from the general theorem that finitely generated torsion-free modules over local rings cancel (Theorem 7.13 in [1]) that the complement, K, of $g(X)$, is locally isomorphic to J_n. Hence K is nearly isomorphic to J_n by Lemma 2.3, which completes the proof. □

Corollary 2.5. *If R has UDI, then R_N has UDI for every maximal ideal N.*

Proof. Evidently, it suffices to consider R_M in the case that M is a nonprincipal maximal ideal for R. Let $I_1',\dots,I_n',J_1',\dots,J_n'$ be ideals of R_M such that $I_1'\oplus\cdots\oplus I_n' \cong J_1'\oplus\cdots\oplus J_n'$. Set $I_j = I_j'\cap R$ and $J_i = J_i'\cap R$. The modules $G=\bigoplus_j I_j$ and $H=\bigoplus_i J_i$ are finitely generated and torsion-free, and satisfy $G_M \cong H_M$. By Proposition 2.4, G and H are isomorphic, and so $I_j \cong J_j$ for all j after reordering. Thus, $I_j' \cong J_j'$ for all j. □

Lemma 2.6. *If there exists a maximal ideal M for R such that every other maximal ideal of R is principal, then every ideal I not contained in M is principal. In addition, every invertible ideal is principal.*

Proof. Let I be an ideal not contained in M. Then $I_M = R_M$ and $I_{[M]} = aR_{[M]}$, with $a = a_1^{e_1}\cdots a_m^{e_m}$, for some $e_i \geq 1$, where $N_i = a_iR$ are all of the maximal ideals of R over I. Then $I = aR$ since this equality holds locally ($a \notin M$). Now let J be invertible, and consider the exact sequence

$$0 \to Hom(J,M) \xrightarrow{g} Hom(J,R) \to Hom(J,R/M) \to 0.$$

Since J is invertible, g cannot be an epimorphism, so there exists a $q \in Hom(J,R)$ such that $qJ \nsubseteq M$. Thus $qJ \subseteq R$ is principal. □

Lemma 2.7. *Suppose R has a unique nonprincipal maximal ideal M. If S is an overring of R and N is a maximal ideal of S lying over M such that N_M is principal, then N is principal.*

Proof. Write $N_M = aS_M$ for some $a \in S$. Because R is h-local (Lemma 2.2), there are only finitely many maximal ideals P of R for which $(aS)_P \neq S$, so that we will produce a generator for N by induction on the number of maximal ideals P of R such that $(cS)_P \neq S$ where $c \in S$ is such that $cS_M = N_M$.

Suppose P is a maximal ideal of R different from M for which $(aS)_P \neq S$; so that, aS is contained in a maximal ideal N' of S such that $P = R\cap N'$ is different from M. Since P is principal, it follows that N' must also be principal. Write $N' = bS$.

Let n be the largest positive integer such that $b^n \mid a$, and write $b^nc = a$ with $c = a/b^n \in S$. Then $b^n \notin N$ but $a = b^n\cdot a/b^n \in N$, so $a/b^n \in N$. Thus $N_M = (a/b^n)S_M$. Also $a/b^n \notin bS$. Because $P = R\cap bS$ is a principal maximal ideal of R, bS is the only maximal ideal of R lying over P, and so $S_P = S_{bS}$ [8, p. 91]. Therefore, $(a/b^n)S_P = S_P$. Because R is h-local, there are only finitely many maximal ideals P of R such that $aR_P \neq R_P$, and so after a finite number of applications of the procedure above, we can produce a generator for N. □

Theorem 2.8. *R has UDI if and only if R is a PID, or, there exists a lone nonprincipal maximal ideal M of R and R_M has UDI.*

Proof. One direction is clear from Lemma 2.1 and Corollary 2.5. For the converse, assume that R has a unique nonprincipal maximal ideal M and R_M has UDI. The proof that R has UDI will follow once we observe that a finitely generated overring S of R has at most one nonprincipal maximal ideal, so let N be a maximal ideal of S. Note that if N is a maximal ideal of S, then $R \cap N$ is a maximal ideal of R due to the fact that S is finitely generated over R. Also, if $N \cap R$ is principal, then so is N, so we assume that N lies over M.

Since R_M has UDI and S_M is a finitely generated overring, Lemmas 2.7 and 2.1 show that S_M has at most one nonprincipal maximal ideal. By Lemma 2.7, if N_M is a principal maximal ideal of S_M, then N is a principal ideal of S, so S has at most one nonprincipal maximal ideal.

Suppose $I_1,\dots,I_n,J_1,\dots,J_n$ are ideals of R for which $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_n$. After localizing at M, we find there is a permutation of the indices such that $(I_j)_M \cong (J_j)_M$ for each j, since R_M has UDI. To simplify notation, fix $j \leq n$ and set $I = I_j$, $J = J_j$ and $S = E(J)$.

Observe that $Hom(I,J)$ is a fractional ideal of S that is locally invertible, since $I_N \cong J_N$ for each maximal ideal N of R. Because S has at most one nonprincipal maximal ideal, $Hom(I,J)$ must be a principal fractional ideal of S (Lemma 2.6). If $Hom(I,J) = g \cdot S$, then $I_N \cong J_N$ for all maximal ideals N and g is an isomorphism. □

It remains to examine the local case.

3. Local domains with UDI

We can readily supply nonintegrally closed, local domains R with UDI in view of the well-known fact that if $\bar{R}$, the integral closure of R in its quotient field, is quasilocal, then the endomorphism ring of each ideal of R is local and hence R has UDI [1, Example 7.5]. Thus what remains to be classified are those local domains without quasilocal integral closure.

Lemma 3.1. *If R is local with UDI, then $\bar{R}$ has at most* 3 *maximal ideals.*

Proof. Suppose $\bar{R}$ has at least 4 distinct maximal ideals. Then there exists a finitely generated overring S of R with least 4 distinct maximal ideals; call them M_1, M_2, M_3, M_4. The map $\sigma : (R + M_1M_2) \oplus (R + M_3M_4) \to S$ with $\sigma(a,b) = a + b$, is split by the map $\phi : S \to (R + M_1M_2) \oplus (R + M_3M_4)$ given as $\phi(t) = (tx, ty)$ where $x \in M_1M_2$ and $y \in M_3M_4$ are such that such that $x + y = 1$. Then ϕ is a splitting map for σ and by UDI, we may assume, without loss of generality, that $S \cong R + M_1M_2$. But since both objects in question are rings, we must have $S = R + M_1M_2$. This is impossible since $M \subseteq M_1 \cap M_2 = M_1M_2$ implies $(R + M_1M_2)/M_1M_2 \cong R/M$ while M_1M_2 is not prime in S. □

Given a ring R, let $|\max(R)|$ represent the number of maximal ideals of R.

Theorem 3.2. *Assume that R is local with maximal ideal M. The following are equivalent for R:*

(1) *R has UDI.*
(2) *There exists a fractional overring R' of R with $|\max(R')| = |\max(\bar{R})|$ such that one of the following occurs:*
  (i) *R' is local.*
  (ii) *R' has exactly 2 distinct maximal ideals M_1', M_2' such that M_1' is principal with $M \nsubseteq (M_1')^2$, and $R'/M_1' \cong R/M$.*
  (iii) *$R' = \bar{R}$ has exactly 3 distinct maximal ideals M_1', M_2', M_3'; all are principal and satisfy $M \nsubseteq (M_j')^2$ and $R'/M_j' \cong R/M$.*
(3) *Every fractional overring R' of R such that $|\max(R')| = |\max(\bar{R})|$ satisfies* (i), (ii) *or* (iii) *above.*

Proof. (1) $\Rightarrow$ (3). Assume that R has UDI and let R' be a fractional overring of R with $|\max(R')| = |\max(\bar{R})|$. This implication breaks down according to the number of maximal ideals in $\bar{R}$, of which there are at most 3 by Lemma 3.1. If $\bar{R}$ is quasilocal then it is easy to check that R' is local.

If $|\max(R')| = 2$ and M_1', M_2' are the two maximal ideals of R', let e_j be the largest positive integer such that $M \subseteq (M_j')^{e_j}$. As above, the map $\sigma : (R + (M_1')^{e_1}) \oplus (R + (M_2')^{e_2}) \to R'$ with $\sigma(a, b) = a + b$, is split by $\phi : R' \to (R + (M_1')^{e_1}) \oplus (R + (M_2')^{e_2})$ given by $\phi(t) = (tx_1, tx_2)$, where $x_j \in (M_j')^{e_j}$, $j = 1, 2$ are such that $x_1 + x_2 = 1$.

Since $Ker\, \sigma$ is isomorphic to an ideal of R, by UDI, we may assume without loss of generality that $R' = R + (M_1')^{e_1}$. Since $M \subseteq (M_1')^{e_1}$, $R'/(M_1')^{e_1} \cong R/M$, clearly implying that $e_1 = 1$ and $R'/M_1' \cong R/M$.

By Lemma 2.1, one of M_1', M_2' is principal. If M_1' is principal, then we have completed case (ii). If M_1' is not principal, then the addition map $M_1' \oplus (R + (M_2')^{e_2}) \to R'$ splits, implying $R' = R + (M_2')^{e_2}$. As before, $e_2 = 1$ and $R'/M_2' \cong R/M$. In this case, (ii) is satisfied after switching indices.

Finally, assume R' has distinct maximal ideals M_1', M_2', M_3', and let e_1, e_2, e_3 be defined as above. Considering the homomorphism $\sigma : (R + (M_1')^{e_1}) \oplus (R + (M_2')^{e_2}) \to R'$ like the one above, we obtain $e_1 = 1$ and $R'/M_1' \cong R/M$ (after reindexing). Next consider the analogous map $(R + (M_2')^{e_2}) \oplus (R + (M_3')^{e_3}) \to R'$ to obtain $e_2 = 1$ and $R'/M_2' \cong R/M$ (after reindexing). The addition map $(R + M_1'M_2') \oplus (R + (M_3')^{e_3}) \to R'$ splits. Since $M \subseteq M_1'M_2' = M_1' \cap M_2'$, $R' = R + M_1'M_2'$ is impossible, so $e_3 = 1$ and $R'/M_3' \cong R/M$ as before. If M_1' is not principal, say, then the splitting of $(R + M_1'M_2') \oplus M_3' \to R'$ implies $R' \neq R + M_1'M_2'$, contradicting UDI.

(3) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1). Let $I_1, \ldots, I_n, J_1, \ldots, J_n$ be ideals of R such that $I_1 \oplus \cdots \oplus I_n \cong J_1 \oplus \cdots \oplus J_n$, and regard $L = I_1 \oplus \cdots \oplus I_n = J_1 \oplus \cdots \oplus J_n$ as two internal decompositions of L into rank-1 finitely generated torsion-free modules. If one of the summands, I_1 say, has

local endomorphism ring $S = E(I_1)$, then consider

$$(\dagger) \qquad 1_{I_1} = \pi_1 \iota_1 = \Sigma_j \pi_1 \iota'_j \pi'_j \iota_1,$$

where $\iota_j : I_j \to A$ and $\iota'_j : J_j \to A$ are the canonical embeddings and $\pi_j : A \to I_j$ and $\pi'_j : A \to J_j$ the coordinate projections.

One of the terms $\pi_1 \iota'_j \pi'_j \iota_1$ must be a unit in S, since S is local; say when $j = 1$. Then $f = \pi_1 \iota'_j : J_1 \to I_1$ is an epimorphism and consequently $J_1 \cong I_1$. By the cancellation result Example 7.5 from [1] mentioned at the beginning of this section, $I_2 \oplus \cdots \oplus I_n \cong J_2 \oplus \cdots \oplus J_n$. Repeating as necessary, we are led to the case that all of $E(I_j)$ and $E(J_i)$ are not local, and so we must now assume $\bar{R}$ has at least two maximal ideals.

Let S be any nonlocal fractional overring of R. Then the fractional overring $S' = R'S$ is readily seen to have properties (ii) or (iii) relative to its maximal ideals. We claim that under the present circumstances, either $S = \bar{R}$ and S is a PID, or, S has two maximal ideals M_1, M_2 with M_1 principal and satisfying $S/M_1 \cong R/M$. There are two cases:

Case 1: R' has property (ii). R' has two maximal ideals M'_1, M'_2 as in (ii), and consequently S has two maximal ideals M_1, M_2 lying under M'_1, M'_2, respectively (R' is integral over S and S is not local). Because M'_j is the unique maximal ideal of R' over M_j, $R_{M'_j} = R_{M_j}$, $j = 1,2$ [8, p. 91]. With M'_1 principal such that $R'/M'_1 \cong R/M$, $S_{M_1} + M_1 R'_{M_1} = R_{M'_1}$ and so $S_{M_1} = R'_{M'_1}$ is a DVR. Also, M_1 is a (locally) principal ideal of S such that $S/M_1 \cong R/M$.

Case 2: R' has property (iii). If S has 3 maximal ideals, $M_j \subseteq M'_j$, $j = 1,2,3$, then as above $S_{M_j} + M_j R'_{M_j} = R'_{M_j}$ implies $S_{M_j} = R'_{M_j}$ for each j. In this case $S = R' = \bar{R}$ is a PID. Otherwise, suppose S has exactly two maximal ideals M_1, M_2. Since R' is finitely generated over S, each $S \cap M'_j$ is a maximal ideal of S. Therefore, after reindexing, we may assume that $M_1 \subseteq M'_1$ and $M_2 \subseteq M'_2 \cap M'_3$. As in the previous case, $S_{M_1} = \bar{R}_{M_1}$ and M_1 is a principal ideal of S such that $S/M_1 \cong R/M$.

Working under the assumption that each $E(I_j)$ and $E(J_i)$ is nonlocal, among all of I_i, J_j, choose one, I_1 say, for which $S = E(I_1)$ is minimal with respect to inclusion (identifying each endomorphism ring as a fractional overring of R in $\bar{R}$). If S has three maximal ideals, then as shown above, $S = \bar{R}$ is a PID and all of the ideals I_j, J_i $j = 1, \ldots, n$ are $\bar{R}$-modules and therefore pairwise isomorphic. It therefore suffices to assume that S has exactly two maximal ideals M_1, M_2, with M_1 principal and $S/M_1 \cong R/M$.

Now consider ($\dagger$): One of the terms $\pi_1 \iota'_j \pi'_j \iota_1$, is a unit in S_{M_2}, say when $j = 1$. For this to be the case, $f(J_1) S_{M_2} = (I_1)_{M_2}$ where $f = \pi_1 \iota'_1$. Write $M_1 = aS$. For some $n \geq 0$, $1/a^n f(J_1) \subseteq (I_1)_{M_1}$ yet $1/a^n f(J_1) \nsubseteq M_1 (I_1)_{M_1}$. Set $h = (1/a^n) f$. Then $h : J_1 \to I_1$ since $h(J_1) \subseteq (I_1)_{M_1} \cap (I_1)_{M_2} = I_1$. Because $(I_1)_{M_1}$ is a principal ideal in the DVR S_{M_1}, and $S_{M_1} = R + M_1 S_{M_1}$, $h(J_1) + M_1 (I_1)_{M_1} = (I_1)_{M_1}$. Also, $h(J_1) S_{M_2} = (I_1)_{M_2}$. Let $U = Hom(J_1, I_1) J_1$, so that U is an S-submodule of I_1 containing $h(J_1)$. From what we have just shown, $U = I_1$ since U agrees with I_1 at each maximal ideal M_1, M_2. We conclude that I_1 is an $E(J_1)$-module and therefore by the minimality of S, $S = E(J_1)$. So h is an S-module map. Using Nakayama's lemma again, since $h(J_1)$ is known to be an S-submodule of I_1, we conclude that $h(J_1)_{M_1} = (I_1)_{M_1}$. Evidently $h(J_1)_{M_2} = (I_1)_{M_2}$.

Therefore $h: J_1 \to I_1$ is an isomorphism, and so by cancellation again, we have that $I_2 \oplus \cdots \oplus I_n \cong J_2 \oplus \cdots \oplus J_n$. The proof now follows by induction on n. □

Corollary 3.3. *Let R be local. If R has UDI then one of the three possibilities occurs*:

(i) $\bar{R}$ *is quasilocal.*
(ii) $\bar{R}$ *has exactly* 2 *distinct maximal ideals* P_1, P_2 *such that* P_1 *is principal*, $M \nsubseteq P_1^2$, *and* $\bar{R}/P_1 \cong R/M$.
(iii) $\bar{R}$ *has exactly* 3 *maximal ideals* P_1, P_2, P_3, *that are principal and satisfy* $\bar{R}/P_i \cong R/M$ *and* $M \nsubseteq P_i^2$ *for each* i.

The converse holds if $\bar{R}$ *is a finite R-module.*

Proof. Let R' be a fractional overring as in Theorem 3.2(2). If M_1' is a principal maximal ideal of R' with $R'/M_1' \cong R/M$, and P_1 is the maximal ideal of $\bar{R}$ lying over M_1', then $R'_{M_1'} = \bar{R}_{P_1}$ has principal maximal ideal $M_1'R'_{M_1'} = P_1\bar{R}_{P_1}$. Thus, P_1 is (locally) principal, and $\bar{R}/P_1 \cong R'/M_1' \cong R/M$. The converse holds since $\bar{R}$ can be used for R' in Theorem 3.2 when $\bar{R}$ is finitely generated over R. □

Corollary 3.4. *The following are equivalent*:

(1) *R has UDI.*
(2) *R is a PID*, *or*, *R has a unique nonprincipal maximal ideal M such that* R_M *satisfies the conditions of Theorem* 3.2.

4. Applications and examples

If R is local with every ideal 2-generated, then either $\bar{R}$ is finitely generated over R or $\bar{R}$ is local [6]. In the first case, with M the maximal ideal of R, $\bar{R}/M\bar{R}$ is at most two dimensional over R/M. From this it follows that $\bar{R}$ is local with maximal ideal $M\bar{R}$, or, $M\bar{R} = P_1P_2$ for distinct ideals P_1, P_2 of $\bar{R}$, with $\bar{R}/P_j \cong R/M$, $j = 1, 2$. By Theorem 3.2, R has UDI. Theorem 4.1 globalizes this observation.

Theorem 4.1. *R has the property that every finitely generated torsion-free R-module uniquely decomposes into the direct sum of ideals if and only if every ideal of R is* 2*-generated*, *and all but at most* 1 *maximal ideal of R is principal.*

Proof. If every finitely generated torsion-free R-module uniquely decomposes into a direct sum of ideals, then, in particular, R has UDI. Assuming that R is not a PID, by Lemma 2.1, R must possess a unique nonprincipal maximal ideal M. If L is a finitely generated torsion-free R_M-module, then it is easy to find a finitely generated torsion-free R-module K such that $K_M = L$. Thus L is a direct sum of ideals and so R_M has the property that every finitely generated torsion-free module uniquely decomposes into the direct sum of ideals by Corollary 2.5. In [7] it is shown that such rings have every ideal 2-generated. Lemma 2.2 shows that R is h-local, so by a theorem of Cohen [7, Theorem 26], every ideal of R is 2-generated.

Conversely, from Theorem 2.8 and the arguments above, R has UDI. Also, from Theorem 57 in [7], R has Krull dimension one. Thus, by a theorem of Rush [9], every finitely generated torsion-free module decomposes into the direct sum of ideals. Because R has UDI, the decompositions are unique. □

Rush shows that when ideals of R are 2-generated, any finitely generated torsion-free module K decomposes into $R_1 \oplus \cdots \oplus R_n \oplus I$ where $R_1 \subseteq \cdots \subseteq R_n \subseteq R_{n+1}$ are fractional overrings of R, and I is an invertible ideal of R_{n+1} [9]. His assertion that the decompositions are unique refers to the fact that if $K \cong S_1 \oplus \cdots \oplus S_n \oplus J$ with $S_1 \subseteq \cdots \subseteq S_n \subseteq S_{n+1}$ fractional overrings of R and J is an invertible ideal of S_{n+1}, then $R_j = S_j$ for $j = 1, \ldots, n$ and $I \cong J$. Subrings of quadratic number fields have every ideal 2-generated but such rings do not usually have UDI (as Example 4.6 will show), so the uniqueness assertion of [9] is the best one could hope for under the circumstances.

Proposition 4.2. *If R is one-dimensional with UDI, then every overring of R has UDI.*

Proof. For the moment assume that R is local and let D be an overring of R. The maximal divisible submodule $h(D/R) = D_0/R$ of D/R is such that D_0 is an overring of R and D/D_0 is finitely generated [7]. It is enough to show that D_0 has UDI, since it easy to see that fractional overrings of UDI domains have UDI. So, without loss of generality, we will assume that D/R is divisible. A consequence of this is that $J = D(J \cap R)$ and $D/J \cong R/(J \cap R)$ for every ideal J of D [7, Theorem 3.4]. In particular, D is local with maximal ideal MD.

Let R' be a finitely generated overring of R with $|\max(R')| = |\max(\bar{R})|$, and set $D' = R'D$. Note that D'/R' is an image of $R' \otimes_R D/R$ and is therefore divisible. Recall that $\bar{R}$ is a PID since R is one dimensional and local, and so $\bar{D}$ must be one of $\bar{R}$, $\bar{R}_P$ for some maximal ideal P of $\bar{R}$, or $\bar{R}_{P_1} \cap \bar{R}_{P_2}$ for maximal ideals P_1, P_2 of $\bar{R}$. If $\bar{D}$ is local, then as discussed at the beginning of Section 3, D has UDI, so we may exclude this case.

A maximal ideal of $\bar{D}$ is $P\bar{D}$ where P is a maximal ideal of $\bar{R}$. Let M' be the maximal ideal of R' lying under P. Then the maximal ideal $N' = P\bar{D} \cap D'$ of D' lies over M'. Thus, since D'/R' is divisible, when M' is principal, $N' = (N' \cap R')D' = M'D'$ is principal. Furthermore, $D'/N' \cong R'/M'$ from which it follows that D' satisfies items (i), (ii) or (iii) from Theorem 3.2 relative to the local ring D, and consequently D has UDI.

For the general case, R has at most one nonprincipal maximal ideal M. If D is any overring of R, then any maximal ideal of D not containing M must be principal (as it must lie over a principal maximal ideal of R). But we have just shown that D_M has UDI, so D_M has at most one nonprincipal maximal ideal. Hence by Lemma 2.7, D has at most one nonprincipal maximal ideal, and if there is a nonprincipal maximal ideal M' of D, then $D_{M'}$ has UDI (from above). Therefore, D has UDI by Theorem 2.8. □

Theorem 4.3. *Let R have Krull dimension* 1. *Then R has UDI if and only if for any rank one R-modules $X_1,\dots,X_n,Y_1,\dots,Y_n$, if $X_1\oplus\cdots\oplus X_n\cong Y_1\oplus\cdots\oplus Y_n$, then after reindexing, $X_j\cong Y_j$ for all j.*

Proof. Our argument is similar to that of the proof the Krull–Schmidt theorem for direct sums of rank one abelian groups. Two rank 1 modules X and Y are called *quasi-isomorphic* when each is isomorphic to a submodule of the other; equivalently, there is an element $0\neq r\in R$ such that X is isomorphic to a submodule X' of Y for which $r(Y/X')=0$. A *type* τ is the quasi-isomorphism class of a rank 1 module X, and the collection of types is a lattice with partial order $\tau_1\leq\tau_2$ when X_1 is isomorphic to a submodule of X_2 (where X_j corresponds to τ_j).

Given a torsion-free module G and $0\neq x\in G$, the *type of* x is the type of the rank 1 pure submodule of G generated by x; i.e., the type of $\{y\in G\mid ry=sx$ for some $r,s\in R\}$, and given a type τ, $G(\tau)\equiv\{x\in G\mid$ type of $x\geq\tau\}$. Suppose $G=X_i\oplus\cdots\oplus X_n\cong H=Y_1\oplus\cdots\oplus Y_n$ for rank-1 modules $X_1,\dots,X_n,Y_1,\dots,Y_n$ in a one-dimensional UDI domain R. Since types are preserved under isomorphisms, given a type τ, $G(\tau)=\bigoplus\{X_j\mid X_j$ has type $\geq\tau\}\cong H(\tau)=\bigoplus\{Y_i\mid Y_i$ has type $\geq\tau\}$, and $G/G(\tau)\cong H/H(\tau)$.

Let X,Y be two submodules of Q containing R. Since $(X/R)/h(X/R)$ is finitely generated, where $h(X/R)$ represents the maximal divisible submodule of X/R, it follows that X and Y are quasiisomorphic if and only if $h(X/R)=h(Y/R)$. Moreover, with $D/R=h(X/R)$, D is an overring of R and X is a module over D [7]. Therefore, if we select a D such that $D/R=h(X_i/R)$ is maximal with respect to inclusion among $h(X_j/R)$, and set τ equal to the type of D, then $G(\tau)\cong H(\tau)$ are D-modules and each is a direct sum of ideals of D. By Proposition 4.2 we can match up the summands, and apply induction to $G/G(\tau)\cong H/H(\tau)$. □

Examples of one dimensional local domains with UDI that conform to the possibilities of Theorem 3.2 can be found readily. The ring of integers is denoted by $\mathbb{Z}$ and its quotient field by $\mathbb{Q}$.

Example 4.4. Let A be the number ring (the integral closure of $\mathbb{Z}$) in a finite-dimensional field extension of $\mathbb{Q}$, and let p be an integral prime. Write $pA=P_1^{e_1}\cdots P_n^{e_n}$:

(i) If $S=A_{P_1}$, then $R=\mathbb{Z}+pS$ has quasilocal integral closure and hence R has UDI.
(ii) If $e_1=1$ and $A/P_1\cong\mathbb{Z}/p\mathbb{Z}$, then for $S=A_{P_1}\cap A_{P_2}$, $R=\mathbb{Z}+pS$ is local with UDI and $\bar{R}=S$ has 2 maximal ideals.
(iii) If $e_1=e_2=e_3=1$ and $A/P_j\cong\mathbb{Z}/p\mathbb{Z}$ for $j=1,2,3$, then for $S=A_{P_1}\cap A_{P_2}\cap A_{P_3}$, $R=\mathbb{Z}+pS$ is local with UDI and $\bar{R}=S$ has 3 maximal ideals.

It follows from our theorems that if R is one dimensional with UDI, then $\bar{R}$ is a PID. For by Proposition 4.2, the Dedekind domain $\bar{R}$ has UDI, and hence by Lemma 2.6, $\bar{R}$ is a PID.

Constructing global UDI rings is delicate. Using the strong results in [11] we can characterize the one-dimensional noetherian domains with module finite integral closure that possess UDI. Given such a ring R, the ideal $C=(R:_R \bar{R})$ is the largest ideal common to both R and $\bar{R}$ and is called the *conductor* of $\bar{R}$ in R. The notation $(\bar{R}/C)^*$ and $(R/C)^*$ constitute the groups of units in the respective rings. Additionally, Λ_R denotes the subgroup of $(\bar{R}/C)^*$ consisting of $u+C$ where u is a unit of $\bar{R}$ [11].

Proposition 4.5. *Suppose R is one dimensional and $\bar{R}$ is a finite R-module such that $\bar{R}\neq R$. Then R has UDI if and only if*

(a) *$\bar{R}$ is a PID,*
(b) *C is a primary ideal of R with radical M, and M obeys one of the splitting conditions* (i), (ii), *or* (iii) *of Theorem* 3.2, *and,*
(c) $(\bar{R}/C)^*=(R/C)^*\cdot\Lambda_R$.

Proof. Item (c) is the characterization from [11] of when the ring R with integral closure a PID has every invertible ideal principal.

Suppose R has UDI. Let $C=C_1\cap\cdots\cap C_n=C_1\cdots C_n$ be a reduced primary decomposition of C in R. If M is not the radical of one of the C_j's, then C is principal by UDI and it follows that $C=R$ and therefore $R=\bar{R}$ is a PID. Otherwise, $C=cC_1$ for some $c\in R$ and M-primary C_1, but because C is the trace of $\bar{R}$ in R and $c^{-1}f$ maps $\bar{R}$ into R when f does, $c=1$.

Conversely, suppose (a)–(c) hold. By Theorem 3.2, it is enough to show that M is singular. Let N be a maximal ideal of R different from M. Locally N is principal since at maximal ideals other than N, N localizes to that of R while at N, R coincides with $\bar{R}$. Therefore N is invertible by (c). Thus M is a singular maximal ideal and Theorem 3.2 is satisfied. □

The proposition allows us to determine when a $\mathbb{Z}$-order in a quadratic number field has UDI. Our first example shows that this rarely happens in the case of nonreal quadratic number fields. As the justification for the next example and its successor follow from the arguments in [10], we will omit the proofs.

Example 4.6. Let $\mathbb{Q}[\sqrt{d}]$ be any nonreal quadratic number field, and let $\bar{R}$ represent the ring of integers in this number field. A $\mathbb{Z}$-order R in $\mathbb{Q}[\sqrt{d}]$ has UDI if and only if $d\in\{-1,-2,-3,-7,-11,-19,-43,-67,-163\}$ and

(a) $R=\bar{R}$, or
(b) $R=\mathbb{Z}+p\bar{R}$ in one of the following situations; $p=3$ and $d=-3$, or $p=2$ and $d=-1,-3,-7$.

An algorithm for determining the real quadratic $\mathbb{Z}$-orders of the form $R=\mathbb{Z}+p\bar{R}$ having UDI is easily obtained. Recall that the fundamental unit u of a real quadratic number ring (a unit such that every unit is of the form $\pm u^k$, $k\in\mathbb{Z}$) can be found as follows: When $d \equiv 2,3 \bmod 4$, choose $b > 0$ minimal such that $db^2 \pm 1$ is a square, a^2 $(a > 0)$, and set $u = a + b\sqrt{d}$. When $d \equiv 1 \bmod 4$, choose $b > 0$ minimal such that $db^2 \pm 4$ is a square, a^2 $(a > 0)$, and set $u = (a/2) + (b/2)\sqrt{d}$ [5]. Assuming one has a PID $\bar{R}$ to begin, we find k minimal such that u^k is congruent to an integer mod p. Then $(R/p\bar{R})^* \cdot \Lambda_R$ has order $k(p-1)$. For R to have UDI, this number must coincide with $p^2 - 1 = (p+1)(p-1)$ when p is inert, $(p-1)^2$ when p splits, and $p(p-1)$ when p ramifies.

Example 4.7. Let $\bar{R}$ be the ring of integers in the real quadratic number field $\mathbb{Q}[\sqrt{d}]$, and assume that $\bar{R}$ is a PID. Let $R = \mathbb{Z} + p\bar{R}$ for some integral odd prime p, and let u be the fundamental unit of $\bar{R}$, $u = a + b\sqrt{d}$ when $d \equiv 2,3 \bmod 4$, and $u = (a + b\sqrt{d})/2$ when $d \equiv 1 \bmod 4$, for integers a, b.

(a) If $p \geq 5$ divides a or b, then R does not have UDI.
(b) If $p \mid d$ and p does not divide a or b, then R has UDI.
(c) (R. Wiegand) If $p \equiv 1 \bmod 4$ or u has norm 1, then R has UDI implies $p \mid d$.
(d) If $3 \mid a$, then R has UDI if and only if 3 splits in $\bar{R}$; if $3 \mid b$ then R does not have UDI.
(e) R has UDI when $p = 2, 3, 5, 7, 11$ and $d = 5$ (here the fundamental unit is $u = (1 + \sqrt{5})/2$).

If R is a Noetherian UDI domain that is not a PID, then R can be written as the intersection of a local UDI ring and a PID; namely, $R = R_{[M]} \cap R_M$, where R_M is a local ring with UDI, and $R_{[M]} = \bigcap_{N \neq M} R_N$ is a PID (here M is the singular ideal of R). This property, however, does not characterize UDI domains.

Example 4.8. Any Dedekind domain R of prime class number is the intersection $R = R_{[M]} \cap R_M$, where M is some maximal ideal of R such that $R_{[M]}$ is a PID. Hence the intersection condition $R = R_{[M]} \cap R_M$, with $R_{[M]}$ a PID and R_M UDI, in general, is insufficient for R to have UDI.

Proof. Since R has class number p for some integral prime p, there exists a maximal ideal M such that every ideal not isomorphic to M is principal. It follows that every ideal of $R_{[M]}$ is either the extension of a principal ideal of R, or is isomorphic to $M \cdot R_{[M]} = R_{[M]}$. $\square$

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