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Constructivist and structuralist foundations: Bishop's and Lawvere's theories of sets

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ABSTRACT

Bishop's informal set theory is briefly discussed and compared to Lawvere's Elementary Theory of the Category of Sets (ETCS). We then present a constructive and predicative version of ETCS, whose standard model is based on the constructive type theory of Martin-Löf. The theory, CETCS, provides a structuralist foundation for constructive mathematics in the style of Bishop.

1. Introduction

Errett Bishop's book *Foundations of Constructive Analysis* from 1967 contains a chapter on set theory. This set theory, apart from being informal, is quite unlike any of the theories of Zermelo–Fraenkel or Gödel–Bernays, which are derived from the iterative concept of a set.

> "A set is not an entity which has an ideal existence: a set exists only when it has been defined. To define a set we prescribe, at least implicitly, what we (the constructing intelligence) must do in order to construct an element of the set, and what we must do to show that two elements are equal." Bishop [3, p. 2]

We find a similar explanation of what a set is also in the type theory of Martin-Löf (1984). Both explanations are aligned to Cantor's early explanation of sets from 1882 in the respect that they mention conditions for equality of elements explicitly; see [24] for a discussion. Bishop [3, p. 74] emphasizes that two elements may not be compared unless they belong to some common set. This indicates a type-theoretic attitude to the foundations. Bishop's version of set theory has, despite its constructiveness, a more abstract character than e.g. ZF set theory in that it does not concern coding issues for basic mathematical objects. It defines a subset of a set X to be a pair (A, i_A) where $i_A : A \longrightarrow X$ is a function so that $a = b$ if, and only if, $i(a) = i(b)$. An element $x \in X$ is a member of the subset if $x = i_A(a)$ for some $a \in A$. That the subset (A, i_A) is included in another subset (B, i_B) of X is defined by requirement that there is a function $f : A \longrightarrow B$ so that $i_A = i_B \circ f$, i.e. that the diagram

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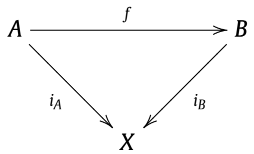


(1)

commutes. The subsets are equal in case f is a bijection. Unions and intersections are only defined when the involved sets are subsets of the same underlying set. These and other features of Bishop's set theory are remarkably reminiscent of Lawvere's *Elementary Theory of the Category of Sets* (ETCS) introduced in 1964. ETCS is obtained by singling out category-theoretic universal properties of various set construction in such a way that they become invariant under isomorphism; see [15] and the introduction [9–16], the full version of the 1964 paper. This invariance is of course fundamental for a *structuralist foundation*. ETCS is an elementary theory in the sense that it uses classical first order logic as a basis, and make no special assumption on existence of second order or higher order objects. The theory is equivalent to the axioms of a well-pointed topos with the axiom of choice [15,11]. It should be emphasized that ETCS was introduced to give an immediate axiomatization of sets, while the Lawvere–Tierney elementary theory of a topos was intended to give axioms for sheaves of sets over an arbitrary topological space.

Bishop [4,5] considered various versions of Gödel's system T as a possible foundation for his set theory. At the basis of the interpretation is a system of computable functions and functionals, which in effect are the core operations of certain modern programming languages. Full-fledged systems suitable for the formalization of constructive mathematics in the style of Bishop emerged later with the constructive type theory of Martin-Löf [14] and the constructive set theories CST [20] and CZF [1]. Of these, the type-theoretic system is the more fundamental from a constructive semantical point of view, since it describes explicitly how the computation of functions are carried out. Indeed, the mentioned set-theoretic system, CZF, can be justified on the grounds of Martin-Löf's type theory (MLTT) as shown by Aczel [1] by a model construction. In MLTT the explanation of when elements of a set (type) are equal halts at the level of definitional equality. There are no quotient constructions, so it is customary to consider a type together with an equivalence relation, as a set-like object, a so-called *setoid*. This gives two possible conceptions of constructive sets based on the formal theories CZF and MLTT, namely iterative sets (sets as trees) and setoids respectively.

In this paper we present a constructive version of ETCS, called CETCS, which is obtained abstracting on category-theoretic properties of CZF sets and of setoids in a universe in MLTT. A first requirement on CETCS is of course that we use intuitionistic first order logic instead of the customary classical logic. CETCS has however the property that by adding the law of excluded middle and the axiom of choice (AC), we get a theory equivalent to ETCS. Furthermore the theories of Aczel–Myhill and Martin-Löf are (generalized) predicative, so that power set principles are not valid. Thus a constructive ETCS cannot be obtained by adding axioms to the elementary theory of toposes. In [18,19] a notion of predicative topos was introduced taking the setoids of MLTT with a hierarchy of universes as a standard model. Other variants of predicative toposes have been introduced and studied [25]; see also [12,2]. A drawback of the category of setoids, as opposed categories of sets, is that there is no canonical choice of pullbacks (Sect. 6, [6]). This makes the formulation of some axioms a bit less concise, but also more general.

We emphasize that ETCS does not deal with the set-class distinction or replacement axioms. ETCS with replacement has however been considered [21,15]. A constructive treatment of the set-class distinction was given by Joyal and Moerdijk [8] by the introduction of notion of a small map. Predicatively acceptable versions of this were developed in [19,26]. It seems rather straightforward to extend CETCS to include axioms for small maps along those lines. Another possible extension of CETCS is to add inductively defined subsets. We leave these investigations for another occasion. A feature of CETCS is that it introduces a constructive version of well-pointedness. Shulman [23] gives a definition of this notion which works for weaker categories.

An outline of the paper is a follows: in Section 2 a standard first-order logic definition of categories is given. We present in Section 3 some notation regarding relations and subobjects for categories where products are not supposed to be chosen. The axioms of ETCS and CETCS are presented in parallel and compared in Section 4. In Section 5 some elementary set-theoretic consequence are drawn from CETCS, which indicates its usefulness for Bishop style constructive mathematics. It is shown that CETCS together with the axiom of choice and classical logic gives the original ETCS. The relation of CETCS to standard category theory notions is given in Sections 6 and 7. This can part can be skipped by the reader that is not particularly interested in categorical logic. Section 7 contains a technical contribution which shows how a "functor free" formulation of locally cartesian closed categories (LCCCs) can be employed in categorical logic.

2. Elementary categories

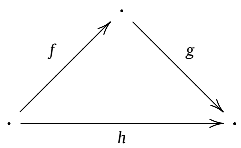
We shall take care to formulate all the axioms so that they may be easily cast in many sorted first-order (intuitionistic) logic. Following the notation of [11], a category $\mathcal{C}$ is specified by an algebraic signature consisting of three collections $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ (for objects, mappings (or arrows), composable mappings) and six functions $\mathsf{id} : \mathcal{C}_0 \longrightarrow \mathcal{C}_1$, $\mathsf{dom}, \mathsf{cod} : \mathcal{C}_1 \longrightarrow \mathcal{C}_0$, comp : $\mathcal{C}_2 \longrightarrow \mathcal{C}_1$, fst, snd : $\mathcal{C}_2 \longrightarrow \mathcal{C}_1$. The intention is that dom gives the domain of the mapping while cod gives its codomain. The collection $\mathcal{C}_2$ is supposed to consist of composable mappings

$$\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$$

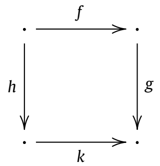
and fst gives the first of these mappings while snd gives the second mapping. Then comp is the composition operation. The axioms for a category are then briefly as follows, where variables ranges are $x \in \mathcal{C}_0, f, g, h, k, \ell \in \mathcal{C}_1, u, v \in \mathcal{C}_2$: (K1) $\mathsf{dom}(\mathsf{id}_x) = x$, (K2) $\mathsf{cod}(\mathsf{id}_x) = x$, (K3) $\mathsf{dom}(\mathsf{comp}(u)) = \mathsf{dom}(\mathsf{fst}(u))$, (K4) $\mathsf{cod}(\mathsf{comp}(u)) = \mathsf{cod}(\mathsf{snd}(u))$ and

(K5) $\mathsf{fst}(u) = \mathsf{fst}(v), \mathsf{snd}(u) = \mathsf{snd}(v) \Longrightarrow u = v$
(K6) $\mathsf{dom}(f) = \mathsf{cod}(g) \Longrightarrow (\exists u : \mathcal{C}_2)\,(\mathsf{snd}(u) = f \,\&\, \mathsf{fst}(u) = g)$.

We introduce abbreviations: for mappings f, g, h write $h \equiv g \circ f$ for $(\exists u \in \mathcal{C}_2)[\mathsf{fst}(u) = f \,\&\, \mathsf{snd}(u) = g \,\&\, \mathsf{comp}(u) = h]$, that is, the diagram



is composable and commutes. Write $k \circ h \equiv g \circ f$ if there is a mapping m so that $m \equiv g \circ f$ and $m \equiv k \circ h$, that is, the following diagram composes and commutes



In terms of these abbreviations we can express the monoid laws: (K7) $f \equiv f \circ (\mathsf{id}_{\mathsf{dom}(f)})$, (K8) $f \equiv (\mathsf{id}_{\mathsf{cod}(f)}) \circ f$, and (K9) if $k \equiv f \circ g$ and $\ell \equiv g \circ h$ then $k \circ h \equiv f \circ \ell$.

$f : a \longrightarrow b$ and $a \xrightarrow{f} b$ are abbreviations for the conjunction $\mathsf{dom} f = a \,\&\, \mathsf{cod} f = b$. We shall often omit $\circ$ and write $h \equiv gf$ for $h \equiv g \circ f$. Moreover $\equiv$ is often replaced by $=$ when there is no danger of confusion.

3. Subobjects and relations

We may define the notion of an n-ary relation in any category. Recall that a mapping $f : A \longrightarrow B$ is *monic* or *is a mono* if for any mappings $h, k : U \longrightarrow A$ with $fh = fk$ it holds that $h = k$. We write in this case $f : A \rightarrowtail B$. This notion can be generalized to several mappings. A sequence of mappings $r_1 : R \longrightarrow X_1, \ldots, r_n : R \longrightarrow X_n$ are *jointly monic*, if for any $f, g : U \longrightarrow R$

$$r_1 f = r_1 g, \ldots, r_n f = r_n g \Longrightarrow f = g.$$

In this case we write $(r_1, \ldots, r_n) : R \rightarrowtail (X_1, \ldots, X_n)$. We regard this as an *n-ary relation between the objects* $X_1, \ldots, X_n$. In particular, a *binary relation between* X_1 *and* X_2 is a pair of mappings $r_1 : R \longrightarrow X_1$ and $r_2 : R \longrightarrow X_2$ which are jointly monic. Another particular case is: if the category has a terminal object $\mathbf{1}$, a 0-ary relation $() : R \rightarrowtail ()$ means that the unique map $R \longrightarrow \mathbf{1}$ is a mono.

Consider a category $\mathcal{C}$ with a terminal object $\mathbf{1}$. An *element* of an object A of $\mathcal{C}$ is a mapping $x : \mathbf{1} \longrightarrow A$. For a monic $m : M \longrightarrow X$ and element x of X write $x \,\epsilon\, m$ if $(\exists a : \mathbf{1} \longrightarrow M) ma = x$. We say that x *is a member of* m. More generally, if $(m_1, \ldots, m_n) : M \rightarrowtail (X_1, \ldots, X_n)$ and $(x_1, \ldots, x_n) : \mathbf{1} \rightarrowtail (X_1, \ldots, X_n)$ we write $(x_1, \ldots, x_n) \,\epsilon\, (m_1, \ldots, m_n)$ if there is $a : \mathbf{1} \longrightarrow M$ so that $m_i a = x_i$ for all $i = 1, \ldots, n$.

To simplify notation we often write $x \in X$ and $(x_1, \ldots, x_n) \in (X_1, \ldots, X_n)$ for $x : \mathbf{1} \longrightarrow X$ and $(x_1, \ldots, x_n) : \mathbf{1} \longrightarrow (X_1, \ldots, X_n)$, respectively. Note the difference between the signs $\in$ (elementhood) and ϵ (membership).

We shall be interested in categories where there is no canonical construction for products, but where it is merely assumed that they exist. Recall that an *n-ary product diagram* in a category is a sequence of mappings $X \xrightarrow[p_i]{} X_i$ $(i = 1, \ldots, n)$ so that for any sequence of mappings $C \xrightarrow[p_i]{} X_i$ $(i = 1, \ldots, n)$ there is a unique $h : C \longrightarrow X$ such that $f_i \equiv hp_i$ for all $i = 1, \ldots, n$. We write

$$h \equiv \langle f_1, \ldots, f_n \rangle_{\bar{p}}$$

when $f_i \equiv hp_i$ for all $i = 1, \ldots, n$, where $\bar{p} = p_1, \ldots, p_n$. It is convenient to drop the subscripts $\bar{p}$ when the product diagrams are obvious from the context.

Proposition 3.1. *Suppose that* $X \xrightarrow[p_i]{} X_i$ *(*$i = 1, \ldots, n$*) is a product diagram. If* $(r_1, \ldots, r_n) : R \longrightarrow (X_1, \ldots, X_n)$, $r' : R \longrightarrow X$ *and* $r' \equiv \langle r_1, \ldots, r_n \rangle_{\bar{p}}$, *then* r' *is monic iff* $(r_1, \ldots, r_n)$ *are jointly monic. Moreover, for* $(x_1, \ldots, x_n) \in (X_1, \ldots, X_n)$, $x' \in X$ *with* $x' \equiv \langle x_1, \ldots, x_n \rangle_{\bar{p}}$, *we have*

$$x' \epsilon r' \iff (x_1, \ldots, x_n) \epsilon (r_1, \ldots, r_n). \quad \square$$

A binary relation $f = (\xi, \upsilon) : R \rightarrowtail\!\!\!\to (X, Y)$ is a *partial function* in case ξ is mono. It is a *total function* in case ξ is iso. A relation

$$f = (\xi_1, \ldots, \xi_n, \upsilon) : R \rightarrowtail\!\!\!\to (X_1, \ldots, X_n, Y)$$

is a *partial function of n variables* if $(\xi_1, \ldots, \xi_n) : R \rightarrowtail\!\!\!\to (X_1, \ldots, X_n)$. We write

$$f : (X_1, \ldots, X_n) \rightharpoonup Y.$$

It is *total function of n variables* if $R \xrightarrow{\xi} X_i$ $(i = 1, \ldots, n)$ is a product diagram. We write

$$f : (X_1, \ldots, X_n) \longrightarrow Y.$$

For $x_1 \in X_1, \ldots, x_n \in X_n$ and $y \in Y$ we write

$$f(x_1, \ldots, x_n) \equiv y$$

in case $(x_1, \ldots, x_n, y) \epsilon f$.

4. Axioms of ETCS and CETCS

Lawvere's theory ETCS [9] has eight axioms: (L1) finite roots exist, (L2) the exponential of any pair of objects exist, (L3) there is a Dedekind-Peano object, (L4) the terminal object is separating, (L5) axiom of choice, (L6) every object not isomorphic to an initial object contains an element, (L7) Each element of a sum is a member of one of its injections, (L8) there is an object with more than one element.

We present a constructive version of ETCS, called CETCS, and some extensions, by laying down axioms for a category $\mathcal{C}$. (It should be evident that the following axioms may be formulated in first-order logic in a language with $\mathcal{C}_0$, $\mathcal{C}_1$, $\mathcal{C}_2$ as sorts and the function symbols id, dom, cod, comp, fst, snd as indicated in Section 2.)

Lawvere's (L1) says that the category is *bicartesian,* i.e. both cartesian and cocartesian.

Recall that $\mathcal{C}$ is *cartesian* if the conditions (C1)–(C3) are satisfied:

(C1) There is a terminal object $\mathbf{1}$ in $\mathcal{C}$.

(C2) Binary products exist: For any pair of objects A and B there exists an object P and two mappings

$$A \xleftarrow{p} P \xrightarrow{q} B$$

which are such that if $A \xleftarrow{f} X \xrightarrow{g} B$ then there exists a unique $h : X \longrightarrow P$ so that $ph \equiv f$ and $qh \equiv g$.

(C3) Equalizers exist: For any parallel pair of mappings $A \underset{g}{\overset{f}{\rightrightarrows}} B$ there exists a mapping $e : E \longrightarrow A$ so that $fe \equiv ge$ and such that whenever $h : X \longrightarrow A$ satisfies $fh \equiv gh$ then there exists a unique $k : X \longrightarrow E$ with $ek \equiv h$.

A category $\mathcal{C}$ is *cocartesian* if it satisfies (D1)–(D3), which are the categorical duals of (C1)–(C3).

(D1) There is an initial object $\mathbf{0}$ in $\mathcal{C}$.

(D2) Binary sums exist: For any pair of objects A, B there is a diagram

$$A \xrightarrow{i} S \xleftarrow{j} B \tag{2}$$

such that if $A \xrightarrow{f} T \xleftarrow{g} B$ then there is a unique $h : S \longrightarrow T$ with $hi \equiv f$ and $hj \equiv g$.

(D3) Coequalizers exist: For any parallel pair of mappings $A \underset{g}{\overset{f}{\rightrightarrows}} B$ there exists a mapping $q : B \longrightarrow Q$ so that $qf \equiv qg$ and such that whenever $h : B \longrightarrow Y$ satisfies $hf \equiv hg$ then there exists a unique $k : Q \longrightarrow E$ with $kq \equiv h$.

The axiom (L2) of ETCS says together with (L1) that the category is cartesian closed. Instead, we take for an axiom the following (Π) which, together with cartesianess and axiom (G) below, states that the category is locally cartesian closed. (This axiom is a theorem of ETCS.)

(Π) Dependent products exist: For any mappings $Y \xrightarrow{g} X \xrightarrow{f} I$ there exists a commutative diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\mathrm{ev}} & P & \xrightarrow{\pi_1} & F \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \varphi} \\ & & X & \xrightarrow[f]{} & I \end{array} \qquad (3)$$

where the square is a pullback, and which is such that for any element $i \in I$ and any partial function $\psi = (\xi, \upsilon) : R \rightharpoonup (X, Y)$ such that

(a) for all $(x, y) \in (X, Y)$, $(x, y) \epsilon \psi$ implies $gy \equiv x$ and $fx \equiv i$,
(b) if $fx \equiv i$, then there is $y \in Y$ with $(x, y) \epsilon \psi$,

then there is a unique $s \in F$ so that $\varphi s = i$ and for all $(x, y) \in (X, Y)$,

$$(s, x, y) \epsilon \alpha \iff (x, y) \epsilon \psi. \qquad (4)$$

Here $\alpha = (\pi_1, \pi_2, \mathrm{ev}) : P \rightarrowtail (F, X, Y)$.

A diagram (3) satisfying these properties is called a *universal dependent product diagram* or shortly a *universal* Π*-diagram* for $Y \xrightarrow{g} X \xrightarrow{f} I$.

The third axiom (L3) of ETCS says, in now common terminology, that there exists a *natural numbers object* (NNO). A category $\mathcal{C}$ has an NNO if there is a sequence of mappings (the NNO) $\mathbf{1} \xrightarrow{0} N \xrightarrow{s} N$ so that for any other sequence of mappings $\mathbf{1} \xrightarrow{b} A \xrightarrow{h} A$ there is a unique $f : N \longrightarrow A$ with $f0 \equiv b$ and $fS \equiv hf$.

Axiom (L4) states in modern terminology that $\mathbf{1}$ is a separating object, i.e. as in Proposition 4.2. We consider instead a stronger axiom (G) which is a theorem of ETCS. A mapping $f : A \longrightarrow B$ of $\mathcal{C}$ is *onto* if for any $y \in B$ there exists an $x \in A$ so that $y \equiv fx$. Our axiom is

(G) Any mapping which is both onto and mono, is an isomorphism.

The fifth axiom (L5) of ETCS states the axiom of choice in peculiar way; see Section 5.2. A more standard way is to first define an object P of $\mathcal{C}$ to be a *choice object*, if for any onto $f : A \longrightarrow P$ there is a $g : P \longrightarrow A$ with $fg = \mathrm{id}_P$. The *axiom of choice* (AC) says that every object is a choice object. This is a far too strong assumption in a constructive setting. There is a constructively acceptable weakening which accords well with Bishop's distinction of operations and functions, the *presentation axiom* [1]:

(PA) For any object A there is an onto mapping $P \longrightarrow A$ where P is a choice object.

Axiom (L6) of ETCS says in contrapositive form: if an object has no elements then it is an initial object. We take instead

(I) The object $\mathbf{0}$ has no elements.

This together with (G) implies (L6).

The Axiom (L7) of ETCS is *each element of a sum is a member of one of its injections.* We adopt this axiom unaltered but call it the *disjunction principle (DP)* as it connects sums to disjunctions:

(DP) In a sum diagram $A \xrightarrow{i} S \xleftarrow{j} B$: for any $z \in S$, $z \epsilon i$ or $z \epsilon j$.

The final axiom (L8) of ETCS states that there exists object with at least two elements. We state this as

(NT, Non-triviality) For any sum diagram $\mathbf{1} \xrightarrow{x} S \xleftarrow{y} \mathbf{1}$ it holds that $x \neq y$.

There are two further axioms that we shall consider, which are in fact theorems of ETCS.

(Fct) Factorization. Any mapping f can be factored as $f \equiv ie$ where i is mono and e is onto.

(Eff) All equivalence relations are effective. For each equivalence relation $(r_1, r_2) : R \rightarrowtail (X, X)$ there is some mapping $e : X \longrightarrow E$ so that

$$(x_1, x_2) \epsilon (r_1, r_2) \iff ex_1 \equiv ex_2$$

for all $(x_1, x_2) \in (X, X)$.

In summary, the theory CETCS consists of the axioms (C1–C3), (D1–D3), (Π), (G), (PA), (I), (DP), (NT), (Fct) and (Eff). Observe that it is a finitely axiomatized theory just as ETCS. We do not know whether this set of axioms is optimal.

Remarks 4.1. Note that it is not assumed that the (co)products or (co)equalizers are given as functions of their data. The axiom (G) is in the terminology of [7] that $\mathbf{1}$ *generates* $\mathcal{C}$. It entails that one can "reason using elements" as the two following results exemplify. This gives a substantial simplification of the internal logic.

Proposition 4.2. *Let $\mathcal{C}$ be a cartesian category which satisfies (G). Then*

(a) *For any pair of mappings $f, g : A \longrightarrow B$, $f = g$ whenever $(\forall x \in A)(fx = gx)$.*
(b) *A mapping $f : A \longrightarrow B$ is monic if and only if $(\forall x, y \in A)(fx = fy \Rightarrow x = y)$.*

Proof. (b) follows easily from (a). To prove the non-trivial direction of (a): assume that $(\forall x \in A)(fx = gx)$. Construct an equalizer $E \xrightarrow{e} A \underset{g}{\overset{f}{\rightrightarrows}} B$ of f and g. Then e is monic. By the assumption and the equalizing property it is also easy to see it is onto. Hence by (G) e is an isomorphism. Since $fe = ge$ we get $f = g$. □

Define an element-wise inclusion relation for monos $m : M \to X$ and $n : N \to X$

$$m \dot{\subseteq} n \iff_{\mathrm{def}} (\forall x \in X)(x \,\epsilon\, m \Rightarrow x \,\epsilon\, n).$$

The standard inclusion relation in a category is given by $m \leq n \iff_{\mathrm{def}} (\exists f : M \to N)(m = nf)$. Compare diagram (1). Their correspondence is given by:

Proposition 4.3. *Let $\mathcal{C}$ be a cartesian category which satisfies (G). Then for all monos $m : M \to X$ and $n : N \to X$,*

$$m \dot{\subseteq} n \iff m \leq n.$$

Proof. ($\Leftarrow$) This is straightforward in any category with a terminal object. ($\Rightarrow$) Suppose that $m : M \rightarrowtail X$ and $n : N \rightarrowtail X$ satisfies $m \dot{\subseteq} n$. Form a pullback square

$$\begin{array}{ccc} P & \xrightarrow{p} & N \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle n} \\ M & \xrightarrow[m]{} & X \end{array}$$

To prove $m \leq n$ it is evidently enough to show that q is an isomorphism. Now q is the pullback of a mono, so it is a mono as well. By (G) it is sufficient to show that q is onto. Let $y \in M$. Thus $my \,\epsilon\, m$ and by assumption also $my \,\epsilon\, n$. There is thus $t \in N$ with $my = nt$. Hence by the pullback square there is a unique $u \in P$ so that $qu = y$ and $pu = t$. In particular, this shows that q is onto. □

Functions as a graphs and as morphisms can be characterized as follows.

Proposition 4.4. *Let $\mathcal{C}$ be a cartesian category which satisfies (G). Let $r = (r_1, r_2) : R \to (X, Y)$ be a relation. Then*

(a) *r is a partial function if and only if*

$$(\forall x \in X)(\forall y, z \in Y)[(x, y) \,\epsilon\, r \;\&\; (x, z) \,\epsilon\, r \Rightarrow y = z]. \quad (5)$$

(b) *r is a total function if and only if*

$$(\forall x \in X)(\exists! y \in Y)(x, y) \,\epsilon\, r. \quad (6)$$

(c) (Unique Choice) *If $(\forall x \in X)(\exists! y \in Y)(x, y) \,\epsilon\, r$, then there is $f : X \to Y$ with*

$$(\forall x \in X)(x, fx) \,\epsilon\, r.$$

Proof. (a): by definition r is a partial function if and only if r_1 is mono. By Proposition 4.2, r is thus a partial function precisely when

$$(\forall s, t \in R)[r_1 s = r_1 t \Rightarrow s = t].$$

This is easily seen to be equivalent to (5).

(b, $\Rightarrow$): Suppose r is a total function. Then r_1 is iso. For $x \in X$, we have $(x, y) \,\epsilon\, r$ with $y = r_2 r_1^{-1} x)$. By (a) it follows that y is unique.

(b, $\Leftarrow$): Suppose (6) holds. By (a) r_1 is mono. For each $x \in X$ there is some $t \in R$ and $y = r_2 t$ so that $(x, y) \,\epsilon\, r$. Thus r_1 is onto, and by (G) r_1 is iso.

(c): This is clear from (b, $\Leftarrow$) since then r_1 is invertible, and we may take $f = r_2 r_1^{-1}$: for $x \in X$, $x = r_1 r_1^{-1} x$ and $fx = r_2 r_1^{-1} x$ so $(x, fx) \,\epsilon\, r$. □

Proposition 4.5. *Let $\mathcal{C}$ be a cartesian category which satisfies (G). Then a commutative diagram*

$$\begin{array}{ccc} P & \xrightarrow{\pi_1} & A \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

is a pullback diagram if, and only if,

$$(\forall x \in A)(\forall y \in B)[fx = gy \Longrightarrow (\exists! t \in P) x = \pi_1 t \;\&\; y = \pi_2 t]. \quad (7)$$

Proof. ($\Rightarrow$) Immediate. ($\Leftarrow$): Assume (7). It follows that π_1 and π_2 are jointly monic. Suppose there is given a commutative square

$$\begin{array}{ccc} Q & \xrightarrow{q_1} & A \\ {\scriptstyle q_2}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & C \end{array}$$

Form the pullback

$$\begin{array}{ccc} R & \xrightarrow{h} & Q \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle \langle q_1,q_2\rangle} \\ P & \xrightarrow[\langle \pi_1,\pi_2\rangle]{} & A \times B \end{array}$$

Clearly h is mono, since it is a pullback of a mono. By (7)

$$(\forall s \in Q)(\exists! t \in P)[q_1 s = \pi_1 t \ \&\ q_2 s = \pi_2 t] \tag{8}$$

but this implies that h is onto. Hence h is iso by (G). Thus $m = kh^{-1} : Q \longrightarrow P$ satisfies $\pi_i m = q_i$ for $i = 1, 2$, and is the desired map. It is unique since π_1 and π_2 are joint monic. □

5. Basic set-theoretic consequences

We mention some easy consequences of the axioms.

Proposition 5.1 (*Quotient Sets*)**.** *Suppose that the bicartesian category $\mathcal{C}$ satisfies (G). For any equivalence relation $r =_{\rm def} (r_1, r_2) : R \rightarrowtail (X, X)$ there exists a mapping $q : X \longrightarrow Q$ so that for all $(x_1, x_2) \in (X, X)$*

$$(x_1, x_2) \epsilon\, r \Longrightarrow qx_1 = qx_2 \tag{9}$$

and if $f : X \longrightarrow Y$ is any mapping with

$$(x_1, x_2) \epsilon\, r \Longrightarrow fx_1 = fx_2. \tag{10}$$

then there exists a unique $h : Q \longrightarrow Y$ with $hq = f$.

In case the category also satisfies (Eff) it follows that (9) is an equivalence.

Proof. Construct a coequalizer diagram

$$R \underset{r_2}{\overset{r_1}{\rightrightarrows}} X \xrightarrow{q} Q.$$

Since the diagram commutes, the implication (9) holds. Let $f : X \longrightarrow Y$ be any mapping satisfying the implication (10). Thus for any $t \in R$, $fr_1 t = fr_2 t$. Thus by Proposition 4.2 (a) we have $fr_1 = fr_2$ and since q is a coequalizer, there is a unique $h : Q \longrightarrow Y$ with $hq = f$.

From Axiom (Eff) it follows that there is some $e : X \longrightarrow E$ such that

$$(x_1, x_2) \epsilon\, r \Longleftrightarrow ex_1 = ex_2 \tag{11}$$

for all $(x_1, x_2) \in (X, X)$. Thus $er_1 = er_2$. Let $e' : Q \longrightarrow E$ be the unique mapping so that $e'q = e$. Thus if $qx_1 = qx_2$, it follows that $ex_1 = ex_2$ and hence $(x_1, x_2) \epsilon\, r$ by (11). □

Proposition 5.2 (*Induction*)**.** *Assume that $\mathcal{C}$ is a cartesian category which satisfies (G) and (NNO). Let $r : R \rightarrowtail N$. Suppose that $0 \epsilon\, r$ and that for each $n \in N$, $n \epsilon\, r$ implies $Sn \epsilon\, r$. Then for all $n \in N$, $n \epsilon\, r$.*

Proof. Since $0 \epsilon\, r$, there is $z : \mathbf{1} \longrightarrow R$ with $0 \equiv rz$. Form a pullback square

$$\begin{array}{ccc} P & \xrightarrow{q} & R \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle r} \\ R & \xrightarrow[S \circ r]{} & N \end{array}$$

As r is mono, so is p. We claim that p is onto. Let $u : \mathbf{1} \longrightarrow R$. Thus $ru \epsilon r$. Hence by assumption $Sru \epsilon r$. There is thus a map $v : \mathbf{1} \longrightarrow R$ with $Sru = rv$. By the pullback property there is $x : \mathbf{1} \longrightarrow P$ so that $px = u$ and $qx = v$. In particular p is onto. By (G) p is an isomorphism. Let p^{-1} be its inverse. Thus $qp^{-1} : R \longrightarrow R$. By the property of the natural number object there is a unique $f : N \longrightarrow R$ with $f0 = z$ and $fS = qp^{-1}f$. Now $(rf)0 = 0$ and

$$(rf)S = rqp^{-1}f = Srf.$$

But id_N instead of $r \circ f$ also satisfies these two equations. Thus $rf = \mathrm{id}$. Thus for any $n \in N$, $rfn = n$, and hence $n \epsilon r$. □

Proposition 5.3 (*Exponential Objects*)**.** *Assume that $\mathcal{C}$ is a cartesian category that satisfies (G) and (Π). Then for any objects X and Y there is an object E and a total function $e : (E, X) \longrightarrow Y$ such that for every morphism $f : X \longrightarrow Y$ there is a unique $s \in E$ such that for $x \in X$ and $y \in Y$:*

$$e(s, x) \equiv y \iff f \circ x \equiv y.$$

Theorem 5.4 (*Dependent Choices*)**.** *Assume that $\mathcal{C}$ is a cartesian category that satisfies (G), (Π), (Fct) and (PA). Then for any object X, any total relation $r = (r_1, r_2) : R \rightarrowtail (X, X)$ and any $x \in X$ there is a morphism $f : N \longrightarrow X$ with $f0 = x$ and for all $n \in N$*

$$(fn, f \circ Sn) \epsilon r. \qquad (12)$$

Proof (*Sketch*)**.** Take a projective cover $p : P \longrightarrow X$ of X. Since r is total, we have thus for each $u \in P$ some $v \in P$ with $(pu, pv) \epsilon r$. As P is a choice object, there is a morphism $g : P \longrightarrow P$ with $(pu, pgu) \epsilon r$ for all $u \in P$. Let $x \in X$. Then $p \circ w \equiv x$ for some $w \in P$. Now $\mathbf{1} \xrightarrow{0} N \xrightarrow{S} N$ is a natural numbers object, so there is $h : N \longrightarrow P$ with $h0 = w$ and $hS = gh$. Now it is easy to check by induction that $f =_{\mathrm{def}} ph$ satisfies (12). □

5.1. Constructing new relations

We review some of the possibilities to construct relations in a bicartesian category satisfying the axioms (G), (Π), (DP), (Fct) and (I). On any object X the identity mapping gives a universally true relation $\mathrm{t}_X = \mathrm{id}_X : X \longrightarrow X$, i.e. for all $x \in X$

$$x \epsilon \mathrm{t}_X.$$

The unique mapping from the initial object $\mathrm{f}_X : \mathbf{0} \longrightarrow X$ gives an universally false relation, i.e. for all $x \in X$,

$$\neg(x \epsilon \mathrm{f}_X).$$

If $E \xrightarrow{e} X \underset{h}{\overset{g}{\rightrightarrows}} Y$ is an equalizer diagram, then for $x \in X$

$$x \epsilon e \iff gx = hx.$$

Given a relation $r = (r_1, \ldots, r_n) : R \rightarrowtail (X_1, \ldots, X_n)$ we can extend it with a variable. Let Y be a object and let $R \xleftarrow{p} R' \xrightarrow{q} Y$ be a product diagram. The extended relation

$$r' = (r_1p, \ldots, r_np, q) : R' \rightarrowtail (X_1, \ldots, X_n, Y)$$

satisfies, for all $(x_1, \ldots, x_n, y) \in (X_1, \ldots, X_n, Y)$ that

$$(x_1, \ldots, x_n, y) \epsilon r' \iff (x_1, \ldots, x_n) \epsilon r.$$

If $\sigma : \{1, \ldots, n\} \longrightarrow \{1, \ldots, n\}$ is a permutation then

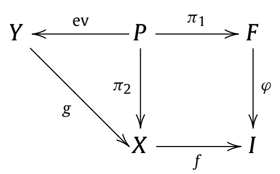
$$r_\sigma = (r_{\sigma(1)}, \ldots, r_{\sigma(n)}) : R \rightarrowtail (X_{\sigma(1)}, \ldots, X_{\sigma(n)})$$

satisfies for all $(x_1, \ldots, x_n) \in (X_{\sigma(1)}, \ldots, X_{\sigma(n)})$

$$(x_1, \ldots, x_n) \in r_\sigma \iff (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) \in r.$$

The following lemma is standard

Lemma 5.5. *If in the universal Π-diagram*



(13)

the mapping g is mono, then so is φ. □

Relations can be combined using the logical operations ($\wedge, \vee, \Rightarrow$) and quantifiers ($\forall$,$\exists$) over fixed objects:

Theorem 5.6. *Let $\mathcal{C}$ be a bicartesian category satisfying the axioms (G), (Π), (DP), (Fct) and (I). Let $r = (r_1, \ldots, r_n) : R \rightarrowtail (X_1, \ldots, X_n)$ and $s = (s_1, \ldots, s_n) : S \rightarrowtail (X_1, \ldots, X_n)$. Then exists $(r \wedge s)$, $(r \vee s)$, $(r \Rightarrow s) : R \rightarrowtail (X_1, \ldots, X_n)$ so that for all $x = (x_1, \ldots, x_n) \in (X_1, \ldots, X_n)$*

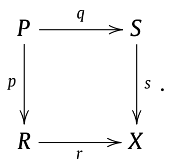
(a) *$x \epsilon (r \wedge s)$ if and only if $x \epsilon r$ and $x \epsilon s$,*
(b) *$x \epsilon (r \vee s)$ if and only if $x \epsilon r$ or $x \epsilon s$,*
(c) *$x \epsilon (r \Rightarrow s)$ if and only if $x \epsilon r$ implies $x \epsilon s$,*

Moreover, if $m : M \rightarrowtail (X_1, \ldots, X_n, Y)$ then there is $\forall(m) : A \rightarrowtail (X_1, \ldots, X_n)$ and $\exists(m) : E \rightarrowtail (X_1, \ldots, X_n)$ so that for all $x = (x_1, \ldots, x_n) \in (X_1, \ldots, X_n)$

(d) *$x \epsilon \forall(m)$ if and only if for all $y \in Y$, $(x_1, \ldots, x_n, y) \epsilon m$,*
(e) *$x \epsilon \exists(m)$ if and only if for some $y \in Y$, $(x_1, \ldots, x_n, y) \epsilon m$.*

Proof. By Proposition 3.1 it is enough to prove (a)–(c) for the case when $n = 1$; write $X = X_1, r = r_1, s = s_1$.

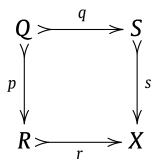
As for (a): form the pullback square



The diagonal, call it $(r \wedge s)$ is a mono. It is straightforward by the pullback property that the equivalence in (a) holds.

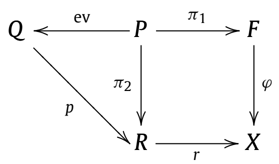
As for (b): form a sum diagram $R \xrightarrow{i} U \xleftarrow{j} S$. Let $f : U \longrightarrow X$ be the unique mapping with $r = fi$ and $s = fj$. Let $U \xrightarrow{e} I \xrightarrow{m} X$ be a factorization of f as an onto mapping followed by a mono (Fct). We claim that $(r \vee s) =_{\text{def}} m$ satisfies the equivalence in (b). Suppose that $x \in X$ satisfies $x \epsilon r$. Then $x = rt$ for some $t \in R$. Thus $x = fit = meit$, and hence $x \epsilon m$. Similarly $x \epsilon s$ implies $x \epsilon m$. Suppose on the other hand that $x \epsilon m$. Now, e is onto so there is some $u \in U$ with $x = fu$. Then by Axiom (DP) we have $u = it$ for some $t \in R$, in which case $x \epsilon r$, or we have $u = jv$ for some $v \in S$, in which case $x \epsilon s$.

As for (c): Form the pullback



(14)

Axiom (Π) yields for $Q \xrightarrow{p} R \xrightarrow{r} X$ a universal Π-diagram



(15)

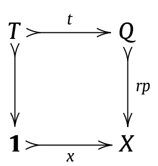
We claim that $(r \Rightarrow s) =_{\text{def}} \varphi$ makes the equivalence in (c) true. Let $x \in X$. To prove ($\Rightarrow$) assume that $x \epsilon \varphi$. Thus $x = \varphi u$ for some $u \in F$. Suppose $x \epsilon r$. Thus $x = rv$ for some $v \in R$. By the pullback in (15) there is $w : \mathbf{1} \longrightarrow w$ so that $\pi_2 w = v$ and $\pi_1 w = u$. We have further by the diagrams

$$x = rv = r\pi_2 w = rp\,\text{ev}\,w = sq\,\text{ev}\,w.$$

This shows $x \epsilon s$. As for the converse ($\Leftarrow$) suppose the implication

$$x \epsilon r \Rightarrow x \epsilon s$$

holds. We aim to show $x \epsilon \varphi$ using the properties of the universal Π-diagram. Form a pullback diagram



(16)

Then $\psi = (pt, t) : T \rightarrowtail (R, Q)$ is evidently a partial function since p and t are monic. If $\psi(u) \equiv v$, then there is $w \in T$, so that $u = ptw$ and $v = tw$, and hence $u = pv$ and $ru = rpv = x$. This verifies condition (a) of (Π). To verify condition (b) of (Π), assume that $ru = x$. Thus $x \epsilon r$, and so by the implication above $x \epsilon s$, i.e. $sw = x$, for some w. By the pullback (14) there is $v : \mathbf{1} \longrightarrow Q$ with $u = pv$ and $w = qv$. Thus $rpv = x$. But, by the pullback (16) there is $z : \mathbf{1} \longrightarrow T$ with $tz = v$. Now $(ptz, tz) = (u, v)$, i.e. $\psi(u) \equiv v$. According to (Π), there is now some $k \in F$ with $\varphi k = x$. Thus $x \epsilon \varphi$.

As for (d): Suppose $m = (m_1, m_2) : M \rightarrowtail (X, Y)$. Using (Fct) factor m_1 into a onto mapping followed by a mono $M \xrightarrow{e} I \xrightarrow{i} X$. Let $\exists(m) = i$. Thus using the fact that e is onto

$$x \epsilon \exists(m) \Leftrightarrow (\exists t \in I) x = it \Leftrightarrow (\exists s \in M) x = ies \Leftrightarrow (\exists s \in M) x = m_1 s.$$

The latter implies that $(x, m_2 s) \epsilon (m_1, m_2)$. Clearly $m_2 s \in Y$. Conversely, suppose that for some $y \in Y$ we have $(x, y) \epsilon (m_1, m_2)$. Thus for some $s \in M$ it holds that $x = m_1 s$ and $y = m_2 s$ and we have $x \epsilon \exists(m)$.

As for (e): Suppose $m = (m_1, m_2) : M \rightarrowtail (X, Y)$. First construct a product diagram $X \xleftarrow{p} U \xrightarrow{q} Y$. Then let $m' \equiv \langle m_1, m_2 \rangle_{p,q}$. Use (Π) to obtain a universal Π-diagram

$$\begin{array}{ccccc} M & \xleftarrow{\text{ev}} & P & \xrightarrow{\pi_1} & F \\ & \searrow^{m'} & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \varphi} \\ & & U & \xrightarrow[p]{} & X \end{array} \tag{17}$$

We let $\forall(m) = \varphi$. Suppose $x \in X$. To prove (e, $\Rightarrow$) suppose $x \epsilon \varphi$ and $y \in Y$. Thus $x = \varphi f$ for some $f \in F$ and moreover there is $u \in U$ with $x = pu$ and $y = qu$. By the pullback in (17) we get $w \in P$ so that $u = \pi_2 w$ and $f = \pi_1 w$. From the triangle of (17) it follows that $m' \text{ ev } w = \pi_2 w$. Hence $u \epsilon m'$ and thus $(x, y) \epsilon m$. To prove (e, $\Leftarrow$) let $x \in X$ be fixed and suppose that for all $y \in Y$, $(x, y) \epsilon m$. Let $n : N \longrightarrow M$ be the pullback of x along m_1:

$$\begin{array}{ccc} N & \xrightarrow{n} & M \\ \downarrow & & \downarrow{\scriptstyle m_1} \\ \mathbf{1} & \xrightarrow[x]{} & X \end{array} \tag{18}$$

Then $(m'n, n) : N \rightarrowtail (U, M)$ is a partial function since both m' and n are mono. As for condition (a): if $(u, v) \epsilon (m'n, n)$ then $u = m'nt$ and $v = nt$ for some $t \in N$. Clearly, $m'v = u$ and $pu = m_1 nt = x$. Regarding condition (b): Suppose that $u \in U$ satisfies $pu = x$. Let $y = qu$. By the first assumption $(x, y) \epsilon m$. Thus for some $s \in M$, $x = m_1 s$ and $y = m_2 s$. By construction of m' we have $m's = u$. Since $x = m_1 s$, the pullback (18) gives a unique $t \in N$ with $s = nt$. Thus t is a witness to $(u, s) \epsilon (m'n, n)$. Since conditions (a)–(b) are now verified, (Π) gives $f \in F$ satisfying, in particular, $\varphi f = x$. Hence $x \epsilon \varphi$. □

5.2. Decidable relations and classical logic

Let $\mathcal{C}$ be a CETCS category. Construct a two element set using the sum axiom $\mathbf{1} \xrightarrow{f} \mathbf{2} \xleftarrow{t} \mathbf{1}$. If $r : P \rightarrowtail X$ is decidable, i.e. for all $x \in X$,

$$x \epsilon r \quad \text{or} \quad \neg x \epsilon r,$$

then we can construct $\chi_r : X \longrightarrow \mathbf{2}$ so that for all $x \in X$

$$x \epsilon r \wedge \chi_r(x) = t \quad \text{or} \quad (\neg x \epsilon r) \wedge \chi_r(x) = f.$$

It follows that χ_r is the unique map $X \longrightarrow \mathbf{2}$ such that $x \epsilon r$ iff $\chi_r(x) = t$. Thus $\mathbf{1} \xrightarrow{t} \mathbf{2}$ classifies decidable relations. In case we take the axioms of CETCS with classical logic every relation is decidable, and hence $\mathbf{1} \xrightarrow{t} \mathbf{2}$ is a full subobject classifier for the category. In this case $\mathcal{C}$ is a topos.

The Lawvere's choice axiom (L5) states: if $f : A \longrightarrow B$ is mapping and A contains at least one element, then there is a mapping $g : B \longrightarrow A$ so that $fgf = f$.

Theorem 5.7. *In CETCS with classical logic (AC) and (L5) are equivalent.*

Corollary 5.8. *ETCS and CETCS + PEM + AC have the same theorems.*

6. Correspondence to standard categorical formulations

Lemma 6.1. *Let $\mathcal{C}$ be a cartesian category which satisfies (G). Then a pullback of an onto mapping is again an onto mapping.*

We recall some basic definitions from [7]:

Definition 6.2. A sequence of mappings $A \xrightarrow{e} I \xrightarrow{m} B$ is an *image factorization* of $f : A \longrightarrow B$ if $f \equiv m \circ e$ and m is a mono, and whenever $f \equiv m' \circ e'$ where $m' : I' \longrightarrow B$ is mono then there is some $t : I \longrightarrow I'$ with $m \equiv m' \circ t$. Such an m is a called an *image of* f.

Definition 6.3. A morphism f is a *cover* if whenever it can be factored as $f \equiv m \circ g$ where m mono, then m be must an isomorphism.

Proposition 6.4. *In any category, if $A \xrightarrow{e} I \rightarrowtail^{i} B$ is an image factorization of $f : A \longrightarrow B$, then e is a cover.*

Theorem 6.5. *Let $\mathcal{C}$ be a cartesian category satisfying (G). If $f : X \longrightarrow Y$ is factored as $X \xrightarrow{g} I \rightarrowtail^{i} Y$, where g is onto, then it is an image factorization.*

Proof. Suppose therefore that $X \xrightarrow{h} J \rightarrowtail^{j} Y$ is another factoring of f. It is sufficient to show that $i \leq j$ as subobjects of Y. By Proposition 4.3 it is equivalent to prove $i \dot{\subseteq} j$. Suppose that $y \in Y$ satisfies $y \,\epsilon\, i$. Then $y = it$ for some $t \in I$. Now g is onto, so there is $x \in X$ with $gx = t$. Now $y = igx = fx = j(hx)$. Hence $y \,\epsilon\, j$. Thus we have $i \dot{\subseteq} j$. □

Lemma 6.6. *Suppose that $\mathcal{C}$ is a cartesian category that satisfies (G). Then*

(a) *every onto mapping is a cover,*
(b) *if $\mathcal{C}$ in addition satisfies (Fct), then every cover is onto.*

Proof. (a): If $f : A \longrightarrow B$ is onto, then $A \xrightarrow{f} B \xrightarrow{\mathrm{id}} B$ is an image factorization, so by Theorem 6.5 and Proposition 6.4 f is a cover.

(b): Let $f : A \longrightarrow B$ be a cover. By (Fct) take a factorization $A \xrightarrow{e} I \xrightarrow{i} B$ of f where e is onto and i is mono. Now since f is a cover, i is an isomorphism. Hence f is onto as well. □

In standard category-theoretic terms [7] various combinations of the CETCS axioms can be characterized by the following theorems. First recall that a *regular category* is a category with finite limits, which has image factorization and where covers are preserved by pullbacks.

Theorem 6.7. *Let $\mathcal{C}$ be a cartesian category satisfying (G). Then $\mathcal{C}$ satisfies (Fct) if, and only if, $\mathcal{C}$ is a regular category where the terminal object is projective.*

Proof. ($\Rightarrow$) According to Theorem 6.5 $\mathcal{C}$ has image factorizations. By Lemma 6.6 onto morphisms are the same as covers. Thus by Lemma 6.1 covers are preserved by pullbacks. This shows that $\mathcal{C}$ is regular. If $A \longrightarrow \mathbf{1}$ is a cover then it is onto, and hence $\mathbf{1}$ is a choice object. Since the category is regular, it follows that $\mathbf{1}$ is projective.

($\Leftarrow$) Suppose $\mathbf{1}$ is projective. Hence any cover is onto. Thus by regularity, any morphism can be factored as an onto morphism followed by a mono. This gives (Fct). □

Theorem 6.8. *Let $\mathcal{C}$ be a cartesian category satisfying (G). Then $\mathcal{C}$ is locally cartesian closed if and only if $\mathcal{C}$ satisfies the axiom (Π).*

Proof. See Section 7. □

Lemma 6.9. *In a CETCS category $\mathcal{C}$ every epi is onto; consequently $\mathcal{C}$ is balanced.*

Proof. Let $f : A \longrightarrow B$ be an epi. Form the sum $\mathbf{1} \xrightarrow{i} S \xleftarrow{j} B$. Let $m : M \rightarrowtail B$ be a subobject so that $y \,\epsilon\, m$ iff $(\exists x \in A) fx = y$. Then form the sum $\mathbf{1} \xrightarrow{r} K \xleftarrow{s} M$ and let $k : K \longrightarrow S$ be the unique mapping so that $kr = i$ and $ks = jm$. Define, using Section 5.1, an equivalence relation $(r_1, r_2) : R \longrightarrow (S, S)$ by

$$(u, z) \,\epsilon\, (r_1, r_2) \iff ((\exists w \in K) kw = u \Leftrightarrow (\exists w \in K) kw = z).$$

By Proposition 5.1 let $q : S \longrightarrow Q$ be such that

$$(u, z) \,\epsilon\, (r_1, r_2) \iff qu = qz.$$

Let $g : B \longrightarrow S$ be given by $g = i \circ !_B$ and $h = j : B \longrightarrow S$. It is straightforward to check that for all $x \in A$, $qgfx = qhfx$. Thus $qgf = qhf$, and since f is epi, $qg = qh$. For each $y \in B$ we have, since $(\exists w \in K) kw = gy$ is true, that

$$(\exists w \in K) kw = hy.$$

By (DP) and disjointness of sums we must have $w = st$ for some $t \in M$. Hence $jmt = kst = kw = hy = jy$. Since j is mono, $mt = y$. Thus $y \,\epsilon\, m$, that is $(\exists x \in A) fx = y$.

The last statement follows by Axiom (G). □

Theorem 6.10. *Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ satisfies CETCS if, and only if, $\mathcal{C}$ has the following properties*

(i) *it is locally cartesian closed,*
(ii) *it is a pretopos,*
(iii) *it has NNO,*
(iv) *its terminal object is projective and generates $\mathcal{C}$,*
(v) $\mathbf{0} \not\cong \mathbf{1}$,
(vi) *it satisfies the disjunction property,*
(vii) *it has enough projectives.*

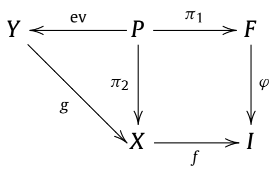
Proof. ($\Rightarrow$): (i) follows from Theorem 6.8. Properties (iii), (vi), (vii) are axioms of CETCS. (iv) follows from Theorem 6.7. (v) is clear by (I). By (Lemma 1.5.13–14, [7]) every locally cartesian closed which is cocartesian and balanced (Lemma 6.9) is a pretopos.

($\Leftarrow$) It is known that in a locally cartesian closed pretopos with NNO has all coequalizers (Remark 2.8, [18]). Using Theorem 6.8 we get axiom (Π). Axioms (G), (NNO), (PA) and (DP) are given. (I) follows easily from (v) using uniqueness of mappings. In a pretopos the pullback object of x and y in a sum diagram $\mathbf{1} \xrightarrow{x} S \xleftarrow{y} \mathbf{1}$ will be $\mathbf{0}$, so (NT) follows from (I). In pretopos every map can be factored as a cover followed by a mono. But using that $\mathbf{1}$ is projective we can show that covers are onto, so (Fct) is verified. In a pretopos all equivalence relations are effective, so (Eff) follows. □

7. Functor-free formulation of LCCCs

The standard way [7] of defining a locally cartesian category $\mathcal{C}$ is to say that it is a cartesian category so that pullbacks along a mapping $f : X \longrightarrow Y$ induces a functor $f^* : \mathcal{C}/Y \longrightarrow \mathcal{C}/X$ and that this functor has a right adjoint $\Pi_f : \mathcal{C}/X \longrightarrow \mathcal{C}/Y$. These functors must, in particular, be defined on the objects of the slice categories. This means that the pullback object must be possible to construct as a function of mappings $g : A \longrightarrow Y$ and $f : X \longrightarrow Y$. This can be forced if one assumes the full axiom of choice in the meta-theory of $\mathcal{C}$, but is not possible if we only use intuitionistic logic. Makkai [17] has developed a theory of functors – anafunctors – by which one can avoid such uses of choice. In [22] we showed how LCCCs could be formulated replacing f^* and Π_f by the appropriate anafunctors, so that Π_f is the right adjoint of f^*. We here extract what is the existence condition for such Π_f and formulate it without functors. Thus a functor-free formulation of LCCC will be given in Definition 7.1.

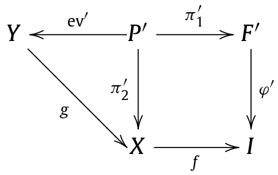
A Π-*diagram* for $Y \xrightarrow{g} X \xrightarrow{f} I$ is a commutative diagram of the form



(19)

where the square on the right is a pullback diagram. The object F is called the *parameter object* of the diagram.

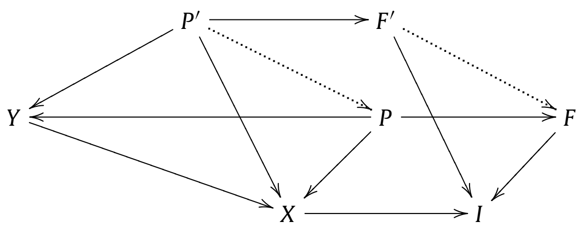
If we have a second Π-diagram for $Y \xrightarrow{g} X \xrightarrow{f} I$



(20)

we say that a mapping $t : F' \longrightarrow F$ is a Π-*diagram morphism* from the second diagram to the first diagram if $\varphi t \equiv \varphi'$ and the unique map $s : P' \longrightarrow P$ such that $\pi_2 s \equiv \pi_2'$ and $\pi_1 s \equiv t\pi_1'$ also satisfies $\mathrm{ev}s \equiv \mathrm{ev}'$.

(21)



It is easily seen that the Π-diagrams and Π-diagram morphisms over fixed mappings $Y \xrightarrow{g} X \xrightarrow{f} I$ forms a category.

A *universal* Π*-diagram for* $Y \xrightarrow{g} X \xrightarrow{f} I$: is a Π-diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\mathrm{ev}} & P & \xrightarrow{\pi_1} & F \\ & \searrow^{g} & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \varphi} \\ & & X & \xrightarrow[f]{} & I \end{array} \tag{22}$$

which is such that for any other Π-diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\mathrm{ev}'} & P' & \xrightarrow{\pi_1'} & F' \\ & \searrow^{g} & \downarrow{\scriptstyle \pi_2'} & & \downarrow{\scriptstyle \varphi'} \\ & & X & \xrightarrow[f]{} & I \end{array} \tag{23}$$

there is a unique mapping $n : F' \longrightarrow F$ so that $\varphi' \equiv \varphi n$ and that the unique mapping $m : P' \longrightarrow P$, with $n\pi_1' \equiv \pi_1 m$ and $\pi_2' \equiv \pi_2 m$, satisfies $\mathrm{ev}' \equiv \mathrm{ev}\, m$.

Definition 7.1. *A cartesian category is* locally cartesian closed, *if it satisfies the* generalized exponential axiom *or the* Π-axiom*: for every composable pair of maps* $Y \xrightarrow{g} X \xrightarrow{f} I$ *there is an universal exponential diagram as in* (22)*. That is, the category of* Π*-diagrams over* $Y \xrightarrow{g} X \xrightarrow{f} I$ *has a terminal object.*

7.1. Characterization of universal Π*-diagrams*

We have the following characterization of Π-diagrams where the parameter object is $F = 1$.

Lemma 7.2. *Consider a cartesian category satisfying (G). Let* $Y \xrightarrow{g} X \xrightarrow{f} I$ *be morphisms and let* $i \in I$ *be an element. For a pair of morphisms* $\psi = (r_1, r_2) : R \longrightarrow (X, Y)$ *the diagram*

$$\begin{array}{ccccc} Y & \xleftarrow{r_2} & R & \longrightarrow & 1 \\ & \searrow^{g} & \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle i} \\ & & X & \xrightarrow[f]{} & I \end{array} \tag{24}$$

is a Π*-diagram if and only if*

(A1) ψ *is a partial function (i.e.* r_1 *is mono)*
(A2) $(\forall x \in X)[fx = i \Longrightarrow (\exists y \in Y)(x, y) \epsilon\, \psi]$
(A3) $(\forall x \in X)(\forall y \in Y)[(x, y) \epsilon\, \psi \Longrightarrow fx = i \wedge gy = x]$

Proof. ($\Rightarrow$) Suppose (24) is a Π-diagram. Since i is mono, the pullback diagram entails that r_1 is mono. Hence ψ is a partial function. Property (A2) follows by the pullback property. (A3) follows since the whole diagram is commutative.

($\Leftarrow$) Suppose that (A1)–(A3) are satisfied. By (A3) it follows that the entire diagram commutes. (A1) and (A2) together yields that the square is a pullback. □

Lemma 7.3. *Consider two* Π*-diagrams in a cartesian category satisfying (G).*

$$\begin{array}{ccccc} Y & \xleftarrow{\mathrm{ev}'} & P' & \xrightarrow{\pi_1'} & F' \\ & \searrow^{g} & \downarrow{\scriptstyle \pi_2'} & & \downarrow{\scriptstyle \varphi'} \\ & & X & \xrightarrow[f]{} & I \end{array} \qquad \begin{array}{ccccc} Y & \xleftarrow{\mathrm{ev}} & P & \xrightarrow{\pi_1} & F \\ & \searrow^{g} & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \varphi} \\ & & X & \xrightarrow[f]{} & I \end{array} \tag{25}$$

Let $\chi : F' \longrightarrow F$ *be such that* $\varphi\chi = \varphi'$*. There is a unique* $\kappa : P' \longrightarrow P$ *so that* $\pi_1\kappa = \chi\pi_1'$ *and* $\pi_2\kappa = \pi_2'$*. For this* κ *it holds that* $\mathrm{ev}\kappa = \mathrm{ev}'$ *if and only if for all* $\upsilon \in F'$, $x \in X$ *and* $y \in Y$

$$(\upsilon, x, y) \epsilon\, (\pi_1', \pi_2', \mathrm{ev}') \Longleftrightarrow (\chi\upsilon, x, y) \epsilon\, (\pi_1, \pi_2, \mathrm{ev}).$$

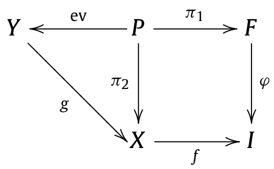
Proof. ($\Leftarrow$): Assume the equivalence. Let $t \in P'$ be arbitrary. We prove $\mathrm{ev}\kappa t = \mathrm{ev}'t$. Clearly $(\pi_1't, \pi_2't, \mathrm{ev}'t) \epsilon (\pi_1', \pi_2', \mathrm{ev}')$, so by the equivalence $(\chi\pi_1't, \pi_2't, \mathrm{ev}'t) \epsilon (\pi_1, \pi_2, \mathrm{ev})$. Thus there is a $u \in P$ with $\chi\pi_1't = \pi_1 u$, $\pi_2't = \pi_2 u$ and $\mathrm{ev}'t = \mathrm{ev}u$. Now we have $\pi_1 u = \chi\pi_1't = \pi_1\kappa t$ and $\pi_2 u = \pi_2't = \pi_2\kappa t$. By the pullback property, π_1 and π_2 are jointly mono, so $u = \kappa t$. Thus $\mathrm{ev}'t = \mathrm{ev}\kappa t$.

($\Rightarrow$): Assume $\mathrm{ev}\kappa = \mathrm{ev}'$. Suppose $(v, x, y) \epsilon (\pi_1', \pi_2', \mathrm{ev}')$. Thus for some $t \in P'$, it holds that $v = \pi_1't$, $x = \pi_2't$ and $y = \mathrm{ev}'t$. Hence $x = \pi_2\kappa t$, $y = \mathrm{ev}\kappa t$. Finally $\pi_1\kappa = \chi\pi_1'$ gives $\chi v = \pi_1\kappa t$, so that $(\chi v, x, y) \epsilon (\pi_1, \pi_2, \mathrm{ev})$. For the converse, assume $(\chi v, x, y) \epsilon (\pi_1, \pi_2, \mathrm{ev})$. Thus $\chi v = \pi_1 s$, $x = \pi_2 s$ and $y = \mathrm{ev}s$ for some $s \in P$. Then

$$f\pi_2 s = \varphi\pi_1 s = \varphi\chi v = \varphi' v.$$

Thus there is a unique $t \in P'$ with $\pi_2't = \pi_2 s$ and $\pi_1't = v$. We have then $\pi_2't = x$, so to prove $(v, x, y) \epsilon (\pi_1', \pi_2', \mathrm{ev}')$ it suffices to show $y = \mathrm{ev}'t$. Now $\mathrm{ev}'t = \mathrm{ev}\kappa t$. We have $\pi_1\kappa t = \chi\pi_1't = \chi v = \pi_1 s$ and $\pi_2\kappa t = \pi_2't = \pi_2 s$. By the pullback property π_1 and π_2 are jointly mono, so $\kappa t = s$. Hence $y = \mathrm{ev}s = \mathrm{ev}\kappa t = \mathrm{ev}'t$ as desired. □

Theorem 7.4. *Let $\mathcal{C}$ be a cartesian category satisfying (G). Let $Y \xrightarrow{g} X \xrightarrow{f} I$ be fixed morphisms. Suppose that the Π-diagram*



(26)

is universal for $Y \xrightarrow{g} X \xrightarrow{f} I$. Then for every $i \in I$ and for every pair of morphisms $\psi = (r_1, r_2) : R \longrightarrow (X, Y)$ satisfying (A1)–(A3), there is a unique $v \in F$ with $\varphi v = i$ such that for all $x \in X$ and $y \in Y$

$$(x, y) \epsilon \psi \iff (v, x, y) \epsilon \alpha. \tag{27}$$

Here $\alpha = (\pi_1, \pi_2, \mathrm{ev}) : P \longrightarrow (F, X, Y)$.

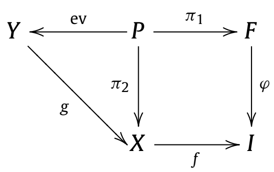
Proof. By Lemma 7.2 (24) is a Π-diagram. Since (26) is a universal diagram, there is a map $v : 1 \longrightarrow F$ such that $\varphi v = i$ and for all $u \in 1$, $x \in X$ and $y \in Y$,

$$(u, x, y) \epsilon (!, r_1, r_2) \iff (vu, x, y) \epsilon \alpha$$

(by Lemma 7.3). But $vu = v$ and $(u, x, y) \epsilon (!, r_1, r_2)$ is equivalent to $(x, y) \epsilon \psi$, so (27) is proved. □

There is a converse.

Theorem 7.5. *Let $\mathcal{C}$ be a cartesian category satisfying (G). Let $Y \xrightarrow{g} X \xrightarrow{f} I$ be fixed morphisms. Consider the Π-diagram*



(28)

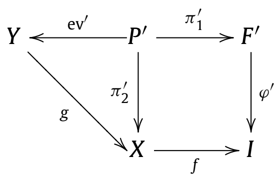
and let $\alpha = (\pi_1, \pi_2, \mathrm{ev}) : P \longrightarrow (F, X, Y)$.

Suppose that for every $i \in I$ and for every pair of morphisms $\psi = (r_1, r_2) : R \longrightarrow (X, Y)$ satisfying (A1)–(A3), there is a unique $v \in F$ with $\varphi v = i$ such that for all $x \in X$ and $y \in Y$

$$(x, y) \epsilon \psi \iff (v, x, y) \epsilon \alpha.$$

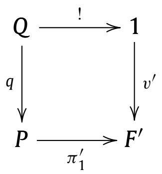
Then (28) is universal for $Y \xrightarrow{g} X \xrightarrow{f} I$.

Proof. Let



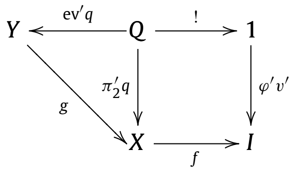
(29)

be an arbitrary Π-diagram. For $v' \in F'$ form the pullback



(30)

Then the composed diagram



(31)

is, obviously, again a Π-diagram. For $x \in X$ and $y \in Y$ we then have

$$(x, y) \epsilon (\pi_2' q, \mathrm{ev}' q) \iff (v', x, y) \epsilon (\pi_1', \pi_2', \mathrm{ev}').$$

Indeed, suppose $x = \pi_2' qu$ and $y = \mathrm{ev}' qu$ for some $u \in Q$. We have by (30) $v' = v'!u = \pi_1' qu$. Hence $(v', x, y) \epsilon (\pi_1', \pi_2', \mathrm{ev}')$. Conversely, suppose $v' = \pi_1' t, x = \pi_2' t$ and $y = \mathrm{ev}' t$ for some $t \in P'$. From $v' = \pi_1' t$ it follows by (30) that there is a unique $s \in Q$ with $t = qs$. Thus $x = \pi_2' qs$ and $y = \mathrm{ev}' qs$ and hence $(x, y) \epsilon (\pi_2' q, \mathrm{ev}' q)$.

Now (31) is a Π-diagram so $\psi = (\pi_2' q, \mathrm{ev}' q) : Q \longrightarrow (X, Y)$ satisfies (A1)–(A3) for $i = \varphi' v'$ (by Lemma 7.3). Hence by assumption we have that there is a unique $v \in F$ with $\varphi v = \varphi' v'$ and for all $x \in X$ and $y \in Y$

$$(v, x, y) \epsilon (\pi_1, \pi_2, \mathrm{ev}) \iff (x, y) \epsilon \psi.$$

In conclusion, for every $v' \in F'$ there is a unique $v \in F$ such that $\varphi v = \varphi' v'$ and

$$(\forall x \in X)(\forall y \in Y)[(v, x, y) \epsilon (\pi_1, \pi_2, \mathrm{ev}) \iff (v', x, y) \epsilon (\pi_1', \pi_2', \mathrm{ev}')]. \tag{32}$$

By unique choice (Proposition 4.4) and Theorem 5.6 there is $\chi : F' \longrightarrow F$ so that for all $v' \in F$ it holds that $\varphi \chi v' = \varphi' v'$ and

$$(\forall x \in X)(\forall y \in Y)[(\chi v', x, y) \epsilon (\pi_1, \pi_2, \mathrm{ev}) \iff (v', x, y) \epsilon (\pi_1', \pi_2', \mathrm{ev}')]. \tag{33}$$

Hence according to Lemma 7.3 the unique map $\kappa : P' \longrightarrow P$ satisfying $\pi_1 \kappa = \chi \pi_1'$ and $\pi_2 \kappa = \pi_2'$ also satisfies $\mathrm{ev}\kappa = \mathrm{ev}'$. To finish the proof we have to show that χ is unique. Suppose that $\theta : F' \longrightarrow F$ satisfies $\varphi\theta = \varphi'$ and that $\lambda : P' \longrightarrow P$ is the unique map with $\pi_1 \lambda = \theta \pi_1'$ and $\pi_2 \lambda = \pi_2'$, and that this λ satisfies $\mathrm{ev}\lambda = \mathrm{ev}'$. Then by Lemma 7.3, it holds for all $v' \in F$

$$(\forall x \in X)(\forall y \in Y)[(\theta v', x, y) \epsilon (\pi_1, \pi_2, \mathrm{ev}) \iff (v', x, y) \epsilon (\pi_1', \pi_2', \mathrm{ev}')].$$

Thus by the uniqueness in (32) for all $v' \in F', \theta v' = \chi v'$. Hence $\theta = \chi$. □

Corollary 7.6 (*Theorem* 6.8)**.** *Let $\mathcal{C}$ be a cartesian category satisfying (G). Then $\mathcal{C}$ is locally cartesian closed if and only if $\mathcal{C}$ satisfies the axiom (Π).*

Proof. ($\Rightarrow$) This is Theorem 7.4. ($\Leftarrow$) Suppose axiom (Π) holds. By Theorem 7.5 this says that every $Y \xrightarrow{g} X \xrightarrow{f} I$ has a universal Π-diagram. □

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