Discrete Applied Mathematics 99 (2000) 413–425

on and similar papers at core.ac.uk

provide

Demidenko matrices

Vladimir G. Deĭneko [*,1], Gerhard J. Woeginger [2]

Institut für Mathematik B, Technische Universitaet Graz, Steyrergasse 30, A-8010 Graz, Austria

Received 8 October 1997; received in revised form 16 February 1999; accepted 9 March 1999

Abstract

It is well-known that the Travelling Salesman Problem (TSP) is solvable in polynomial time, if the distance matrix fulfills the so-called *Demidenko* conditions. This paper investigates the closely related *Maximum Travelling Salesman Problem* (MaxTSP) on symmetric Demidenko matrices. Somewhat surprisingly, we show that — in strong contrast to the minimization problem — the maximization problem is NP-hard to solve. Moreover, we identify several special cases that are solvable in polynomial time. These special cases contain and generalize several predecessor results by Quintas and Supnick and by Kalmanson. © 2000 Elsevier Science B.V. All rights reserved.

MSC: primary: 90C27

Keywords: Travelling salesman problem; Demidenko condition; Kalmanson condition; Supnick condition; Combinatorial optimization; Computational complexity; Polynomial algorithm

1. Introduction

In the *Travelling Salesman Problem* (TSP), the objective is to find for a given $n \times n$ distance matrix $C = (c_{i,j})$ a cyclic permutation τ of the set $\{1, 2, \ldots, n\}$ that *minimizes* the function

$$c(\tau) = \sum_{i=1}^{n} c_{i,\tau(i)}. \tag{1}$$

* Corresponding author.

E-mail address: deineko@opt.math.tu-graz.ac.at (V.G. Deĭneko)

[1] On leave from Department of Applied Mathematics, Dnepropetrovsk University, Gagarin Av. 72, 320625 Dnepropetrovsk, Ukraine.

[2] Supported by the START program Y43-MAT of the Austrian Ministry of Science.

0166-218X/00/$ - see front matter © 2000 Elsevier Science B.V. All rights reserved.

PII: S0166-218X(99)00148-1

The cyclic permutations are also called *tours*, the elements of $\{1,2,\ldots,n\}$ are also called *cities*, and $c(\tau)$ will be called the *length* of the permutation τ. In other words, the salesman must visit the cities 1 to n in arbitrary order and he wants to minimize his total travel length while doing this. In this paper, we mainly consider the closely related maximization version of the TSP where the salesman wants to *maximize* his travel length; this problem is called the MaxTSP.

It is well known that the TSP and the MaxTSP both are NP-hard problems (see e.g. the book by Lawler et al. [7]). Hence, one branch of research started to investigate specially structured cases of the TSP which can be solved in polynomial time. For comprehensive information on solvable special cases, the reader is referred to the surveys by Gilmore et al. [5] and by Burkard et al. [2]. Although the MaxTSP reduces to the TSP (and vice versa), the special combinatorial structure that leads to a well solvable case for the TSP does not necessarily yield a well solvable case for the MaxTSP. In this paper, we discuss questions of this flavor.

Let us shortly review some of the most prominent well solvable cases of the TSP that will be relevant for the rest of this paper. A symmetric $n\times n$ matrix $C=(c_{i,j})$ is called a *Supnick* matrix if

$$c_{i,j}+c_{j+1,l}\leqslant c_{i,j+1}+c_{j,l}\leqslant c_{i,l}+c_{j,j+1}\quad \text{for } 1\leqslant i<j<j+1<l\leqslant n. \tag{2}$$

It was shown by Supnick [10] that the TSP restricted to matrices with the property (2) is solved by the tour $\langle 1,3,5,7,9,11,\ldots,8,6,4,2,1\rangle$, i.e. the tour first visits the odd-numbered cities in increasing order and then it visits the even-numbered cities in decreasing order (cf. Section 2 for notation on tours). Moreover, Supnick [10] showed that the MaxTSP on Supnick matrices is solved by the tour $\langle 1,n-1,3,n-3,5,n-5,\ldots,n-2,2,n,1\rangle$.

A symmetric $n\times n$ matrix C is called a *Kalmanson* matrix if it fulfills the *Kalmanson conditions* [6]

$$c_{i,j}+c_{s,t}\leqslant c_{i,s}+c_{j,t}\quad \text{for } 1\leqslant i<j<s<t\leqslant n, \tag{3}$$

$$c_{i,t}+c_{j,s}\leqslant c_{i,s}+c_{j,t}\quad \text{for } 1\leqslant i<j<s<t\leqslant n. \tag{4}$$

Kalmanson [6] proved that the TSP restricted to matrices with the properties (3)–(4) is solved by the tour $\langle 1,2,3,4,\ldots,n-1,n,1\rangle$. Kalmanson also analyzed the MaxTSP on these matrices, and he showed that in the case $n=2k+1$ the MaxTSP is solved by the tour $\langle 1,k+2,2,k+3,3,k+4\ldots,k,n,k+1,1\rangle$. If $n=2k$ holds, then an optimum tour can be found among the n tours τ_t, $t=1\ldots n$, where τ_t is given by

$$\tau_t=\langle \phi_t(1),\phi_t(k+1),\phi_t(2),\phi_t(k+3),\ldots,\phi_t(k+4),\phi_t(3),\phi_t(k+2)\rangle, \tag{5}$$

and where the values ϕ_t are cyclic shifts defined by $\phi_t(i)=(t+i-1)\bmod n$ for $i=1,2,\ldots,n$. Observe that if the cities of a convex point set are numbered along the convex hull, the resulting distance matrix is a Kalmanson matrix. Therefore, Kalmanson's results can be considered as generalizations of the purely geometric investigations of Quintas and Supnick [8,9] on the convex Euclidean TSP.

Supnick matrices and Kalmanson matrices are subclasses of the class of symmetric *Demidenko* matrices: A symmetric matrix $C=(c_{i,j})$ is called a *Demidenko matrix* if

$$c_{i,j}+c_{s,t}\leqslant c_{i,s}+c_{j,t} \quad \text{for } 1\leqslant i<j<s<t\leqslant n. \tag{6}$$

In 1976, Demidenko [3] proved in his celebrated paper that the TSP on Demidenko distance matrices can be solved in $O(n^2)$ time (see also [5]). More precisely, Demidenko proved that there always exists an optimum tour that is *pyramidal*: A tour τ is pyramidal if it is of the form $\tau=\langle 1,i_1,i_2,\dots,i_r,n,j_1,\dots,j_{n-r-2}\rangle$ where $i_1<i_2<\cdots<i_r$ and $j_1>\cdots>j_{n-r-2}$ hold. A minimum cost pyramidal tour can be determined in $O(n^2)$ time by a dynamic programming approach.

Blokh and Gutin [1] investigate a well-solvable special case of the MaxTSP where all non-zero entries of the distance matrix are close to the main diagonal.

To recapitulate the above paragraphs, solvable special cases of the TSP have attracted a lot of attention in the literature over the last decade. Inspired by the survey paper by Gilmore et al. [5], numerous papers dealt with finding conditions on the distance matrix that would guarantee that there is an optimal tour that is pyramidal. In this paper, we look at the MaxTSP instead of the standard TSP. We investigate tours that are kind of diametrically opposite to pyramidal tours. Our paper provides a formal framework and enhances the understanding of the issues involved.

Results of this paper: Section 2 summarizes some of the notation that is used throughout the paper. In Section 3, we start our investigations on the MaxTSP on symmetric Demidenko matrices. Note that the MaxTSP on Demdidenko matrices is equivalent to the standard TSP on *reverse* Demdidenko matrices, i.e. on matrices which fulfill inequalities (6) with reversed inequality signs. We describe a strongly structured subset $\mathcal{M}_n$ of permutations that always contains a longest tour on symmetric Demidenko matrices. To our surprise, we had to find out that the problem of finding a longest tour on symmetric Demidenko matrices is NP-hard; the proof is given in Section 3.

In the rest of the paper, we derive some positive results on the MaxTSP on symmetric Demidenko matrices. First, Section 4 deals with a subclass of symmetric Demidenko matrices on which the MaxTSP is trivial to solve: The optimum MaxTSP tour on these matrices can be given in advance and without regarding the precise numerical values of the data (of course, only in case the matrix is known to belong to this class). Secondly, in Section 5, we generalize Kalmanson's results from [6] by introducing the class of so-called *relaxed Kalmanson* matrices. The optimum MaxTSP tour in a relaxed Kalmanson can always be found among a set of $n/2$ tours with a very simple combinatorial structure. Clearly, this yields a polynomial time solution for this class. Finally, Section 6 gives the conclusion.

2. Notation

The set of all permutations of $\{1,2,\dots,n\}$ is denoted by S_n. For $\tau \in S_n$, we denote by τ^{-1} the *inversion* of τ, i.e. the permutation for which $\tau^{-1}(i)$ is the predecessor of i in the tour τ, for $i=1,\dots,n$. For $k>1$, we define $\tau^k(i)$ as $\tau(\tau^{(k-1)}(i))$ and $\tau^{-k}(i)$ as $\tau^{-1}(\tau^{-(k-1)}(i))$. We also use a cyclic representation of a cyclic permutation in the form

$$\tau = \langle i, \tau(i), \tau^2(i), \tau^3(i), \dots, \tau^{-2}(i), \tau^{-1}(i), i\rangle,$$

and we refer to it as a *tour*. A pair (i,j) with $j=\tau(i)$ is referred as an *arc* of the tour τ.

3. The MaxTSP on Demidenko matrices: first results

Let us define a subset $\mathcal{M}_n$ of tours on the set of n vertices by

$$\mathcal{M}_n = \{\tau \in S_n \mid \tau^{-1}(i), \tau(i) \geqslant (n+1)/2 \quad \text{for all } i<(n+1)/2 \text{ and}$$
$$\tau^{-1}(i), \tau(i) \leqslant (n+1)/2 \quad \text{for all } i>(n+1)/2\}.$$

Intuitively speaking, a tour $\tau \in \mathcal{M}_n$ always jumps from the first half of the cities to the second half, then back to the first half, to the second half, and so on. Another way of defining $\mathcal{M}_n$ is to use the concepts of *peaks* (city i is a peak if $i > \max\{\tau^{-1}(i), \tau(i)\}$) and *valleys* (city i is a valley if $i < \min\{\tau^{-1}(i), \tau(i)\}$). In this language, a tour τ is in $\mathcal{M}_n$ if and only if all cities $1,2,\dots,\lfloor n/2 \rfloor$ are valleys and all cities $\lceil n/2\rceil + 1,\dots,n$ are peaks. Note that for even n, every city in a tour in $\mathcal{M}_n$ must be a valley or a peak, whereas for odd n, the city $\lceil n/2 \rceil$ need not be a valley or a peak.

In Sections 4 and 5, we will often use the following observation. For any even number n of the form $n=2k$, the set $\mathcal{M}_n$ contains tours of the following two types.

- *Tours of type* (I): A tour of type (I) neither contains the pair of arcs $(i,k),(k,j)$ nor does it contain the pair of arcs $(s,k+1),(k+1,t)$ with $i<k<j$ and $s>k+1>t$.
- *Tours of type* (II): A tour type (II) contains a pair of arcs $(i,k),(k,j)$ and a pair of arcs $(s,k+1),(k+1,t)$ with $i<k<j$ and $s>k+1>t$.

Theorem 3.1. *For the MaxTSP on a symmetric Demidenko matrix C, there always exists an optimum tour that belongs to $\mathcal{M}_n$. Moreover, if every inequality in* (6) *is strict then all optimum tours belong to $\mathcal{M}_n$.*

Proof. A pair of arcs (i,j) and (s,t) in a tour τ is called a *non-crossing pair* if $i<j<s<t$ or $i>j>s>t$ holds. First we prove the following assertion: Given an arbitrary tour τ, there exists a tour τ_T without non-crossing pairs of arcs such that $c(\tau) \leqslant c(\tau_T)$. Suppose that τ contains a non-crossing pair of arcs (i,j) and (s,t) with $i<j<s<t$. Transform τ into a tour τ_1 by reversing the subpath $\langle j, \tau(j), \dots, \tau^{-1}(s), s\rangle$,

i.e. by deleting the non-crossing arcs (i,j) and (s,t), by reversing all the arcs $(j,\tau(j)),\ldots,(\tau^{-1}(s),s)$ into arcs $(\tau(j),j),\ldots,(s,\tau^{-1}(s))$, and by introducing two new arcs (i,s) and (j,t). Clearly,

$$c(\tau_1)-c(\tau)=c_{i,s}+c_{j,t}-c_{i,j}-c_{s,t}.$$

Together with (6) this yields $c(\tau)\leqslant c(\tau_1)$, and if every inequality in (6) is strict then $c(\tau)<c(\tau_1)$. Now consider the tour τ_1, and delete the next pair of non-crossing arcs and so on. Perform a symmetric procedure for non-crossing pairs with $i>j>s>t$. In order to prove that the process will stop after a finite number of steps consider the potential function $p(\tau)=\sum_{i=1}^{n}|i-\tau(i)|$. Since $p(\tau)<n^2$ for any tour τ and since $p(\tau_1)>p(\tau)$, the convergence of the process is guaranteed.

What remains to be proved is that the set $\mathscr{M}_n$ is exactly the set of all tours without non-crossing pairs of arcs. Clearly, every tour in $\mathscr{M}_n$ does not possess non-crossing pairs of arcs. Suppose now that a tour τ does not contain a non-crossing pair of arcs. This implies that (a) there exists an integer m_1 such that $i\leqslant m_1$ and $m_1\leqslant\tau(i)$ holds for all arcs $(i,\tau(i))$ with $i<\tau(i)$, and that (b) there exists an integer m_2 such that $i\geqslant m_2$ and $m_2\geqslant\tau(i)$ holds for all arcs $(i,\tau(i))$ with $i>\tau(i)$. If n is odd with $n=2k+1$, it follows that $m_1=m_2=k+1$, and if n is even with $n=2k$, it follows that $\{m_1,m_2\}=\{k,k+1\}$. It is easy to see that any tour τ with such a structure belongs to $\mathscr{M}_n$. □

Theorem 3.2. *The MaxTSP on symmetric Demidenko matrices is an* NP-*hard problem.*

Proof. The proof is done by a reduction from the NP-hard Hamiltonian Cycle Problem in Bipartite Graphs (cf. [4]). Let $G=(A\cup B,E)$ be a bipartite graph with $E\subseteq A\times B$ and $|A|=|B|$. Let $A=\{a_1,\ldots,a_k\}$ and $B=\{b_1,\ldots,b_k\}$. From G, we construct a $2k\times 2k$ symmetric Demidenko matrix $C=(c_{i,j})$ as follows: For $i,j=1,\ldots,k$ we set

$$c_{i,j}=-2k(2k+2-i-j) \quad\text{and}\quad c_{k+i,k+j}=-2k(i+j).$$

Moreover, if there is an edge between a_i and b_j, then we set $c_{i,k+j}=c_{k+j,i}=1$ and otherwise we set $c_{i,k+j}=c_{k+j,i}=0$. This completes the description of the symmetric matrix C. If the reader prefers distance matrices with non-negative entries, he should simply add a large positive constant to every entry of C; this will not change the argument.

First we argue that the constructed matrix C indeed is a symmetric Demidenko matrix: Let $1\leqslant i<j<s<t\leqslant 2k$ be four indices as in (6). If $i,j,s,t\leqslant k$, then

$$c_{i,j}+c_{s,t}=-2k(4k+4-i-j-s-t)=c_{i,s}+c_{j,t}$$

and inequality (6) is fulfilled. A similar equation holds in case $i,j,s,t\geqslant k+1$. If $i\leqslant k$ and $j,s,t\geqslant k+1$, then

$$c_{i,j}+c_{s,t}\leqslant 1-2k(s+t)\leqslant 0-2k(j+t)\leqslant c_{i,s}+c_{j,t},$$

where the middle inequality follows from $j < s$. A similar equation holds in case $i, j, s \leqslant k$ and $t \geqslant k+1$. Finally, if $i, j \leqslant k$ and $s, t \geqslant k+1$, then

$$c_{i,j} + c_{s,t} = -2k(2k+2-i-j) - 2k(s+t) \leqslant 0 \leqslant c_{i,s} + c_{j,t}.$$

Hence, in all possible cases inequality (6) is fulfilled, and matrix C indeed is a symmetric Demidenko matrix. Next we claim that there exists a tour τ for C with length $c(\tau) = 2k$ if and only if the bipartite graph G possesses a Hamiltonian cycle.

Proof of the (if)-part: Let $a_{i_1}, b_{j_1}, a_{i_2}, b_{j_2}, \ldots, a_{i_k}, b_{j_k}$ denote the Hamiltonian cycle in G. Then every arc of the tour $\langle i_1, k+j_1, i_2, k+j_2, \ldots, i_k, k+j_k \rangle$ has length 1, and thus this tour has length $2k$.

Proof of the (only if)-part: Suppose that there exists a tour τ for C with length $c(\tau) = 2k$. Since every entry of C is less or equal to 1, every one of the $2k$ arcs of τ must have length 1. By the definition of C, the tour must alternate between $\{1, \ldots, k\}$ and $\{k+1, \ldots, 2k\}$ and hence must be of the form $\langle i_1, k+j_1, i_2, \ldots, i_k, k+j_k \rangle$. Then $a_{i_1}, b_{j_1}, a_{i_2}, \ldots, b_{j_k}$ is a Hamiltonian cycle in G. □

Corollary 3.3. *The problem of finding for a given input matrix C a longest tour in the set* $\mathscr{M}_n$ *is NP-hard.*

4. The MaxTSP on Demidenko matrices: a special case

This section deals with a subclass of symmetric Demidenko matrices on which the MaxTSP is trivial to solve. For these matrices, the optimum tour can be given in advance and without regarding the precise numerical values of the data. For even n of the form $n = 2k$, the optimum tour will be

$$\tau^\star = \langle 1, k+1, 2, k+3, 4, k+5, 6, \ldots, 7, k+6, 5, k+4, 3, k+2, 1 \rangle$$

and for odd n of the form $n = 2k+1$, the optimum tour will be

$$\sigma^\star = \langle 1, k+2, 2, k+3, 3, k+4, 4, k+5, 5, \ldots, k, n, k+1, 1 \rangle.$$

These two results are proved in Theorems 4.1 and 4.2, respectively.

Theorem 4.1. *Let C be a symmetric* $n \times n$ *Demidenko matrix with* $n = 2k$ *that additionally fulfills the conditions*

$$c_{i,k} + c_{k+1,j} \leqslant c_{k+1,k} + c_{i,j}, \quad i = 1, \ldots, k-1, \; j = k+1, \ldots, n, \tag{7}$$

$$c_{1,k+1} + c_{i,j} \geqslant c_{1,j} + c_{i,k+1}, \quad i = 2, \ldots, k, \; j = k+2, \ldots, n, \tag{8}$$

$$c_{p+1,k+p} + c_{i,j} \geqslant c_{p+1,j} + c_{i,k+p}, \quad i = p+2, \ldots, k, \; j = k+p+1, \ldots, n, \tag{9}$$

$$c_{p,k+p+1} + c_{i,j} \geqslant c_{p,j} + c_{i,k+p+1}, \quad i = p+1, \ldots, k, \; j = k+p+2, \ldots, n, \; p = 1, \ldots, k-2 \tag{10}$$

then the tour $\tau^\star$ *is a tour of maximum length.*

Proof. Observe that for $n=2k$, it easily follows from conditions (7) that a tour of maximum length can be found among the tours of type (I): Any tour of type (II) with two pairs of arcs $(i,k),(k,j)$ and $(t,k+1),(k+1,s)$ with $i<k<j$ and $t>k+1>s$ can be transformed into a tour of type (I) by reversing the path $\langle k,j,\dots,t\rangle$, without decreasing the length of the tour.

Now let τ be an arbitrary tour of type (I). Our goal is to prove that conditions (8)–(10) imply that $c(\tau^\star)\geqslant c(\tau)$ holds. Again, we will use a tour improvement technique: We look through the cities $\tau(1),\tau^2(1),\tau^{-1}(1),\tau^{-2}(1),\tau^3(1),\tau^4(1),\dots$ and compare them with the cities in the corresponding places in the tour $\tau^\star$. If we find an index i with $\tau(i)\neq\tau^\star(i)$, then we transform τ into another tour τ_1 by defining $\tau_1(i)=\tau^\star(i)$ in the way as described below.

First, suppose that $\tau(1)=l\neq k+1$ holds, that is that the tour τ is of the form $\langle 1,l,q_1,\dots,q_t,k+1,i,\dots,1\rangle$ with $i<k+1<l$. Define the new tour $\tau_1=\langle 1,k+1,q_t,\dots,q_1,l,i,\dots,1\rangle$. Then $c(\tau_1)-c(\tau)=c_{1,k+1}+c_{i,l}-c_{1,l}-c_{i,k+1}$ holds. Together with (8) this yields $c(\tau_1)\geqslant c(\tau)$. Next, suppose that $\tau_1(k+1)=i\neq 2$ and $\tau_1(2)=l$ with $2<i<k+1<l$. Then the tour $\tau_2=\langle 1,k+1,2,\dots,i,l,\dots,1\rangle$ which is obtained from τ_1 by reversing the subpath $\langle i,\tau_1(i),\dots,2\rangle$ fulfills $c(\tau_2)-c(\tau_1)=c_{2,k+1}+c_{i,l}-c_{2,l}-c_{i,k+1}$. With this, inequality (9) yields $c(\tau_2)\geqslant c(\tau_1)$.

In the next step we check the placements of cities j for $j=k+2,3,k+3,4,k+4,\dots,k,2k$ in the tour τ_2 and compare their positions with the corresponding positions in $\tau^\star$. Denote by $j_{\min}$ the first city which is placed at different positions in $\tau^\star$ and in τ_2. We distinguish four cases (where $2m\leqslant k$ holds): (a) $j_{\min}=k+2m$, (b) $j_{\min}=2m+1$, (c) $j_{\min}=k+2m+1$, and (d) $j_{\min}=2m+2$.

Case a: If $j_{\min}=k+2m$ holds, then

$$\tau_2=\langle 1,k+1,2,\dots,i,k+2m,\dots,l,2m-1,\dots,1\rangle$$

with $2m-1<i<k+1$ and $l>k+2m$. By reversing the subpath $\langle k+2m,\dots,l\rangle$ in τ_2, we obtain the new tour

$$\tau_3=\langle 1,k+1,2,\dots,i,l,\dots,k+2m,2m-1\dots,1\rangle.$$

Clearly, $c(\tau_3)-c(\tau_2)=c_{i,l}+c_{2m-1,k+2m}-c_{i,k+2m}-c_{2m-1,l}$, and it follows from (10) with $p=2m-1$ that $c(\tau_3)\geqslant c(\tau_2)$.

Case b: If $j_{\min}=2m+1$ holds, then

$$\tau_2=\langle 1,k+1,\dots,k+2m-1,2m,\dots,t,2m+1,\dots,s,k+2m,2m-1,\dots,k+2,1\rangle$$

with $2m+1<s<k+1$ and $t>k+2m$. By reversing the subpath $\langle 2m+1,\dots,s\rangle$ in τ_2 we obtain a new tour

$$\tau_3=\langle 1,k+1,\dots,2m,\dots,t,s,\dots,2m+1,k+2m,2m-1,\dots,k+2,1\rangle.$$

In this case $c(\tau_3)-c(\tau_2)=c_{2m+1,k+2m}+c_{st}-c_{2m+1,t}-c_{s,k+2m}$, and it follows from (9) with $p=2m$ that $c(\tau_3)\geqslant c(\tau_2)$.

Case c: If $j_{\min}=k+2m+1$ holds, then

$$\tau_2=\langle 1,k+1,\dots,2m,l\dots,k+2m+1,i,\dots,2m+1,k+2m,\dots,k+2,1\rangle$$

with $2m < i < k+1$ and $l > k+2m+1$. By reversing the subpath $\langle l,\dots,k+2m+1\rangle$ in τ_2 we obtain a new tour

$$\tau_3 = \langle 1, k+1,\dots,2m, k+2m+1\dots,l,i,\dots,2m+1,k+2m,\dots,k+2,1\rangle.$$

In this case $c(\tau_3)-c(\tau_2)=c_{2m,k+2m+1}+c_{i,l}-c_{2m,l}-c_{i,k+2m+1}$, and it follows from (10) with $p=2m$ that $c(\tau_3)\geqslant c(\tau_2)$.

Case d: If $j_{\min}=2m+2$ holds, then

$$\tau_2 = \langle 1, k+1,\dots,2mkk+2m+1,s,\dots,2m+2,t,\dots,k+2,1\rangle$$

with $2m+2 < s < k+1$ and $t > k+2m+1$. By reversing the subpath $\langle s,\dots,2m+2\rangle$ in τ_2, we obtain a new tour

$$\tau_3 = \langle 1, k+1,\dots,2m, k+2m+1, 2m+2,\dots,s,t,\dots,k+2,1\rangle.$$

In this case $c(\tau_3)-c(\tau_2)=c_{2m+2,k+2m+1}+c_{st}-c_{s,k+2m+1}-c_{2m+2,t}$, and it follows from (9) with $p=2m+1$ that $c(\tau_3)\geqslant c(\tau_2)$.

Summarizing, after a finite number of steps we have transformed tour τ into tour $\tau^\star$, and in all intermediate steps the length of the tour is non-decreasing. This completes the proof of the theorem. □

Theorem 4.2. *Let C be a symmetric $n\times n$ Demidenko matrix with $n=2k+1$ that additionally fulfills the conditions*

$$c_{1,k+1}+c_{i,j}\geqslant c_{1,j}+c_{i,k+1},\quad i=2,\dots,k,\ \ j=k+2,\dots,n, \tag{11}$$

$$c_{p+1,k+1+p}+c_{i,j}\geqslant c_{p+1,j}+c_{i,k+1+p},\quad i=p+2,\dots,k,\ \ j=k+p+2,\dots,n, \tag{12}$$

$$c_{p,k+1+p}+c_{i,j}\geqslant c_{p,j}+c_{i,k+p+1},\quad i=p+1,\dots,k,\ \ j=k+p+2,\dots,n,\ \ p=1,\dots,k-1 \tag{13}$$

then the tour $\sigma^\star$ is a tour of maximum length.

The proof of Theorem 4.2 can be done by a tour improvement technique that is very similar to that applied in the proof of Theorem 4.1, and hence is omitted. Theorem 4.2 contains as a special case the well-known result of Kalmanson [6] for the longest tour in a convex set of odd cardinality in the Euclidean plane; cf. the paragraph following inequalities (3) and (4) in the introduction. Summarizing, these matrices form a subset of the Demidenko matrices and a superset of the Kalmanson matrices.

5. The MaxTSP on relaxed Kalmanson matrices

In this section, we introduce and analyze the class of *relaxed Kalmanson* matrices. This class is a proper subset of the symmetric Demidenko matrices and a proper superset of the Kalmanson matrices: An $n \times n$ Demidenko matrix $C=(c_{i,j})$ is a *relaxed Kalmanson* matrix if it fulfills conditions

$$c_{i,t} + c_{j,s} \leqslant c_{i,s} + c_{j,t} \tag{14}$$

for all $i = 1,\dots,k$; $s = \max\{k, i+2\},\dots,n-1$; $j = i+1,\dots,\min\{s-1, k+1\}$; and $t = s+1,\dots,n$, where $k = \lfloor n/2 \rfloor$. The class of relaxed Kalmanson matrices is similar to, but incomparable with the class of matrices discussed in Section 4:

$$A = \begin{pmatrix} - & 4 & 4 & 4 & 3 & 3 \\ 4 & - & 4 & 4 & 3 & 3 \\ 4 & 4 & - & 1 & 3 & 3 \\ 4 & 4 & 1 & - & 2 & 2 \\ 3 & 3 & 3 & 2 & - & 1 \\ 3 & 3 & 3 & 2 & 1 & - \end{pmatrix}, \quad B = \begin{pmatrix} - & 1 & 2 & 2 & 2 & 1 \\ 1 & - & 1 & 0 & 0 & 0 \\ 2 & 1 & - & 1 & 1 & 1 \\ 2 & 0 & 1 & - & 0 & 0 \\ 2 & 0 & 1 & 0 & - & 0 \\ 1 & 0 & 1 & 0 & 0 & - \end{pmatrix}.$$

It can be verified that matrix A is a relaxed Kalmanson matrix, but does not fulfill condition (7) of Theorem 4.1 since $a_{41} + a_{25} > a_{23} + a_{45}$. On the other hand, matrix B does fulfill all conditions in Theorem 4.1, but is not a relaxed Kalmanson matrix. We will show that the MaxTSP on relaxed Kalmanson matrices can be solved in polynomial time.

Let us briefly discuss relaxed Kalmanson matrices C whose dimension n is odd and of the form $n = 2k+1$. In this case, it can easily be verified that C fulfills conditions (11)–(13) in Theorem 4.2. Consequently, the tour $\sigma^\star$ is an optimum tour for the MaxTSP on relaxed Kalmanson matrices with odd n.

Hence, throughout the remainder of this section we will consider the case that n is even, i.e. there is an integer k such that $n = 2k$. In this case, it can be verified that C fulfills the three conditions (8)–(10) in Theorem 4.1. However, conditions (7) may be violated in C, and therefore there is no a priori guarantee that the tour $\tau^\star$ is an optimum tour for the MaxTSP on relaxed Kalmanson matrices with even n.

To simplify further definitions, we introduce certain subpaths with a special combinatorial structure on the set of indices $\{i_1, i_2, \dots, i_p\} \cup \{j_1, j_2, \dots, j_q\}$ with $1 \leqslant i_1 < i_2 < \cdots < i_p < k$, $k+1 < j_1 < j_2 < \cdots < j_q \leqslant n$ and $p = q+1$. Define

$$\Lambda(\{i_1, i_2, \dots, i_p; j_1, j_2, \dots, j_q\}) \doteq \langle i_2, j_2, i_4, j_4, \dots, j_3, i_3, j_1, i_1 \rangle$$

and

$$V(\{i_1, i_2, \dots, i_p; j_1, j_2, \dots, j_q\}) \doteq \langle i_{p-1}, j_{q-1}, i_{p-3}, j_{q-3}, \dots, j_{q-2}, i_{p-2}, j_q, i_p \rangle.$$

By using the above notation, we now define a sequence of specially structured tours $\tau_1^\star, \dots, \tau_{k-1}^\star$ of type (II) as follows:

$$\tau_1^\star = \langle 1, k, n, k+1, \Lambda(1, 2, \dots, k-1; k+2, k+3, \dots, n-1) \rangle,$$
$$\tau_2^\star = \langle 1, k, n-1, V(k-2, k-1; n-2), n, k+1,$$
$$\Lambda(1, 2, \dots, k-3; k+2, \dots, n-3) \rangle,$$

$$\tau_3^\star = \langle 1, k, n-1, V(k-4,\dots,k-1; n-4, n-3, n-2), n, k+1,$$
$$A(1,\dots,k-5; k+2,\dots,n-5)\rangle,$$

$\dots \quad \dots$

$$\tau_{k-2}^\star = \langle 1, k, n-1, V(3,\dots,k-1; k+3,\dots,n-2), n, k+1, A(1,2; k+2)\rangle,$$
$$\tau_{k-1}^\star = \langle 1, k, n-1, V(2,3,\dots,k-1; k+2,\dots,n-2), k-1, n, k+1, 1\rangle.$$

Theorem 5.1. *Let C be a symmetric $n \times n$ relaxed Kalmanson matrix with $n = 2k$. Then a tour of maximum length can be found among the tours $\tau^\star, \tau_1^\star, \tau_2^\star, \dots, \tau_{k-1}^\star$.*

Theorem 5.1 will be proved via the two intermediate Lemmas 5.2 and 5.3. First, observe that by Theorem 3.1, an optimum tour for the MaxTSP on relaxed Kalmanson matrices can be found in the set $\mathcal{M}_n$. Theorem 4.1 implies that if a distance matrix is a relaxed Kalmanson matrix, then the tour $\tau^\star$ is the longest tour among all tours of type (I). Hence, it remains to consider the tours of type (II). We start our investigations with the following lemma.

Lemma 5.2. *Let C be an $n \times n$ relaxed Kalmanson matrix with $n = 2k$ and let τ be a tour of type* (II) *that contains a subpath $\langle i, k, j, k+1, l\rangle$ with $i < k < j$ and $j > k+1 > l$. Then $c(\tau_1^\star) \geqslant c(\tau)$.*

Proof. The proof is similar to the proof of Theorem 4.1. First suppose that $\tau(k) = j \neq n$. Then $\tau(s) = n$ and $\tau(n) = t$ for some $s, t < k$. By exchanging j and n in τ we obtain a new tour τ_1 with

$$c(\tau_1) - c(\tau) = c_{kn} + c_{k+1,n} + c_{sj} + c_{tj} - c_{kj} - c_{k+1,j} - c_{sn} - c_{tn}$$
$$= (c_{sj} + c_{kn} - c_{sn} - c_{kj}) + (c_{tj} + c_{k+1,n} - c_{t,n} - c_{k+1,j}).$$

Since $s \neq t$, $s, t < k$ and $k+1 < j < n$, it follows from (14) that $c(\tau_1) \geqslant c(\tau)$.

Next, let us check the placement of the cities j, $j = 1, 2, k+2, k+3, 3, 4, k+4, k+5, \dots$ in the tour τ_2 and let us compare their positions with their positions in the tour $\tau_1^\star$. Denote by $j_{\min}$ the first city which is placed in $\tau_1^\star$ and in τ_2 at different positions. Distinguish four cases: (a) $j_{\min} = 2m-1$; (b) $j_{\min} = 2m$; (c) $j_{\min} = k+2m$, and (d) $j_{\min} = k+2m+1$.

Case a: If $j_{\min} = 2m-1$ holds, then τ_2 contains a subpath $\langle t, 2m-1, \dots, i, k+2m-2\rangle$ with $2m-1 < i < k$ and $k+2m-2 < t < n$. By reversing the subpath $\langle 2m-1, \dots, i\rangle$ in τ_2, we obtain a tour τ_3 with

$$c(\tau_3) - c(\tau_2) = c_{2m-1,k+2m-2} + c_{i,t} - c_{2m-1,t} - c_{i,k+2m-2}.$$

It follows from (14) that $c(\tau_3) \geqslant c(\tau_2)$.

Case b: If $j_{\min} = 2m$ holds, then in τ_2 contains a subpath $\langle k+2m-1, i, \dots, 2m, t\rangle$ with $2m < i < k$ and $k+2m-1 < t < n$. We transform τ_2 into τ_3 by reversing the subpath $\langle i, \dots, 2m\rangle$. We conclude from (14) that $c(\tau_3) \geqslant c(\tau_2)$ holds.

Case c: If $j_{\min} = k+2m$ holds, then in τ_2 there is a subpath $\langle i, k+2m \ldots, t, 2m-1\rangle$ with $2m-1 < i < k$ and $k+2m < t < n$. We transform τ_2 into τ_3 by reversing the subpath $\langle k+2m, \ldots, t\rangle$, and we conclude that $c(\tau_3) \geqslant c(\tau_2)$.

Case d: If $j_{\min} = k+2m+1$ holds, then there is a subpath $\langle 2m, t, \ldots, k+2m+1, i\rangle$ with $2m < i < k$ and $k+2m+1 < t < n$. We get τ_3 by reversing the subpath $\langle t, \ldots, k+2m+1\rangle$; Again, $c(\tau_3) \geqslant c(\tau_2)$.

Summarizing, after a finite number u of steps we end up with a tour $\tau_u = \tau_1^\star$ that fulfills $c(\tau_1^\star) \geqslant c(\tau)$. □

Lemma 5.3. *Let C be an $n \times n$ relaxed Kalmanson matrix with $n = 2k$ and let τ be a tour of type* (II) *that contains a subpath $\langle i, k+1, j, k, l\rangle$ with $i > k+1 > j$ and $j < k < l$. Then $c(\tau_{k-1}^\star) \geqslant c(\tau)$.*

Proof. This lemma can be proved by using a similar transformation technique as in the proof of the previous lemma. □

Proof of Theorem 5.1. Consider a tour of type (II) from the set $\mathscr{M}_n$ that has the form

$$\tau_{(m)} = \langle k, s_1, s_2, \ldots, s_{2m+1}, k+1, t_1, \ldots, t_{n-2m-3}, k\rangle,$$

where $1 \leqslant m \leqslant k-3$, $t_{n-2m-3} < k < s_1$, and $s_{2m+1} > k+1 > t_1$. By using transformations similar to the ones used in the proofs of Lemmas 5.2 and 5.3, and by taking into account the inequalities (14), it can be shown that only tours $\tau_{(m)}$ with

$$s_{2m+1} > s_1 > s_{2m-1} > s_3 > \cdots > k+1,$$
$$k > s_{2m} > s_2 > s_{2m-2} > s_4 > \cdots,$$
$$t_{n-2m-3} < t_1 < t_{n-2m-5} < t_3 < \cdots < k,$$
$$k+1 < t_{n-2m-4} < t_2 < t_{n-2m-6} < t_4 < \cdots$$

need to be considered. The tours $\tau_{m+1}^\star$ $(m = 1, 2, \ldots, k-3)$ defined above fulfill all these conditions together with the additional property $t_{2x+1} < s_{2y}$ and $t_{2y} < s_{2x+1}$ for all $x, y = 1, 2, \ldots$. By once more using a tour improvement technique, we will transform the tour $\tau_{(m)}$ into the tour $\tau_{m+1}^\star$ without decreasing its length, i.e. we show that $c(\tau_{m+1}^\star) \geqslant c(\tau_{(m)})$. In doing this, we will prove the theorem.

We will check the cities in the subpath $\langle k+1, \ldots, k\rangle$ in $\tau_{(m)}$ in the following order:

$$t_{n-2m-3}, t_1, t_{n-2m-4}, t_2, t_{n-2m-5}, t_3, t_{n-2m-6}, t_4 \ldots .$$

If necessary, we will exchange these cities with some cities s_i with smaller number. After every exchange step, the transformation procedure will be repeated for the resulting tour.

Let $t^{(1)}$ be the first city whose position does not agree with the corresponding position in the tour $\tau_{m+1}^\star$. Hence, there is a city $s^{(1)}$ with $s^{(1)} < t^{(1)}$ that should be placed instead of $t^{(1)}$. Suppose that in the subpath $\langle t^{(1)}, \ldots, \tau_{(m)}^{-1}(k)\rangle$, all elements but $t^{(1)}$ have already been checked (the symmetric case is analyzed in a symmetric way). That means that $\tau(t^{(1)}) = t^{(2)} < \tau(s^{(1)}) = s^{(2)}$ holds. We analyze the two sequences

$t^{(1)}, \tau_{(m)}^{-1}(t^{(1)}), \tau_{(m)}^{-2}(t^{(1)}),\ldots$ and $s^{(1)}, \tau_{(m)}^{-1}(s^{(1)}), \tau_{(m)}^{-2}(s^{(1)}),\ldots$. We determine the first two edges $(s^{(4)}, s^{(3)})$ and $(t^{(4)}, t^{(3)})$ (where $s^{(4)} = \tau_{(m)}^{-1}(s^{(3)}) = \tau_{(m)}^{-x}(s^{(1)})$ and where $t^{(4)} = \tau_{(m)}^{-1}(t^{(3)}) = \tau_{(m)}^{-x}(t^{(1)})$) such that $t^{(3)} > s^{(3)}$ and $t^{(4)} < s^{(4)}$ holds. The existence of such a pair follows from the fact $s^{(1)} < t^{(1)}$ and from the inequality $t^{(4)} < s^{(4)}$ for $s^{(4)} = k$ or for $t^{(4)} = k+1$. We transform $\tau_{(m)}$ into $\tau'_{(m)}$ by exchanging the subpaths $\langle s^{(3)},\ldots,s^{(1)}\rangle$ and $\langle t^{(3)},\ldots,t^{(1)}\rangle$. More precisely, the tour

$$\tau_{(m)} = \langle 1,\ldots,s^{(4)},s^{(3)},\ldots,s^{(1)},s^{(2)},\ldots,t^{(4)},t^{(3)},\ldots,t^{(1)},t^{(2)},\ldots\rangle$$

is transformed into the tour

$$\tau'_{(m)} = \langle 1,\ldots,s^{(4)},t^{(3)},\ldots,t^{(1)},s^{(2)},\ldots,t^{(4)},s^{(3)},\ldots,s^{(1)},t^{(2)},\ldots\rangle.$$

Clearly,

$$c(\tau'_{(m)}) - c(\tau_{(m)}) = c_{s^{(1)}t^{(2)}} + c_{t^{(1)}s^{(2)}} + c_{s^{(3)}t^{(4)}} + c_{t^{(3)}s^{(4)}} \\ - c_{s^{(1)}s^{(2)}} - c_{t^{(1)}t^{(2)}} - c_{s^{(3)}s^{(4)}} + c_{t^{(3)}t^{(4)}} \geqslant 0.$$

Hence, the length of the tour does not decrease. Moreover, the tour $\tau'_{(m)}$ is closer to $\tau^\star_{m+1}$ than $\tau_{(m)}$, since $\tau'_{(m)}$ and $\tau^\star_{m+1}$ also agree in the position of city $s^{(1)}$. We define $\tau_{(m)} = \tau'_{(m)}$ and repeat the transformation step. Since every transformation step brings one more city into the right position, the whole procedure will terminate after at most n transformation steps with $\tau'_{(m)} = \tau^\star_{m+1}$. The proof of the theorem is complete. □

6. Conclusion

In this paper we showed that the MaxTSP on symmetric Demidenko distance matrices is NP-hard. Moreover, we described two special subclasses of symmetric Demidenko matrices for which the MaxTSP can be solved in $O(n^2)$ time. These two subclasses are incomparable; both contain the matrices of Kalmanson [6] as special cases.

As an open question we pose to decide the computational complexity of the MaxTSP on *Van der Veen matrices* and on *monotone Toeplitz* matrices: A symmetric $n \times n$ matrix $(c_{i,j})$ is called a Van der Veen matrix if $c_{i,j} + c_{s,t} \leqslant c_{i,t} + c_{s,j}$ holds for all $1 \leqslant i < j < s < t \leqslant n$ (cf. [12]). The class of Van der Veen matrices and the class of symmetric Demidenko matrices are incomparable (cf. [12]). A symmetric $n \times n$ matrix $(c_{i,j})$ is called a monotone Toeplitz matrix if there is a function f: $\{1,\ldots,n-1\} \to \mathbb{R}$ such that $c_{i,j} = f(|i-j|)$ and such that $c_{1,2} \leqslant c_{1,3} \leqslant \cdots \leqslant c_{1,n}$ holds (i.e. matrix C is constant on every diagonal, and the further a diagonal is away from the main diagonal, the larger are the values on this diagonal). It is easy to see that monotone Toeplitz matrices form a subclass of the symmetric Demidenko matrices. For more information on Hamiltonian properties of Toeplitz matrices, we refer the reader to van Dal et al. [11].

Acknowledgements

We acknowledge the helpful comments of an unknown referee; this referee also suggested the open problem on Van der Veen matrices.

References

[1] D. Blokh, G. Gutin, Maximizing traveling salesman problem for special matrices, Discrete Appl. Math. 56 (1995) 83–86.

[2] R.E. Burkard, V.G. Deineko, R. van Dal, J.A.A. van der Veen, G.J. Woeginger, Well-solvable special cases of the TSP: a survey, SIAM Rev. 40 (1998) 496–546.

[3] V.M. Demidenko, A special case of travelling salesman problems, Izv. Akad. Nauk. BSSR, Ser. Fiz.-mat. Nauk 5 (1976) 28–32 (in Russian).

[4] M.R. Garey, D.S. Johnson, Computers and Intractability, Freeman and Company, San Francisco, 1979.

[5] P.C. Gilmore, E.L. Lawler, D.B. Shmoys, Well-solved special cases, The Travelling Salesman Problem, Wiley, Chichester, 1985, pp. 87–143.

[6] K. Kalmanson, Edge-convex circuits and the travelling salesman problem, Canad. J. Math. 27 (1975) 1000–1010.

[7] E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan, D.B. Shmoys, The Travelling Salesman Problem, Wiley, Chichester, 1985.

[8] L.V. Quintas, F. Supnick, Extreme Hamiltonian circuits: resolution of the convex-odd case, Proc. Amer. Math. Soc. 15 (1964) 454–459.

[9] L.V. Quintas, F. Supnick, Extreme Hamiltonian circuits: resolution of the convex-even case, Proc. Amer. Math. Soc. 16 (1965) 1058–1061.

[10] F. Supnick, Extreme Hamiltonian lines, Ann. Math. 66 (1957) 179–201.

[11] R. Van Dal, G. Tijssen, J.A.A. Van der Veen, C. Zamfirescu, T. Zamfirescu, Hamiltonian properties of Toeplitz graphs, Discrete Math. 159 (1996) 69–81.

[12] J.A.A. Van der Veen, A new class of pyramidally solvable symmetric traveling salesman problems, SIAM J. Discrete Math. 7 (1994) 585–592.