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Plane four-regular graphs with vertex-to-vertex unit triangles

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In memory of Egmont Köhler.

Abstract

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For the smallest number of non-overlapping vertex-to-vertex unit triangles in the plane it is proved ≤42 in general, and ≤3800 if additional triangles are not allowed.

It is asked for the following plane unit triangle graphs T_n: Realizations of planar four-regular graphs with n vertices in the plane, with straight line edges of unit length, and with the properties that every edge is a side of exactly one of $2n/3$ unit triangles which have in pairs the vertices of T_n as common vertex points, and which do not overlap one another. These realizations do not forbid an unusual border-line case for plane graphs, namely, that a common vertex point of two triangles is an inner point of an edge of another triangle, that means, the common vertex point of two triangles can touch a third triangle (see D in Fig. 2). Such plane triangle graphs T_n may be thought of as a result of a puzzle with $2n/3$ non-overlapping unit triangles on a plane board, where always exactly two vertex points of the triangles have to meet at the vertices. Hence touching points may occur as in Fig. 2.—Among other things the author was inspired to ask for these graphs by the figure on page 133 in [5].

The existence of graphs T_n is proved by T_{63} of Fig. 1 with 42 triangles. This graph was independently and in an other context discovered by Grünbaum and Shephard [1]. It is unknown whether there exist graphs T_n for $n < 63$ (see also [3]).

Conjecture 1. The smallest number of non-overlapping vertex-to-vertex unit triangles is 42 (Fig. 1).

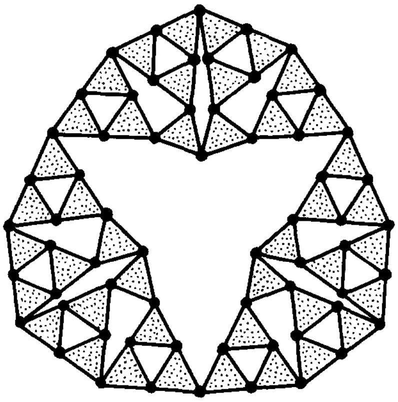

Fig. 1.

If one puzzles to find other graphs T_n it seems that always some three of the $2n/3$ unit triangles occur which have pairwise vertex points in common and thus surround an additional unit triangle not belonging to those $2n/3$ unit triangles of T_n. Do graphs T_n exist without such additional triangles?

The following ring graphs $T_n = T_{3s(4r+3)}$ give an affirmative answer: Choose a rhomb surrounded by four unit triangles. Repeated copies of this quadruple of unit triangles determine a string of such quadruples where only two vertices at each end of the string are vertices of only one unit triangle. Fit two strings at one end (A in Fig. 2) together such that the vertex D of the upper string in Fig. 2 lies on (B, E), and choose this second string such that (D, E) is of length $1/(r+1)$ with a positive integer r. Then $r + \frac{1}{2}$ quadruples of unit triangles in the lower, and $r + 1$ quadruples of unit triangles in the upper string of Fig. 2 force two vertices of unit triangles at the end of the two strings to coincide. Let α, β, γ be the angles at A of the triangles (A, B, E), (A, D, C), (A, D, E), respectively, in Fig. 2. If now the first rhomb, that means α, and r are chosen such that:

$$\gamma = \pi/s \text{ with a positive integer } s, \tag{1}$$

$$0 < \alpha < \beta < \pi/6, \tag{2}$$

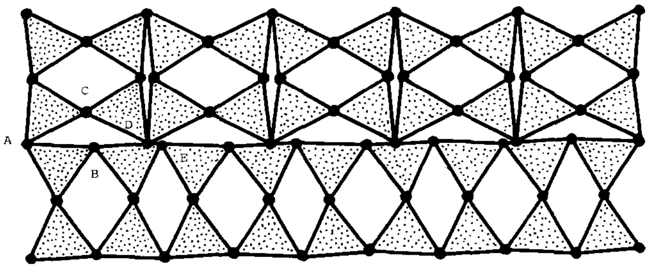

Fig. 2.

then s of the above pairs of strings, that means, alternating a lower and an upper string of Fig. 2, can be put together to form a cyclic ring of

$$t = 2s(4r + 3) \tag{3}$$

non-overlapping unit triangles which determine the desired graph $T_{3s(4r+3)}$. By (1) it is guaranteed that in closing the ring the last two pairs of vertex points do coincide, and (2) guarantees that the unit triangles within the strings do not overlap. By the way, these graphs $T_{3s(4r+3)}$ are rigid examples as in [1].

Theorem 1. *The minimum number of unit triangles in ring graphs* (*without additional triangles*) *is* 3800.

Proof. From triangle (A, D, E) in Fig. 2 the following equations can be deduced,

$$\tan \alpha = (2r + 1)\tan \gamma, \tag{4}$$

$$4(r + 1)^2 \cos^2 \beta = 1 + 4r(r + 1)\cos^2 \alpha. \tag{5}$$

With $\alpha > 0$ from (5) follows $\cos \beta < (2r + 1)/(2r + 2)$, and this for $r = 1$ and $r = 2$ yields $\beta > \pi/6$ in contradiction to (2). With $\beta < \pi/6$ from (2) it holds

$$\cos^2 \alpha > \frac{3(r + 1)^2 - 1}{4r(r + 1)}. \tag{6}$$

Equations (6), (4), (1), (3) in this sequence imply

for $r = 3$: $\alpha < 8.299$, $\gamma < 1.194$, $s \geqslant 151$, $t \geqslant 4530$;

for $r = 4$: $\alpha < 15.895$, $\gamma < 1.813$, $s \geqslant 100$, $t \geqslant 3800$;

for $r = 5$: $\alpha < 19.22$, $\gamma < 1.82$, $s \geqslant 99$, $t \geqslant 4554$.

For $r \geqslant 6$ with $\alpha < \pi/6$ from (2), equations (4), (1), (3) imply $\gamma < 2.55$, $s \geqslant 71$, $t \geqslant 3834$. Thus $t = 3800$ is the minimum which is attained for $r = 4$ and $s = 100$. □

In general it may be conjectured that no other plane graphs T_n do exist, neither with touching points as D in Fig. 2 being allowed, nor without.

Conjecture 2. Besides ring graphs no other plane unit triangle graphs T_n without additional triangles do exist.

The truth of this conjecture would imply the truth of the following.

Conjecture 3. The smallest number of non-overlapping vertex-to-vertex unit triangles without additional triangles is 3800.

Generalizations. (i) Plane unit triangle graphs with always three vertex points meeting at a vertex do not exist, since a six-regular plane graph contradicts Euler's polyhedron formula.

(ii) Plane graphs where always two vertex points of regular unit m-gons, $m \geqslant 4$, meet at a vertex do not exist, since all angles of the surrounding unit polygon are greater than π.

(iii) In $\mathbb{R}^d$, $d \geqslant 2$, the smallest number $t(d)$ of non-overlapping vertex-to-vertex unit tetrahedra fulfills $t(d) \leqslant 21 \cdot 2^{d-1}$, since the tetrahedra in $\mathbb{R}^{d-1}$ can be completed to d-dimensional tetrahedra, and then two copies can be put together in $\mathbb{R}^d$. However, $t(3) \leqslant 78$ can be achieved by a construction similar to that of Fig. 1.

(iv) If overlapping of the unit triangles in the plane is allowed, and always k vertex points meet at a vertex, then the smallest number t_k of unit triangles is known to be $t_k \leqslant 3^k$, and $t_2 = 9$ (see [2]).

References

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