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Note

Crossing-critical graphs with large maximum degree

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ABSTRACT

A conjecture of Richter and Salazar about graphs that are critical for a fixed crossing number k is that they have bounded bandwidth. A weaker well-known conjecture of Richter is that their maximum degree is bounded in terms of k. In this note we disprove these conjectures for every $k \geqslant 171$, by providing examples of k-crossing-critical graphs with arbitrarily large maximum degree.

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A graph is *k-crossing-critical* (or simply *k-critical*) if its crossing number is at least k, but every proper subgraph has crossing number smaller than k. Using the Excluded Grid Theorem of Robertson and Seymour [9], it is not hard to argue that k-crossing-critical graphs have bounded tree-width [2]. However, all known constructions of crossing-critical graphs suggested that their structure is "path-like". Salazar and Thomas conjectured (cf. [2]) that they have bounded path-width. This problem was solved by Hliněný [3], who proved that the path-width of k-critical graphs is bounded above by $2^{f(k)}$, where $f(k) = (432\log_2 k + 1488)k^3 + 1$.

In the late 1990s, two other conjectures were proposed and made public in 2003 at the Bled'03 conference [7] (see also [8,6]).

Conjecture 1. *(See Richter [7].) For every positive integer k, there exists an integer $D(k)$ such that every k-crossing-critical graph has maximum degree less than $D(k)$.*

The second conjecture was proposed as an open problem in the 1990s by Carsten Thomassen and formulated as a conjecture by Richter and Salazar.

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Conjecture 2. *(See Richter and Salazar [7,8].) For every positive integer k, there exists an integer $B(k)$ such that every k-crossing-critical graph has bandwidth at most $B(k)$.*

Conjecture 2 would be a strengthening of Hliněný's theorem about bounded path-width and would also imply Conjecture 1.

Hliněný and Salazar [5] recently made a step towards Conjecture 1 by proving that k-crossing-critical graphs cannot contain a subdivision of $K_{2,N}$ with $N = 30k^2 + 200k$.

In this note we give examples of k-crossing-critical graphs of arbitrarily large maximum degree, thus disproving both Conjectures 1 and 2.

A *special graph* is a pair (G, T), where G is a graph and $T \subseteq E(G)$. The edges in the set T are called *thick edges* of the special graph. A *drawing* of a special graph (G, T) is a drawing of G such that the edges in T are not crossed. The crossing number $\mathrm{cr}(G, T)$ of a special graph is the minimum number of edge crossings in a drawing of (G, T) in the plane. (We set $\mathrm{cr}(G, T) = \infty$ if a thick edge is crossed in every drawing of G.) An edge $e \in E(G) \setminus T$ is *k-critical* if $\mathrm{cr}(G, T) \geqslant k$ and $\mathrm{cr}(G - e, T) < k$. Let $\mathrm{crit}_k(G, T)$ be the set of k-critical edges of (G, T). If $T = \emptyset$, then we write just $\mathrm{cr}(G)$ for the crossing number of G and $\mathrm{crit}_k(G)$ for the set of k-critical edges of G. Note that the graph G is k-critical if $\mathrm{crit}_k(G) = E(G)$.

A standard result (see, e.g., [1]) is that we can eliminate the thick edges by replacing them with sufficiently dense subgraphs. (In fact, one can replace every edge xy by $t = \mathrm{cr}(G, T) + 1$ parallel edges or by $K_{2,t}$ if multiple edges are not desired.)

Lemma 3. *For every special graph (G, T) with $\mathrm{cr}(G, T) < \infty$ and for any k, there exists a graph $\tilde{G} \supseteq G$ such that $\mathrm{cr}(G, T) = \mathrm{cr}(\tilde{G})$ and $\mathrm{crit}_k(G, T) \subseteq \mathrm{crit}_k(\tilde{G})$.*

Furthermore, note the following:

Lemma 4. *Let k be an integer. Any graph G with $\mathrm{cr}(G) \geqslant k$ contains a k-crossing-critical subgraph H such that $\mathrm{crit}_k(G) \subseteq E(H)$.*

Proof. For a contradiction, suppose that G is a smallest counterexample. If G were k-critical, then we would set $H = G$, hence G contains a non-k-critical edge e. It follows that $\mathrm{cr}(G - e) \geqslant k$. Let f be a k-critical edge in G, i.e., $\mathrm{cr}(G - f) < k$. As $\mathrm{cr}((G - e) - f) \leqslant \mathrm{cr}(G - f) < k$, f is a k-critical edge in $G - e$. Therefore, $\mathrm{crit}_k(G) \subseteq \mathrm{crit}_k(G - e)$. Since G is the smallest counterexample, $G - e$ has a k-critical subgraph H with $\mathrm{crit}_k(G - e) \subseteq E(H)$. However, $H \subseteq G$ and $\mathrm{crit}_k(G) \subseteq E(H)$, which is a contradiction. □

Let us now proceed with the main result. Two paths P_1 and P_2 in a special graph are *almost edge-disjoint* if all the edges in $E(P_1) \cap E(P_2)$ are thick.

Lemma 5. *For any d, there exist a special graph (G, T) and a vertex $v \in V(G)$ such that $\mathrm{crit}_{171}(G, T)$ contains at least d edges incident with v.*

Proof. Let (G, T) be the special graph drawn as follows: we start with $d+1$ thick cycles $C_0, C_1, \ldots, C_d$ intersecting in a vertex v, i.e., $C_i \cap C_j = \{v\}$ for $0 \leqslant i < j \leqslant d$. Their lengths are $|C_0| = 28$, $|C_d| = 24$ and $|C_i| = 7$ for $1 \leqslant i < d$. They are drawn in the plane so that all their vertices are incident with the unbounded face and their clockwise order around v is $C_0, C_1, \ldots, C_d$. See Fig. 1 illustrating the case $d = 5$. Let $C_0 = va_1a_2 \ldots a_{19}b_1b_2b_3c_1^0c_2^0 \ldots c_5^0$, $C_d = vt^db_3'b_2'b_1'a_1'a_2' \ldots a_{19}'$ and $C_i = vt^ic_1^ic_2^i \ldots c_5^i$ for $1 \leqslant i < d$. Furthermore, add d vertices $s^1, \ldots, s^d$ adjacent to v. The clockwise cyclic order of the neighbors of v is $a_1, c_5^0, s^1, t^1, c_5^1, s^2, t^2, c_5^2, \ldots, s^{d-1}, t^{d-1}, c_5^{d-1}, s^d, t^d, a_{19}'$. For $1 \leqslant i \leqslant d$, add thick cycles K_i whose vertices in the clockwise order are t^i, s^i, and five new vertices $\tilde{c}_5^{i-1}, \tilde{c}_4^{i-1}, \ldots, \tilde{c}_1^{i-1}$. Finally, add the following edges: $c_j^i\tilde{c}_j^i$ for $0 \leqslant i < d$ and $1 \leqslant j \leqslant 5$, a_ia_i' for $1 \leqslant i \leqslant 19$ and b_ib_i' for $1 \leqslant i \leqslant 3$. As described, $T = \bigcup_{i=0}^d E(C_i) \cup \bigcup_{i=1}^d E(K_i)$. Let $M = \{a_1a_1', a_2a_2', \ldots, a_{19}a_{19}', b_1b_1', b_2b_2', b_3b_3'\}$.

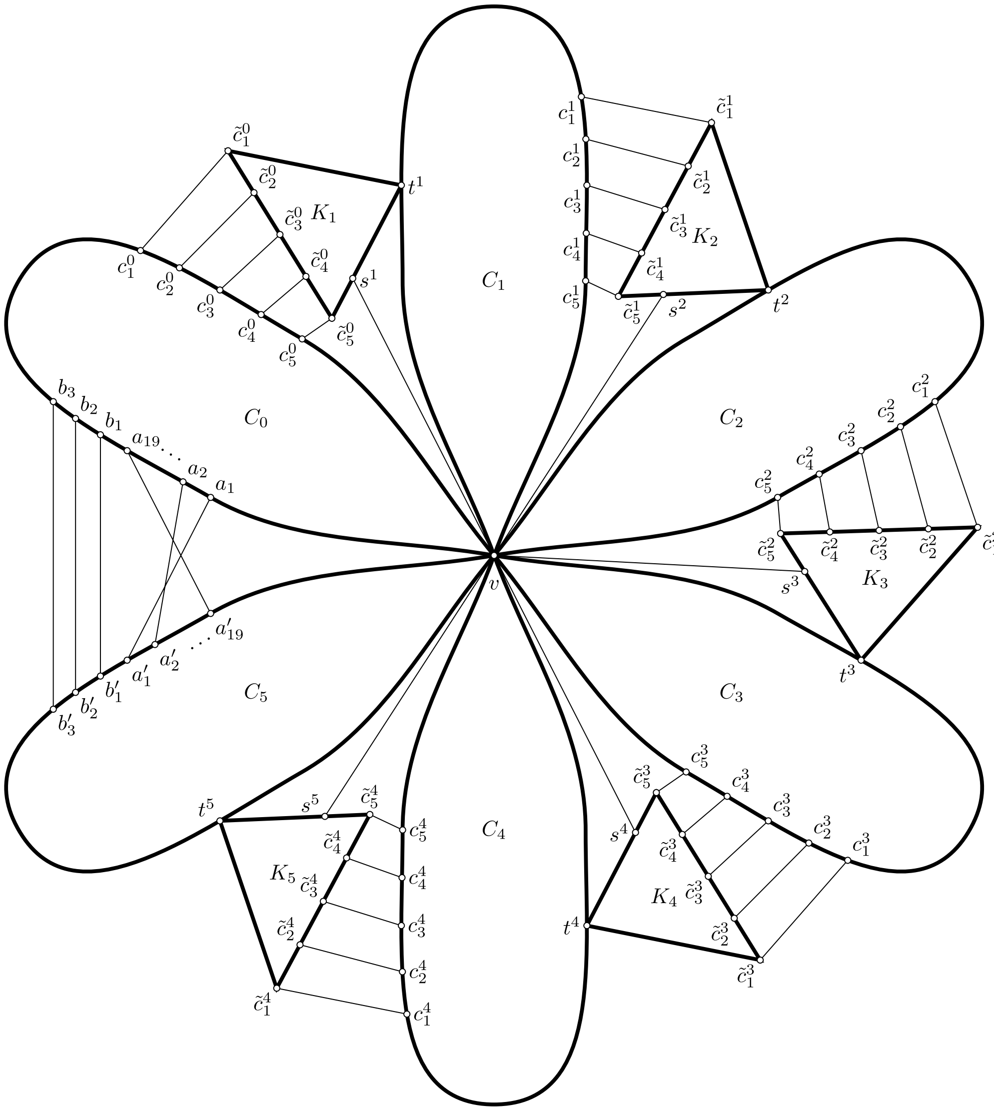

Fig. 1. A special graph with critical edges vs^i.

This drawing $\mathcal{G}$ of (G,T) has $\binom{19}{2} = 171$ crossings, as the edges $a_i a'_i$ and $a_j a'_j$ intersect for each $1 \leqslant i < j \leqslant 19$, and there are no other crossings. Let us show that $\mathrm{cr}(G,T) = 171$. Let $\mathcal{G}'$ be an arbitrary drawing of (G,T), and for a contradiction assume that it has less than 171 crossings. Let us first observe that every thick cycle C_i and K_j is an induced nonseparating cycle of G. Therefore it bounds a face of $\mathcal{G}'$. Consider the cyclic clockwise order of the neighbors of v according to the drawing $\mathcal{G}'$. For each cycle C_i $(0 \leqslant i \leqslant d)$, the two edges of C_i incident with v are consecutive in this order, since C_i bounds a face. Without loss of generality, we assume that each cycle C_i bounds a face distinct from the unbounded one. If the cyclic order of the vertices around the face C_i is the same as in the drawing $\mathcal{G}$, we say that C_i is drawn *clockwise*, otherwise it is drawn *anti-clockwise*. We may assume that C_0 is drawn clockwise. If C_d were drawn clockwise as well, then each pair of edges $a_i a'_i$ and $a_j a'_j$ with $1 \leqslant i < j \leqslant 19$ would intersect, and the drawing $\mathcal{G}'$ would have at least

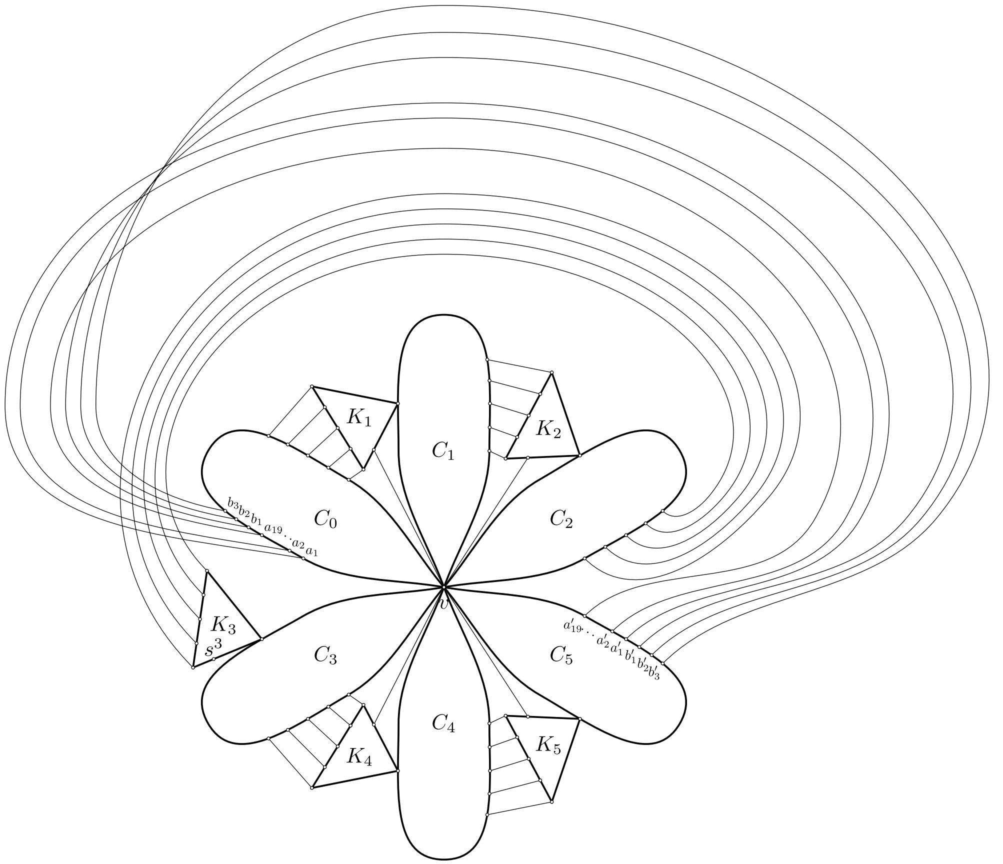

Fig. 2. A drawing of the graph $G - vs^3$ with 170 intersections.

171 crossings. Therefore, C_d is drawn anti-clockwise. It follows that the edges $a_i a'_i$ and $b_j b'_j$ intersect for $1 \leqslant i \leqslant 19$ and $1 \leqslant j \leqslant 3$, and the edges $b_i b'_i$ and $b_j b'_j$ intersect for $1 \leqslant i < j \leqslant 3$, giving 60 crossings. For $1 \leqslant i \leqslant 5$, let P_i be the path $c_i^0 \tilde{c}_i^0 \tilde{c}_{i-1}^0 \ldots \tilde{c}_1^0 t^1 c_1^1 c_2^1 \ldots c_i^1 \tilde{c}_i^1 \ldots \tilde{c}_1^1 t^2 \ldots t^d$. These paths are mutually almost edge-disjoint and each of them intersects all edges of M in the drawing $\mathcal{G}'$, thus contributing at least 110 crossings all together. Therefore, the drawing $\mathcal{G}'$ has at least 170 crossings. Since we assume that this drawing has fewer than 171 crossings, we conclude that there are no other crossings.

The cycle $va_1 a'_1 a'_2 \ldots a'_{19} v$ splits the plane into two regions R_1 and R_2, such that R_1 contains the face bounded by C_0 and R_2 contains the face bounded by C_d. For $j = 1, 2$, let A_j be the set of cycles C_i ($0 \leqslant i \leqslant d$) such that the face bounded by C_i lies in the region R_j. As P_1 intersects the edge $a_1 a'_1$ only once, $A_1 = \{C_0, C_1, \ldots, C_{k-1}\}$ and $A_2 = \{C_k, C_{k+1}, \ldots, C_d\}$ for some n with $1 \leqslant n \leqslant d$. As the path P_1 does not intersect itself, all cycles in A_1 are drawn clockwise and their clockwise order around v is $C_0, C_1, \ldots, C_{n-1}$. Similarly, all cycles in A_2 are drawn anti-clockwise and their clockwise order around v is $C_d, C_{d-1}, \ldots, C_n$.

Let us now consider the cycle K_n. Since the edges $c_4^{n-1} \tilde{c}_4^{n-1}$ and $c_5^{n-1} \tilde{c}_5^{n-1}$ do not intersect, the thick path $c_5^{n-1} v t^k s^n \tilde{c}_5^{n-1}$ is not intersected, and C_{n-1} is drawn clockwise, K_n is drawn clockwise as well. Since C_n lies in the region R_2, the vertex t^n and thus the whole thick cycle K_n lie in R_2. However, that means that the edge $s^k v$ intersects either the path P_1 or the edge $a_1 a'_1$, which is a contradiction. We conclude that $\mathrm{cr}(G, T) = 171$.

On the other hand, $\mathrm{cr}(G - vs^n, T) < 171$, for $1 \leqslant n \leqslant d$ (in fact, $\mathrm{cr}(G - vs^n, T) = 170$). To see that, consider the drawing of $(G - vs^n, T)$ in which the cycles $C_0, C_1, \ldots, C_{n-1}$ are drawn clockwise, the cycles $C_n, C_{n+1}, \ldots, C_d$ are drawn anti-clockwise, and the cyclic order of the neighbors of v is $a_1 c_5^0 s^1 t^1 c_5^1 \ldots s^{n-1} t^{n-1} c_5^{n-1} a'_{19} t^d c_5^{d-1} s^{d-1} t^{d-1} \ldots c_5^n t^n$, see Fig. 2. The intersections of this drawing are of edges $a_i a'_i$ with $b_j b'_j$ for $1 \leqslant i \leqslant 19$ and $1 \leqslant j \leqslant 3$, the edges $b_i b'_i$ with $b_j b'_j$ for $1 \leqslant i < j \leqslant 3$, and the edges $c_i^{n-1} \tilde{c}_i^{n-1}$ with all edges of M for $1 \leqslant i \leqslant 5$. Therefore, the edge vs^n is 171-critical for each n, so v is incident with d critical edges. □

We are ready for our main result.

Theorem 6. *For every $k \geqslant 171$ and every d, there exists a k-crossing-critical graph H containing a vertex of degree at least d.*

Proof. Let (G, T) be the special graph constructed in Lemma 5. By Lemma 3, there exists a graph $H' \supseteq G$ such that $\mathrm{cr}(H') = \mathrm{cr}(G, T) \geqslant 171$ and $\mathrm{crit}_{171}(G, T) \subseteq \mathrm{crit}_{171}(H')$. Let H be the 171-critical subgraph of H' obtained by Lemma 4. As $\mathrm{crit}_{171}(G, T) \subseteq \mathrm{crit}_{171}(H') \subseteq E(H)$, H contains at least d edges incident with one vertex, hence $\Delta(H) \geqslant d$. For $k > 171$ we add to H $k - 171$ disjoint copies of K_5 in order to get a k-crossing-critical graph. □

Actually, in the proof of Theorem 6, we can take $t = \lfloor \frac{k}{171} \rfloor$ copies of the graph H and $k - 171t$ copies of K_5. This gives rise to a k-critical graph with $t = \Omega(k)$ vertices of (arbitrarily) large degree. We conjecture that this is best possible in the following sense:

Conjecture 7. *For every positive integer k there exists an integer $D = D(k)$ such that every k-crossing-critical graph contains at most k vertices whose degree is larger than D.*

It is not even obvious that there exist k-crossing-critical graphs with arbitrarily many vertices of degree more than 6. Surprisingly, such examples have been constructed recently by Hliněný [4]. His examples may contain arbitrarily many vertices of any even degree smaller than $2k - 1$.

Let us also remark that the use of Lemma 4 means that we do not present an explicit counterexample to Conjecture 1, but only its supergraph. Consequently, we cannot ensure that the counterexample has some particular properties, e.g., we cannot prove that Conjecture 1 fails for 3-connected simple graphs. It might be of interest to rectify this problem by carrying out the construction explicitly, replacing the cycles of thick edges by some suitable planar graphs.

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