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Note(s)

Uniqueness of large solutions

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ABSTRACT

Given a nondecreasing nonlinearity f, we prove uniqueness of large solutions to Eq. (1) below, in the following two cases: the domain is the ball or the domain has nonnegative mean curvature and the nonlinearity is asymptotically convex.

1. Introduction

In this paper, we are interested in the so-called large solutions of a certain class of partial differential equations. Let us recall what they are: given a bounded domain Ω of $\mathbb{R}^D$, $D \geq 1$ and $f \in C^1(\mathbb{R})$, a large solution is a function $u \in C^2(\Omega)$ satisfying

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where the boundary condition is understood in the sense that

$$\lim_{x \to x_0, x \in \Omega} u(x) = +\infty \quad \text{for all } x_0 \in \partial\Omega$$

and where f is assumed to be positive at infinity, in the sense that

$$\exists\, a \in \mathbb{R} \quad \text{s.t.} \quad f(a) > 0 \quad \text{and} \quad f(t) \geq 0 \quad \text{for } t > a. \tag{2}$$

When the boundary of Ω is smooth enough, the existence of a solution of (1) is equivalent to the so-called Keller–Osserman condition:

$$\int^{+\infty} \frac{dt}{\sqrt{F(t)}} < +\infty, \quad \text{where } F(t) = \int_a^t f(s)\, ds. \tag{3}$$

For a proof of this fact, see the seminal works of Keller [1] and Osserman [2] for the case of monotone f, as well as [3] for the general case. From here on, we always assume that (3) holds.

Uniqueness of solutions of (1) turns out to be delicate. As one might expect, it fails in the presence of oscillations. For example, if $f(u) = u^2 \sin^2(u)$, the equation has infinitely many solutions (see e.g. [3]). It is also known (see e.g. the remark on p. 325 in [4]) that uniqueness fails for a nonlinearity of the form $f(u) = u^p$, $p > 1$, if the domain is not smooth enough.

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Proposition 1.1. *Assume that $\Omega = B \setminus \{0\}$ is the punctured unit ball of $\mathbb{R}^D$, $D \geq 2$. Let $p \in (1, \frac{D}{D-2})$ if $D \geq 3$ (respectively, $p \in (1, +\infty)$ if $D = 2$) and $f(u) = u^p$. Then, there exist infinitely many solutions of (1).*

However, one could hope that uniqueness holds under the simple assumptions that f is a nondecreasing function and that Ω has smooth boundary (see [5] for a slightly more general conjecture, due to P.J. McKenna). As of today, this question remains open. In the case where Ω is a ball, uniqueness is known (see part (c) of Corollary to Theorem 6 on p. 69 in [6] or Corollary 1.4 in [5]).

Theorem 1.2. *Assume that Ω is the unit ball in $\mathbb{R}^D$, $D \geq 1$. Assume that f is a nondecreasing function such that (2) and (3) hold. Then, there exists a unique solution of (1).*

In this paper, we give a shorter proof of this fact. Under extra convexity assumptions, we obtain the following answer for a more general class of domains.

Theorem 1.3. *Assume that $\partial\Omega$ is of class C^3 and that its mean curvature is nonnegative. Assume that f is a nondecreasing function such that (2) and (3) hold. Assume in addition that there exists $M \in \mathbb{R}$ such that $\sqrt{F}$ is convex in $(M, +\infty)$. Then, there exists a unique solution of (1).*

Remark 1.4. For arbitrary smoothly bounded domains, the best known results that we are aware of are contained in [7,8].

Remark 1.5. If f is asymptotically convex, then so is $\sqrt{F}$.

Let us turn to the proofs.

2. Proof of Theorem 1.2

Step 1. Reduction to the radial case.

Assume Ω is the ball. It is well-known (see e.g. Lemma 2.4 in [5]) that the equation has a minimal and a maximal solution, each of which is radial. That is, there exist two large radial solutions U_1, U_2 such that any large solution u satisfies $U_1 \leq u \leq U_2$. In particular, it suffices to prove that $U_1 \geq U_2$.

Step 2. Let u be a large radial solution. There exists $r_0 \in (0, 1)$ such that in $(r_0, 1)$, u is strictly increasing and

$$\frac{1}{2D}F(u) \leq \left(\frac{du}{dr}\right)^2 \leq 4F(u) \tag{4}$$

This is essentially Keller's classical argument (see [1]): let u be a large radial solution. Using (2), it follows that for r close to 1,

$$r^{1-D}\frac{d}{dr}\left(r^{D-1}\frac{du}{dr}\right) = \Delta u = f(u) \geq 0. \tag{5}$$

Since u is unbounded, there exists r_1 close to 1 such that $du/dr(r_1) > 0$. By (5), $du/dr > 0$ in $[r_1, 1)$. Integrating (5), we also have for $r \in (r_1, 1)$,

$$\begin{aligned} r^{D-1}\frac{du}{dr} &= r_1^{D-1}\frac{du}{dr}(r_0) + \int_{r_1}^{r} s^{D-1} f(u(s))\, ds \\ &\leq r_1^{D-1}\frac{du}{dr}(r_1) + f(u(r))\frac{r^D}{D}. \end{aligned}$$

Since f is nondecreasing and satisfies the Keller–Osserman condition (3), $\lim_{+\infty} f = +\infty$. Using this in the above, given $\epsilon > 0$, we find $r_2 \in [r_1, 1)$ such that for $r \in (r_2, 1)$,

$$\frac{1}{r}\frac{du}{dr} \leq \left(\frac{1}{D} + \epsilon\right) f(u).$$

Taking $\epsilon = \frac{1}{2D(D-1)}$ and recalling that

$$\frac{d^2u}{dr^2} + \frac{D-1}{r}\frac{du}{dr} = f(u),$$

we deduce that

$$\frac{1}{2D}f(u) \leq \frac{d^2u}{dr^2} \leq f(u) \quad \text{in } [r_2, 1).$$

Multiplying by $2du/dr$, integrating and letting $c = du/dr(r_2)^2 - F(u(r_2))$, we obtain

$$\frac{1}{D}F(u) + c \leq \left(\frac{du}{dr}\right)^2 \leq 2F(u) + c \quad \text{for } r \in [r_2, 1)$$

and so we find $r_0 \in [r_2, 1)$ such that (4) holds in $[r_0, 1)$.

Step 3. Change of independent variable.

Thanks to Step 2, for r close to 1, given $i \in \{1, 2\}$, we may perform the change of variable $u = U_i(r)$. Let $r = r_i(u)$ denote the inverse mapping of U_i and $V_i = \frac{dU_i}{dr} \circ r_i$. By the chain rule,

$$V_i \frac{dV_i}{du} + \frac{D-1}{r_i} V_i = f, \tag{6}$$

while $dr_i/du = 1/V_i$, so that

$$1 - r_i = \int_u^{+\infty} \frac{1}{V_i} du'. \tag{7}$$

Step 4. There exists $u_0 > 0$ such that $r_1 \geq r_2$ and $V_1 \geq V_2$ in $[u_0, +\infty)$.

Since r_i is the inverse mapping of U_i and $U_1 \leq U_2$, we have $r_1 \geq r_2$. By (6), the function $z = V_2 - V_1$ satisfies

$$\frac{dz}{du} + (D-1)\left\{\frac{1}{r_2} - \frac{1}{r_1}\right\} = \left(\frac{1}{V_2} - \frac{1}{V_1}\right) f = -\frac{f}{V_1 V_2} z.$$

Since $r_1 \geq r_2$, we deduce that z satisfies the differential inequality

$$\frac{dz}{du} + az \leq 0, \tag{8}$$

where $a = \frac{f}{V_1 V_2} \geq 0$ for large u. By (7), we also have

$$\int_u^{+\infty} \frac{1}{V_2} du' \geq \int_u^{+\infty} \frac{1}{V_1} du'.$$

So, there must exist u_0 such that $1/V_2(u_0) \geq 1/V_1(u_0)$ i.e. $w(u_0) \leq 0$. Using this together with (8), we deduce that $z \leq 0$ in $[u_0, +\infty)$, as desired.

Step 5. The function $w = r_1^{2D-2} V_1^2 - r_2^{2D-2} V_2^2$ is bounded.

To see this, observe first that

$$\frac{dw}{du} = 2(r_1^{2D-2} - r_2^{2D-2}) f. \tag{9}$$

Hence, w is a nonnegative nondecreasing function and

$$\frac{dw}{du} \leq 4(D-1)(r_1 - r_2) f = 4(D-1)\left(\int_u^{+\infty} \left(\frac{1}{V_2} - \frac{1}{V_1}\right) du'\right) f$$

Now, if u_0 is chosen so large that $\frac{1}{2} \leq r_2$ in $[u_0, +\infty)$,

$$\frac{1}{V_2} - \frac{1}{V_1} = \frac{V_1^2 - V_2^2}{V_1 V_2 (V_1 + V_2)} \leq \frac{2^{2D-2} w}{V_1 V_2 (V_1 + V_2)}. \tag{10}$$

Integrating (9) and using (4), it follows that for $u \geq u_0$,

$$w(u) \leq w(u_0) + C(D) \int_{u_0}^{u} \left(\int_{u'}^{+\infty} \frac{w}{F^{\frac{3}{2}}} du''\right) f \, du'.$$

Integrating by parts

$$w(u) \leq w(u_0) + C(D)\left(F(u) \int_u^{+\infty} \frac{w}{F^{\frac{3}{2}}} du' + \int_{u_0}^{u} \frac{w}{F^{\frac{1}{2}}} du'\right).$$

Thanks to the Keller–Osserman condition (3), if u_0 is chosen large enough,

$$\int_{u_0}^{u} \frac{w}{F^{\frac{1}{2}}} du' \leq w(u) \int_{u_0}^{+\infty} \frac{1}{\sqrt{F}} \leq \frac{1}{2C(D)} w(u).$$

We have then obtained

$$w(u) \leq 2w(u_0) + 2C(D) F(u) \int_u^{+\infty} \frac{w}{F^{\frac{3}{2}}} du'. \tag{11}$$

Introduce $G(u) = \int_u^{+\infty} \frac{w}{F^{\frac{3}{2}}} du'$. Thanks to (4) and (3), we have $G(+\infty) = 0$. In addition, letting $c = 2C(D)$, (11) can be rewritten as

$$-\frac{dG}{du} \leq \frac{2w(u_0)}{F^{\frac{3}{2}}} + \frac{c}{F^{\frac{1}{2}}} G.$$

That is,

$$-\frac{d}{du}\left(G\exp\left(-c\int_u^{+\infty}\frac{1}{\sqrt{F}}du'\right)\right)\le\frac{2w(u_0)}{F^{\frac{3}{2}}}\exp\left(-c\int_u^{+\infty}\frac{1}{\sqrt{F}}du'\right)\le\frac{2w(u_0)}{F^{\frac{3}{2}}}.$$

Integrating between u and $+\infty$, we then obtain, using once again (3),

$$G(u)\le C\int_u^{+\infty}\frac{1}{F^{\frac{3}{2}}}=o\left(\frac{1}{F}\right).$$

Going back to (11), we deduce that w is bounded above.

Step 6. The difference $U_2(r)-U_1(r)$ converges to 0 as $r\to 1$.

Given r close to 1 and $i\in\{1,2\}$, let $u_i=U_i(r)$. Then,

$$\int_{u_1}^{+\infty}\frac{1}{V_1}du=1-r=\int_{u_2}^{+\infty}\frac{1}{V_2}du.$$

That is,

$$\int_{u_1}^{u_2}\frac{1}{V_1}du=\int_{u_2}^{+\infty}\left(\frac{1}{V_2}-\frac{1}{V_1}\right)du.$$

Using (10), (4), and the previous step, we deduce that

$$\int_{u_1}^{u_2}\frac{1}{\sqrt{F}}du\le C\int_{u_2}^{+\infty}\frac{1}{F^{3/2}}du.$$

It follows that

$$0\le\frac{u_2-u_1}{\sqrt{F(u_2)}}\le\frac{C}{\sqrt{F(u_2)}}\int_{u_2}^{+\infty}\frac{1}{F}du$$

and the claim follows promptly.

Step 7. End of proof.

Let $w=U_2-U_1$. Since $U_2\ge U_1$ and f is nondecreasing, we see from the previous step that

$$\begin{cases}\Delta w=f(U_2)-f(U_1)\ge 0 & \text{in } B,\\ w=0 & \text{on } \partial B.\end{cases}$$

By the maximum principle, $w\le 0$ in B, as desired.

3. Proof of Theorem 1.3

Take a solution u to (1) and a real number b such that $u>b$. Without loss of generality, we may assume that $f(b)=0$ (otherwise, replace f by any C^1 function g that agrees with f on the range of u and such that $g(b)=0$). Let $\underline{u}$ denote the minimal large solution of (1) relative to b (see [3] for the existence of such a solution). In particular, $\underline{u}\le u$ and there exists a nondecreasing sequence of solutions to

$$\begin{cases}\Delta\underline{u}_N=f(\underline{u}_N) & \text{in } \Omega,\\ \underline{u}_N=N & \text{on } \partial\Omega,\end{cases}\tag{12}$$

converging to $\underline{u}$.

Let a be the constant appearing in (2), M the constant beyond which $\sqrt{F}$ is convex, and fix $\tilde{M}>\max(0,a,M)$. Fix $\varepsilon>0$ so small that $\underline{u}>\tilde{M}$ in $\Omega_\varepsilon=\{x\in\Omega:\mathrm{dist}(x,\partial\Omega)<\varepsilon\}$.

Step 1. We begin by proving that there exists a sequence of functions $(u_N)_{N\in\mathbb{N}}$ solving

$$\begin{cases}\Delta u_N=f(u_N) & \text{in } \Omega_\varepsilon,\\ u_N=N & \text{on } \partial\Omega,\\ u_N=u & \text{on } \{x\in\Omega:\mathrm{dist}(x,\partial\Omega)=\varepsilon\},\end{cases}\tag{13}$$

such that

$$\underline{u}_N\le u_N\le u\quad\text{in } \Omega_\varepsilon.\tag{14}$$

Observe that $\underline{u}_N$ and u are respectively a sub and a supersolution of (13) and that they are ordered. It follows that there exists a minimal solution u_N to (13) such that (14) holds.

By minimality, (u_N) is a nondecreasing sequence. Thanks to (14) and elliptic regularity, we may also assert that (u_N) converges in $C^2_{loc}(\overline{\Omega_\varepsilon} \setminus \partial\Omega)$ to a function $\tilde{u}$ solving

$$\begin{cases} \Delta\tilde{u} = f(\tilde{u}) & \text{in } \Omega_\varepsilon, \\ \tilde{u} = +\infty & \text{on } \partial\Omega, \\ \tilde{u} = u & \text{on } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = \varepsilon\}, \end{cases} \tag{15}$$

Step 2. There holds

$$|\nabla u_N|^2 - 2F(u_N) \le K_N \quad \text{in } \Omega_\varepsilon, \tag{16}$$

where

$$K_N = \sup_{\operatorname{dist}(x,\partial\Omega)=\varepsilon} \left[|\nabla u_N|^2 - 2F(u_N)\right]. \tag{17}$$

The proof is a straightforward adaptation of an argument due to Bandle and Marcus [9], which uses the method of P-functions. We give the full argument here for the convenience of the reader. Let

$$P_N = |\nabla u_N|^2 - 2F(u_N).$$

By a result of Payne and Stackgold ([10], see also Chapter 5 in [11]), there exists a bounded continuous vector field A, such that

$$\Delta P_N - \frac{A \cdot \nabla P_N}{|\nabla u_N|^2} \ge 0$$

at every point in Ω_ε where $\nabla u_N \ne 0$. Hence, P_N attains its maximum over $\overline{\Omega_\varepsilon}$ either on $\partial\Omega$, on $\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = \varepsilon\}$, or at a critical point of u_N. It only remains to prove that the first case cannot occur. We claim that $\partial P_N/\partial n \le 0$ on $\partial\Omega$, where n is the outward unit normal to $\partial\Omega$. The boundary-point lemma then implies that P_N cannot attain its maximum on $\partial\Omega$. It remains to prove our claim. Since u_N is a constant on $\partial\Omega$, we have

$$\frac{\partial P_N}{\partial n} = 2\frac{\partial u_N}{\partial n}\frac{\partial^2 u_N}{\partial n^2} - 2f(N)\frac{\partial u_N}{\partial n}, \quad \text{on } \partial\Omega.$$

Furthermore, letting H denote the mean curvature of $\partial\Omega$,

$$\Delta u_N = \frac{\partial^2 u_N}{\partial n^2} + (D-1)H\frac{\partial u_N}{\partial n} \quad \text{on } \partial\Omega.$$

Hence,

$$\frac{\partial P_N}{\partial n} = -2(D-1)H\left(\frac{\partial u_N}{\partial n}\right)^2 \quad \text{on } \partial\Omega.$$

Since $H \ge 0$, this implies that $\partial P_N/\partial n \le 0$, as desired. We have just proved (16).

Step 3. The function $\tilde{u} = \lim_{N\to+\infty} u_N$ coincides with u in Ω_ε.

The proof of this fact bears resemblances with a trick due to Nirenberg given in [12]. By (14), we already have $\tilde{u} \le u$ in Ω_ε and it remains to prove the reverse inequality. Thanks to (14) and elliptic regularity, there exists a constant K such that

$$2K \ge K_N,$$

where K_N is given by (17). Now let $\tilde{F} = F + K$ and define

$$v_N = \int_{u_N}^{+\infty} \frac{dt}{\sqrt{2\tilde{F}(t)}}.$$

Then, (16) can be rewritten as

$$|\nabla v_N| \le 1 \quad \text{in } \Omega_\varepsilon$$

from which it easily follows that

$$|\nabla \tilde{v}| \le 1 \quad \text{in } \Omega_\varepsilon, \tag{18}$$

where we defined similarly

$$\tilde{v} = \int_{\tilde{u}}^{+\infty} \frac{dt}{\sqrt{2\tilde{F}(t)}}.$$

Let at last

$$v = \int_u^{+\infty} \frac{dt}{\sqrt{2\tilde{F}(t)}}.$$

It remains to prove that $u \le \tilde{u}$, i.e. $\tilde{v} \le v$ in Ω_ε. Using the equations satisfied by u and $\tilde{u}$, we see that $w = v - \tilde{v}$ solves

$$\begin{aligned} -\Delta w &= \frac{f}{\sqrt{2\tilde{F}}}(u)\left(1 - |\nabla v|^2\right) - \frac{f}{\sqrt{2\tilde{F}}}(\tilde{u})\left(1 - |\nabla \tilde{v}|^2\right) \\ &= \left[\frac{f}{\sqrt{2\tilde{F}}}(u) - \frac{f}{\sqrt{2\tilde{F}}}(\tilde{u})\right]\left(1 - |\nabla \tilde{v}|^2\right) + \frac{f}{\sqrt{2\tilde{F}}}(u)\left(|\nabla \tilde{v}|^2 - |\nabla v|^2\right). \end{aligned}$$

By (14), we have $M < \underline{u} \le \tilde{u} \le u$ in Ω_ε. Since $\sqrt{2F}$ is convex in $(M, +\infty)$, $\frac{f}{\sqrt{2\tilde{F}}}$ is nondecreasing in the same interval. Using this and (18), we deduce that

$$\begin{cases} -\Delta w + b(x) \cdot \nabla w \ge 0, & \text{in } \Omega_\varepsilon \\ w = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $b(x) = \frac{f}{\sqrt{2\tilde{F}}}(u)\nabla(v + \tilde{v})$ is locally bounded in Ω. We may now apply the maximum principle to conclude that $w \ge 0$ in Ω, as desired.

Step 4. End of proof. The rest of the proof is similar to an argument due to García-Melián [8]. We take two arbitrary solutions $u, \overline{u}$ of our Eq. (1). We let $u_N, \overline{u}_N$ be the corresponding solutions to the approximated problem (13). In particular, $w_N = u_N - \overline{u}_N$ solves

$$\begin{cases} \Delta w_N = f(u_N) - f(\overline{u}_N) & \text{in } \Omega_\varepsilon, \\ w_N = 0 & \text{on } \partial\Omega, \\ w_N = u - \overline{u} & \text{on } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = \varepsilon\}, \end{cases} \tag{19}$$

By the maximum principle,

$$w_N \le \sup_{\operatorname{dist}(x,\partial\Omega)=\varepsilon} (u - \overline{u}) \quad \text{in } \Omega_\varepsilon,$$

with equality at some point x_N such that $\operatorname{dist}(x_N, \partial\Omega) = \varepsilon$. Extracting a sequence if necessary, we deduce that $w = u - \overline{u}$ satisfies

$$w \le \sup_{\operatorname{dist}(x,\partial\Omega)=\varepsilon} (u - \overline{u}) \quad \text{in } \Omega_\varepsilon, \tag{20}$$

with equality at some point z such that $\operatorname{dist}(z, \partial\Omega) = \varepsilon$. Now, we also have

$$\begin{cases} \Delta w = f(u) - f(\overline{u}) & \text{in } \Omega \setminus \Omega_\varepsilon, \\ w = u - \overline{u} & \text{on } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) = \varepsilon\}. \end{cases}$$

By the maximum principle, we deduce that inequality (20) holds throughout Ω, with equality at the point z. The strong maximum principle implies that w is equal to a constant c. Since $u, \overline{u}$ solve (1), we deduce that $f(u) = f(u + c)$, which is possible only if $c = 0$. □

4. Proof of Proposition 1.1

We thank Laurent Véron [13] for the following proof. Given $p \in (1, D/(D-2))$, $k \in \mathbb{N}$ and $\lambda > 0$, we begin by solving

$$\begin{cases} -\Delta u + u^p = \lambda\delta_0 & \text{in } B, \\ u = k & \text{on } \partial B. \end{cases} \tag{21}$$

Since 0 is a subsolution, while a large constant multiple of the fundamental solution is a supersolution, we deduce from the method of sub and supersolutions (see e.g. [14] for the appropriate statement) that there exists a solution $u = u_k$ to (21). By the maximum principle, u_k is the unique solution to (21), and the sequence (u_k) is nondecreasing. Thanks to the Keller–Osserman estimate (see e.g. [1]), the sequence (u_k) is uniformly bounded on compact subsets of the punctured ball $B \setminus \{0\}$. It follows from elliptic regularity that u_k converges to a solution $u = u_\lambda$ of

$$\begin{cases} -\Delta u + u^p = \lambda\delta_0 & \text{in } B, \\ u = +\infty & \text{on } \partial B. \end{cases}$$

By the results of [15], u_λ behaves like a constant multiple of the fundamental solution near the origin. In particular, each u_λ is a large solution in the punctured ball.

There exists yet another large solution. Simply note that for an appropriate constant $c = c(D, p) > 0$, the function $u_1(x) = c|x|^{-2/(p-1)}$ solves $\Delta u = u^p$ in $\mathbb{R}^D \setminus \{0\}$. Let also u_2 be the unique solution to

$$\begin{cases} \Delta u = u^p & \text{in } B, \\ u = +\infty & \text{on } \partial B, \end{cases}$$

Then, $\underline{u} = \max(u_1, u_2)$ and $\overline{u} = u_1 + u_2$ form an ordered pair of sub and supersolutions to the equation in the punctured ball. The method of sub and supersolutions implies the existence of a new large solution u_∞ which behaves like $c|x|^{-2/(p-1)}$ near the origin, hence distinct from u_λ.

Finally, observe that for the nonlinearity $f(u) = u^p$, if u is a large solution and $\epsilon > 0$, then $(1 + \epsilon)u$ is a supersolution. From this, the classification of singularities both at the origin (see [15]) and on the boundary (see e.g. [9]), and the maximum principle, it easily follows that the set of positive large solutions in the punctured ball is exactly $\{u_\lambda\}_{\lambda\in(0,+\infty]}$.

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