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Inequalities Involving Bessel and Modified Bessel Functions

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Some inequalities for the ratios $J_{\nu+1}(x)/J_\nu(x)$ and $I_{\nu+1}(x)/I_\nu(x)$ of Bessel and modified Bessel functions of the first kind and order $\nu > -1$ are given. The inequalities are found easily from a different definition of the ratios and improve many results obtained recently by several authors. © 1990 Academic Press, Inc.

1. INTRODUCTION

The function

$$\Phi_\nu(x) = \frac{2(\nu+1)}{x} \cdot \frac{J_{\nu+1}(x)}{J_\nu(x)}, \tag{1.1}$$

where $J_\nu(x)$ is the Bessel function of the first kind and order $\nu > -1$ may be defined in terms of a scalar product in a separable Hilbert space. In this definition of $\Phi_\nu(x)$ a self-adjoint and compact operator S_ν is involved. Many lower and upper bounds for $\Phi_\nu(x)$ and $\Phi_\nu(ix)$ $(i^2 = -1)$ can be found easily by several operator techniques and lead obviously to lower and upper bounds for the ratios $J_{\nu+1}(x)/J_\nu(x)$ and $I_{\nu+1}(x)/I_\nu(x)$. ($I_\nu(x)$ is the modified Bessel function of the first kind.) We use these bounds in this article to improve some inequalities, involving Bessel functions which have been found recently by several authors. More precisely: For $0 < x < j_{\nu,1}$, where $j_{\nu,1}$ is the first positive zero of $J_\nu(x)$, the lower bound

$$\Phi_\nu(x) > 1 + \frac{x^2}{4(\nu+1)(\nu+2)}, \qquad \nu > -1, \tag{1.2}$$

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consisting of the first two terms of a lower bound for $\Phi_\nu(x)$, improves the inequalities

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} > \frac{x}{2(\nu+1)}, \qquad 0<x<j_{\nu,1} \tag{1.3}$$

$$\frac{J_\nu(xt)}{J_\nu(x)t^\nu} > \exp\left\{\frac{x^2(1-t^2)}{4(\nu+1)}\right\}, \qquad 0<x<j_{\nu,1} \quad 0<t<1, \tag{1.4}$$

which were proved in [11] only for $\nu>0$. Some other results consist of some lower and upper bounds of the ratios $J_\nu(xt)/J_\nu(x)t^\nu$, $0<t<1$, which are sharper than those given in [9, 11, and 14].

Also the inequality

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} < \frac{x}{2(\nu+1)} + \frac{x^3 j^2_{\nu,1}}{8(\nu+1)^2\,(2+\nu)(j^2_{\nu,1}-x^2)}, \qquad 0<x<j_{\nu,1}, \quad \nu>-1 \tag{1.5}$$

resulting from an upper bound of $\Phi_\nu(x)$ is more stringent than the inequality

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} < \frac{x}{2\nu+1}, \qquad \nu>0, \quad 0<x\leqslant\frac{\pi}{2} \tag{1.6}$$

given in [15, p. 670].

The inequality

$$J_\nu(x) < \left(\frac{x}{2}\right)^\nu \Gamma(\nu+1)^{-1} \exp\left\{-\frac{x^2}{4(\nu+1)} - \frac{x^4}{32(\nu+1)^2\,(\nu+2)}\right\}, \qquad 0<x<j_{\nu,1}, \quad \nu>-1 \tag{1.7}$$

derived from a lower bound of $\Phi_\nu(x)$ is sharper than the inequality

$$J_\nu(x) < \left(\frac{x}{2}\right)^\nu \Gamma(\nu+1)^{-1} e^{-x^2/4(\nu+1)}, \qquad x>0, \quad \nu\geqslant 0 \tag{1.8}$$

given in [16, p. 16]. Moreover this inequality for $\nu=\alpha-\frac{1}{2}$ improves the inequality

$$J_{\alpha-1/2}(x) < \left(\frac{x}{2}\right)^{\alpha-1/2} \Gamma\left(\alpha+\frac{1}{2}\right)^{-1} \left\{1+\frac{x^2}{(2\alpha+1)\alpha}\right\}^{-\alpha/2}, \qquad 0.065\leqslant\alpha<1, \quad x>0 \tag{1.9}$$

given recently in [2].

Finally a lower bound for the ratio $I_{\nu+1}(x)/I_\nu(x)$, which follows from the expansion of $\Phi_\nu(x)$ in terms of the eigenelements and eigenvalues of S_ν, leads to the inequality

$$\frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^\nu \left(\frac{j_{\nu,1}^2 + x^2}{j_{\nu,1}^2 + y^2}\right)^{j_{\nu,1}^2/4(\nu+1)}, \qquad y > x > 0, \quad \nu > -1, \tag{1.10}$$

which is an improvement of the inequality

$$\frac{I_\nu(x)}{I_\nu(x)} < \left(\frac{x}{y}\right)^\nu, \qquad y > x > 0, \quad \nu > -\frac{1}{2} \tag{1.11}$$

given in [14]. The inequality

$$\Gamma(\nu+1)\left(\frac{2}{y}\right)^\nu I_\nu(y) > \left(1 + \frac{y^2}{j_{\nu,1}^2}\right)^{j_{\nu,1}^2/4(\nu+1)}, \qquad y > 0, \quad \nu > -1, \tag{1.12}$$

which follows from (1.10) for $x \to 0$, is more stringent then the inequality

$$\Gamma(\nu+1)\left(\frac{2}{y}\right)^\nu I_\nu(y) > 1, \qquad y > 0, \quad \nu > -\frac{1}{2},$$

given in [12, p. 29].

All these improvements are given in Section 3. In Section 2 we give the definition of the function $\varphi_\nu(x)$ in its abstract form, prove Eq. (1.1), and present the bounds that we use.

2. The Function $\varphi_\nu(x)$

In the following e_n, $n = 1, 2, \ldots$, is used to denote an orthonormal basis in an abstract Hilbert space $\mathscr{H}$. By V we mean the shift operator ($Ve_n = e_{n+1}$), by V^* its adjoint and by L_ν the diagonal operator $L_\nu e_n = (1/(\nu+n))e_n$, which for $\nu > -1$ is nonnegative. The self-adjoint and compact operator

$$S_\nu = L_\nu^{1/2}(V + V^*)L_\nu^{1/2}, \qquad \nu > -1, \tag{2.1}$$

has been used in many papers for the study of the zeros of Bessel functions $J_\nu(x)$, $\nu > -1$ [4–6]. The eigenvalues of S_ν are precisely the values $\pm 2/j_{\nu,n}$, where $j_{\nu,n}$ is the nth positive zero of $J_\nu(x)$. The function $\varphi_\nu(x)$ is defined for every $x \neq \pm j_{\nu,n}$ as

$$\varphi_\nu(x) = \left\langle \left(I - \frac{x}{2} S_\nu\right)^{-1} e_1, e_1 \right\rangle, \tag{2.2}$$

where the symbol $\langle\, ,\, \rangle$ means scalar product in H and $(I-(x/2)S_\nu)^{-1}$ means the inverse of $I-(x/2)S_\nu$. This function has been used in [7] for the study of the zeros of mixed Bessel functions and it can be expressed as

$$\varphi_\nu(x)=2\sum_{n=1}^{\infty}\frac{j_{\nu,n}^2}{j_{\nu,n}^2-x^2}\,|\langle e_1, x_n(\nu)\rangle|^2, \qquad x\neq j_{\nu,n}, \tag{2.3}$$

where

$$x_n(\nu)=\alpha_n\sum_{k=1}^{\infty}\sqrt{\nu+k}\,J_{\nu+k}(j_{\nu,n})e_k, \qquad n=1, 2, ... \tag{2.4}$$

are the eigenvectors of S_ν normalized by $\|x_n(\nu)\|=1$ which correspond to the positive eigenvalues $2/j_{\nu,n}$ and

$$\alpha_n=\left[\sum_{k=1}^{\infty}(\nu+k)\,J_{\nu+k}^2(j_{\nu,n})\right]^{-1/2} \tag{2.5}$$

are the normalization factors [6].

In particular [7, p. 95], for $0<x<j_{\nu,1}$ the function $\varphi_\nu(x)$ takes the form

$$\varphi_\nu(x)=1+\sum_{n=1}^{\infty}\frac{\|S_\nu^n e_1\|^2}{2^{2n}}x^{2n}, \qquad 0\leqslant x<j_{\nu,1}, \quad \nu>-1. \tag{2.6}$$

It can be proved [8, 10] that the normalization factor α_n has the simple form

$$\alpha_n=\sqrt{2}\,[j_{\nu,n}J_{\nu+1}(j_{\nu,n})]^{-1/2}, \qquad n=1, 2, ... \tag{2.7}$$

Consequently from (2.4) it follows that

$$|\langle e_1, x_n(\nu)\rangle|^2=\frac{2(\nu+1)}{j_{\nu,n}^2}, \qquad n=1, 2, ... \tag{2.8}$$

$$\sum_{n=1}^{\infty}|\langle e_1, x_n(\nu)\rangle|^2=\frac{1}{2} \tag{2.8a}$$

and the relationship (1.1) follows from (2.3) and the well-known [16, p. 497] Mittag–Leffler expansion

$$\frac{J_{\nu+1}(x)}{J_\nu(x)}=2x\sum_{n=1}^{\infty}\frac{1}{j_{\nu,n}^2-x^2}, \qquad x\neq j_{\nu,n}. \tag{2.9}$$

We give below an alternative proof of (1.1) and (2.8) from which (2.9) and (2.7) follow immediately. In fact we prove the following:

THEOREM 2.1. *The function $\varphi_\nu(x)$ given by* (2.2) *is equal to the function $\Phi_\nu(x)$ given by* (1.1).

Proof. Consider the function

$$H_{\nu,\alpha}(z)=\alpha J_\nu(z)+zJ'_\nu(z). \tag{2.10}$$

We know from a general result [7, Theorems 3.1 and 4.1] that ρ is a zero of (2.10) if and only if it is a zero of the equation

$$\nu+\alpha=\frac{z^2}{2(\nu+1)}\varphi_\nu(z), \qquad \nu>-1.$$

On the other hand, from (2.10) and the well-known [16] relation

$$zJ'_\nu(z)=-zJ_{\nu+1}(z)+\nu J_\nu(z)$$

it follows immediately that ρ is a zero of (2.10) if and only if it satisfies the equation

$$\nu+\alpha=z\frac{J_{\nu+1}(z)}{J_\nu(z)}.$$

So the function

$$N(z)=\frac{z^2}{2(\nu+1)}\varphi_\nu(z)-z\frac{J_{\nu+1}(z)}{J_\nu(z)}$$

vanishes for all the positive zeros of $H_{\nu,\alpha}(z)$.

We can choose a sequence $\alpha_k\to\alpha$, $k\to+\infty$, such that the sequence of the first positive zeros ρ_k of $H_{\nu,\alpha_k}(z)$ converges to $\rho\neq j_{\nu,n}$. So, $N(z)$ vanishes for a sequence which converges in its domain of analyticity. This implies that $N(z)=0$ for every $z\neq\pm j_{\nu,n}$, $n=1, 2, ...,$ and

$$\frac{z\varphi_\nu(z)}{2(\nu+1)}=\frac{J_{\nu+1}(z)}{J_\nu(z)}, \qquad \nu>-1, \quad z\neq j_{\nu,n}. \tag{2.11}$$

This proves the theorem.

Remark 2.1. From (2.11) and (2.3) we obtain

$$\frac{J_{\nu+1}(z)}{J_\nu(z)}=\frac{z}{\nu+1}\sum_{n=1}^{\infty}\frac{j^2_{\nu,n}}{j^2_{\nu,n}-z^2}|\langle e_1, x_n(\nu)\rangle|^2. \tag{2.12}$$

Integration of (2.12) on a closed rectifiable curve which contains only the positive zero $j_{\nu,n}$ for some n gives

$$2\pi i \operatorname{Res}\frac{J_{\nu+1}(z)}{J_\nu(z)}\bigg|_{z=j_{\nu,n}}=2\pi i j^2_{\nu,n}\frac{|\langle e_1, x_n(\nu)\rangle|^2}{\nu+1}\left(-\frac{1}{2}\right). \tag{2.13}$$

Since

$$\operatorname{Res} \frac{J_{\nu+1}(z)}{J_\nu(z)}\bigg|_{z=j_{\nu,n}}$$

$$= \lim_{z \to j_{\nu,n}} (z - j_{\nu,n}) \cdot \frac{J_{\nu+1}(z)}{j_\nu(z)}$$

$$= J_{\nu+1}(j_{\nu,n}) \cdot \lim_{z \to j_{\nu,n}} \frac{z - j_{\nu,n}}{J_\nu(z)} = \frac{J_{\nu+1}(j_{\nu,n})}{J'_\nu(j_{\nu,n})} = -1,$$

we obtain from (2.13) the relation (2.8).

Note that (2.8) together with (2.3) and (1.1) prove the Mittag–Leffler expansion (2.9) and (2.8) together with (2.4) prove relation (2.7).

Using the relation $J_\nu(ix) = i^\nu I_\nu(x)$ we find from (1.1)

$$\frac{x\varphi_\nu(ix)}{2(\nu+1)} = \frac{I_{\nu+1}(x)}{I_\nu(x)}, \qquad \nu > -1, \tag{2.14}$$

where, by (2.9),

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} = 2x \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2 + x^2}, \qquad \nu > -1. \tag{2.15}$$

Since

$$S_\nu e_1 = \frac{1}{\sqrt{(\nu+1)(\nu+2)}} e_2,$$

$$S_\nu e_2 = \frac{e_1}{\sqrt{(\nu+2)(\nu+1)}} + \frac{e_3}{\sqrt{(\nu+3)(\nu+2)}},$$

$$S_\nu e_3 = \frac{e_2}{\sqrt{(\nu+3)(\nu+2)}} + \frac{e_4}{\sqrt{(\nu+4)(\nu+3)}},$$

$$S_\nu^2 e_1 = \frac{1}{(1+\nu)(2+\nu)} e_1 + \frac{1}{(2+\nu)\sqrt{(1+\nu)(3+\nu)}} e_3,$$

$$S_\nu^3 e_1 = \frac{2}{(1+\nu)(3+\nu)\sqrt{(1+\nu)(2+\nu)}} e_2 + \frac{1}{(2+\nu)(3+\nu)\sqrt{(1+\nu)(4+\nu)}} e_4,$$

we find from (2.6)

$$\varphi_\nu(x) > 1 + \frac{1}{4(1+\nu)(2+\nu)} x^2 + \frac{1}{8(1+\nu)^2(2+\nu)(3+\nu)} x^4 + \frac{5\nu+11}{64(1+\nu)^3(2+\nu)^2(3+\nu)(4+\nu)} x^6, \qquad 0 < x < j_{\nu,1}, \quad \nu > -1$$

which by (1.1) gives the lower bound

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} > \frac{x}{2(\nu+1)} + \frac{x^3}{8(1+\nu)^2(2+\nu)} + \frac{x^5}{16(1+\nu)^3(2+\nu)(3+\nu)} + \frac{(5\nu+11)}{128(1+\nu)^4(2+\nu)^2(3+\nu)} \frac{x^7}{(4+\nu)}, \qquad 0 < x < j_{\nu,1}, \quad \nu > -1. \tag{2.16}$$

The upper bound

$$\varphi_\nu(x) < \frac{j_{\nu,1}^2}{j_{\nu,1}^2 - x^2}, \qquad 0 < x < j_{\nu,1}, \quad \nu > -1 \tag{2.17}$$

can be obtained directly from (2.3), by the inequality $j_{\nu,n}^2/(j_{\nu,n}^2 - x^2) < j_{\nu,1}^2/(j_{\nu,1}^2 - x^2)$ and the equality (2.8α).

Another upper bound better than (2.17) can be found as follows: Since $S_\nu e_1 = [(1+\nu)(2+\nu)]^{-1/2} e_2$ and $\|S_\nu\| = 2/j_{\nu,1}$ [6] we obtain from (2.6)

$$\varphi_\nu(x) = 1 + \frac{x^2}{2^2(1+\nu)(2+\nu)} \left\{ 1 + \frac{\|S_\nu e_2\|^2}{2^2} x^2 + \frac{\|S_\nu^2 e_2\|^2}{2^4} x^4 + \cdots \right\}$$

$$< 1 + \frac{x^2}{2^2(1+\nu)(2+\nu)} \left(1 + \frac{x^2}{j_{\nu,1}^2} + \frac{x^4}{j_{\nu,1}^4} + \cdots \right)$$

which for $0 < x < j_{\nu,1}$ gives

$$\varphi_\nu(x) < 1 + \frac{x^2 j_{\nu,1}^2}{4(1+\nu)(2+\nu)(j_{\nu,1}^2 - x^2)}$$

or

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} < \frac{x}{2(1+\nu)} + \frac{x^3 j_{\nu,1}^2}{8(1+\nu)^2(2+\nu)(j_{\nu,1}^2 - x^2)}, \qquad 0 < x < j_{\nu,1}, \quad \nu > -1. \tag{2.18}$$

This inequality improves the inequality $J_{\nu+1}(x)/J_\nu(x) < x/(2\nu+1)$, $\nu > 0$, $0 < x \leqslant \pi/2$ given in [15, p. 670], with respect to ν and for all x such that

$$x < 2j_{\nu,1} \left[\frac{(\nu+1)(\nu+2)}{(2\nu+1) j_{\nu,1}^2 + 4(\nu+1)(\nu+2)} \right]^{1/2}.$$

Now we give some inequalities which follow easily from the expansion of

$\varphi_\nu(x)$ in terms of the eigenelements and eigenvalues of S_ν. So, from (2.3) and (2.11) we have

$$\frac{2(\nu+1)}{x}\frac{I_{\nu+1}(x)}{I_\nu(x)}=2\sum_{n=1}^{\infty}\frac{j_{\nu,n}^2}{j_{\nu,n}^2+x^2}|\langle e_1, x_n(\nu)\rangle|^2.$$

Applying the inequality $j_{\nu,n}^2/(j_{\nu,n}^2+x^2)>j_{\nu,1}^2/(j_{\nu,1}^2+x^2)$, for $j_{\nu,n}>j_{1,n}$, hence by (2.8α) we find

$$\frac{I_{\nu+1}(x)}{I_\nu(x)}>\frac{j_{\nu,1}^2}{2(\nu+1)}\cdot\frac{x}{j_{\nu,1}^2+x^2},\qquad x>0,\quad \nu>-1. \tag{2.19}$$

Also

$$2(\nu+1)\frac{I_{\nu+1}(x)}{I_\nu(x)}=2x\sum_{n=1}^{\infty}\frac{j_{\nu,n}^2}{j_{\nu,n}^2+x^2}\cdot|\langle e_1, x_n(\nu)\rangle|^2$$

$$=2x\sum_{n=1}^{\infty}|\langle e_1, x_n(\nu)\rangle|^2-2x^3\sum_{n=1}^{\infty}\frac{|\langle e_1, x_n(\nu)\rangle|^2}{j_{\nu,n}^2+x^2}.$$

On using (2.8α) we obtain

$$x-2(\nu+1)\frac{I_{\nu+1}(x)}{I_\nu(x)}=2x^3\sum_{n=1}^{\infty}\frac{|\langle e_1, x_n(\nu)\rangle|^2}{j_{\nu,n}^2+x^2}. \tag{2.20}$$

From (2.20) it follows that

$$\frac{I_{\nu+1}(x)}{I_\nu(x)}<\frac{x}{2(\nu+1)},\qquad x>0,\quad \nu>-1. \tag{2.21}$$

The inequality (2.21) was proved also by I. Nasell in [13], with a different method and improves an inequality given by D. K. Ross [15].

Also from (2.20) it follows that

$$x-2(\nu+1)\frac{I_{\nu+1}(x)}{I_\nu(x)}<x^3\sum_{n=1}^{\infty}\frac{1}{j_{\nu,n}^2+x^2}=\frac{x^2}{2}\frac{I_{\nu+1}(x)}{I_\nu(x)}$$

or

$$\frac{I_{\nu+1}(x)}{I_\nu(x)}>\frac{2x}{4(\nu+1)+x^2},\qquad x>0,\quad \nu>-1. \tag{2.22}$$

Many lower bounds for the ratio $I_{\nu+1}(x)/I_\nu(x)$ are given in [13] with a different method. Among them the most simple is

$$\frac{I_{\nu+1}(x)}{I_\nu(x)} > \frac{x}{2(\nu+1)+x}, \qquad x>0, \quad \nu>-1,$$

which for $0<x<2$ is weaker than (2.22). More sophisticated bounds for $I_{\nu+1}(x)/I_\nu(x)$ can be found in [1].

3. Improvements

1. In [11] the inequalities (1.3) and (1.4) have been proved both for $\nu>0$. Obviously (2.16) is an essential improvement of (1.3). In particular we obtain from (2.16) the inequality

$$\frac{J_{\nu+1}(x)}{J_\nu(x)} > \frac{x}{2(\nu+1)} + \frac{x^3}{8(\nu+1)^2(\nu+2)}. \tag{3.1}$$

Integration of the well-known recurrence relation [16]

$$\frac{J_\nu'(x)}{J_\nu(x)} = \frac{\nu}{x} - \frac{J_{\nu+1}(x)}{J_\nu(x)} \tag{3.2}$$

between xt and x, $0<t<1$, gives

$$\frac{J_\nu(xt)}{J_\nu(x)t^\nu} = \exp\left\{\int_{xt}^{x} \frac{J_{\nu+1}(\omega)}{J_\nu(\omega)}\, d\omega\right\}. \tag{3.3}$$

Inequality (1.4) follows from (3.3), (1.3) and holds not only for $\nu>0$ but for $\nu>-1$.

2. Since $\nu<j_{\nu,1}$, for $\nu>0$ we can take from (3.1) for $x=\nu$ and from (3.3) that

$$\frac{J_\nu(\nu t)}{J_\nu(\nu)t^\nu} > \exp\left\{\frac{\nu^2(1-t^2)}{4(\nu+1)} + \frac{\nu^4(1-t^4)}{32(1+\nu)^2(2+\nu)}\right\}, \qquad \nu>0, \quad 0<t<1 \tag{3.4}$$

which is sharper than the inequality

$$\frac{J_\nu(\nu t)}{J_\nu(\nu)t^\nu} > \exp\left\{\frac{\nu^2(1-t^2)}{4(\nu+1)}\right\}, \qquad \nu>0, \quad 0<t<1$$

given in [9, 11, 14].

3. Let $\rho_{\nu,1,\alpha}$ be the first positive zero of the mixed Bessel function $\alpha J_\nu(x)+xJ'_\nu(x)$, $\alpha\in\mathbb{R}$, $\nu>-1$. Since $\rho_{\nu,1,\alpha}<j_{\nu,1}$ for $\nu>\max\{-\alpha,-1\}$ [7] we can take from (3.1) for $x=\rho_{\nu,1,\alpha}$ and from (3.3) that

$$\frac{J_\nu(\rho_{\nu,1,\alpha}t)}{J_\nu(\rho_{\nu,1,\alpha})t^\nu}>\exp\left\{\frac{\rho^2_{\nu,1,\alpha}(1-t^2)}{4(\nu+1)}+\frac{\rho^4_{\nu,1,\alpha}(1-t^4)}{32(\nu+1)^2(\nu+2)}\right\},\quad 0<t<1,\quad \nu>\max\{-\alpha,-1\}. \tag{3.5}$$

In particular for $\alpha=0$, where $\rho_{\nu,1,0}=j'_{\nu,1}$ is the first positive zero of $J'_\nu(x)$, we have

$$\frac{J_\nu(j'_{\nu,1}t)}{J_\nu(j'_{\nu,1})t^\nu}>\exp\left\{\frac{j'^2_{\nu,1}(1-t^2)}{4(\nu+1)}+\frac{j'^4_{\nu,1}(1-t^4)}{32(\nu+1)^2(\nu+2)}\right\},\qquad 0<t<1,\quad \nu>0, \tag{3.6}$$

which is more stringent than the inequality

$$\frac{J_\nu(j'_{\nu,1}t)}{J_\nu(j'_{\nu,1})t^\nu}>\exp\left\{\frac{j'^2_{\nu,1}(1-t^2)}{4(\nu+1)}\right\},\qquad 0<t<1,\quad \nu>0 \tag{3.7}$$

given in [11].

4. From (3.3) and (2.18), after some calculations we get

$$\frac{J_\nu(xt)}{J_\nu(x)t^\nu}<\left(\frac{j^2_{\nu,1}-x^2t^2}{j^2_{\nu,1}-x^2}\right)^{j^4_{\nu,1}/16(\nu+1)^2(\nu+2)}\times\exp\left\{\frac{x^2(1-t^2)}{4(\nu+1)}-\frac{j^2_{\nu,1}(1-t^2)x^2}{16(\nu+1)^2(\nu+2)}\right\},\quad 0<x<j_{\nu,1},\quad 0<t<1,\quad \nu>-1. \tag{3.8}$$

Replacing x by ν in (3.8) we find

$$\frac{J_\nu(\nu t)}{J_\nu(\nu)t^\nu}<\left(\frac{j^2_{\nu,1}-\nu^2t^2}{j^2_{\nu,1}-\nu^2}\right)^{j^4_{\nu,1}/16(\nu+1)^2(\nu+2)}\times\exp\left\{\frac{\nu^2(1-t^2)}{4(\nu+1)}-\frac{j^2_{\nu,1}\nu^2(1-t^2)}{16(\nu+1)^2(\nu+2)}\right\}. \tag{3.9}$$

Numerical calculations indicate that this inequality is more stringent than the inequality

$$\frac{J_\nu(\nu t)}{J_\nu(\nu)t^\nu}<\left(\frac{j^2_{\nu,1}-\nu^2t^2}{j^2_{\nu,1}-\nu^2}\right)\exp\left\{\frac{(\nu+1)\,\nu^2(1-t^2)}{2j^2_{\nu,1}}\right\},\qquad 0<t<1,\quad \nu>-1 \tag{3.10}$$

given in [9]. Note again that numerical evidence indicates that (3.10) is sharper than the inequality

$$\frac{J_\nu(\nu t)}{J_\nu(\nu)t^\nu} < \exp\{\nu(1-t)\}, \qquad \nu > 0, \quad 0 < t < 1, \tag{3.11}$$

given in [14].

For example for $\nu = \frac{1}{2}$, $t = \frac{1}{2}$ we have respectively by (3.9), (3.10), and (3.11) the upper bounds, 1.0330146, 1.0792831, and 1.2840252.

5. Since

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{\Gamma(\nu+1)\,\Gamma(\nu+n+1)},$$

we find from (3.3) for $t \to 0$ that

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \exp\left\{-\int_0^x \frac{J_{\nu+1}(\omega)}{J_\nu(\omega)}\,d\omega\right\} \tag{3.12}$$

and using (2.18) we obtain

$$J_\nu(x) \geqslant \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \left(1 - \frac{x^2}{j_{\nu,1}^2}\right)^{j_{\nu,1}^4/16(\nu+1)^2(\nu+2)} \times \exp\left\{-x^2\left(\frac{4(\nu+1)(\nu+2) - j_{\nu,1}^2}{16(\nu+1)^2(\nu+2)}\right)\right\}, \qquad 0 < x < j_{\nu,1}, \quad \nu > -1.$$

Since $\nu < j_{\nu,1}$ the above inequality leads to

$$J_\nu(\nu) > \frac{\nu^\nu}{2^\nu \Gamma(\nu+1)} \left(1 - \frac{\nu^2}{j_{\nu,1}^2}\right)^{j_{\nu,1}^4/16(\nu+1)^2(\nu+2)} \times \exp\left\{-\nu^2\left(\frac{4(\nu+1)(\nu+2) - j_{\nu,1}^2}{16(\nu+1)^2(\nu+2)}\right)\right\}, \qquad \nu > 0. \tag{3.13}$$

Numerical calculations show that the bound (3.13) is sharper in the interval (0, 5) than the bound

$$J_\nu(\nu) \geqslant \frac{\Gamma(\frac{1}{3})}{2^{2/3} 3^{1/6} \pi (\nu + \alpha_0)^{1/3}}, \qquad \alpha_0 = 0.0943498 \tag{3.14}$$

given in [3].

Also, from (3.12) and the inequality (3.1) for $x \in (0, j_{\nu,1})$ we obtain the upper bound

$$J_\nu(x) < \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \exp\left\{-\frac{x^2}{4(\nu+1)} - \frac{x^4}{32(\nu+1)^2(\nu+2)}\right\}, \quad 0 < x < j_{\nu,1}, \quad \nu > -1, \tag{3.15}$$

which is more stringent than the upper bound

$$J_\nu(x) < \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \exp\left\{-\frac{x^2}{4(\nu+1)}\right\}, \qquad x > 0, \quad \nu \geqslant 0$$

given in [16, p. 16].

For $\nu = \alpha - \frac{1}{2}$, in (3.15) we obtain the inequality

$$J_{\alpha-1/2}(x) < \left(\frac{x}{2}\right)^{\alpha-1/2} \cdot \Gamma\left(\alpha+\frac{1}{2}\right)^{-1} \exp\left\{-\frac{x^2}{4(\alpha+\frac{1}{2})} - \frac{x^4}{32(\alpha+\frac{1}{2})^2(\alpha+\frac{3}{2})}\right\}, \quad \alpha > -\frac{1}{2}, \quad 0 < x < j_{\alpha-1/2,1} \tag{3.16}$$

It can be readily proved that the inequality (3.16) is more stringent than the inequality

$$J_{\alpha-1/2}(x) < \left(\frac{x}{2}\right)^{\alpha-1/2} \Gamma\left(\alpha+\frac{1}{2}\right)^{-1} \left\{1 + \frac{x^2}{(2\alpha+1)\alpha}\right\}^{-\alpha/2}, \quad 0.065 \leqslant \alpha < 1, \quad x > 0$$

given recently by A. K. Common in [2].

Also numerical evidence shows that the upper bound which follows from (3.15) for $x = \nu$ is better than the upper bound

$$J_\nu(\nu) < \frac{\Gamma(\frac{1}{3})}{2^{2/3} 3^{1/6} \pi \cdot \nu^{1/3}}, \qquad \nu > 0$$

given in [16, p. 259], in the interval (0, 4).

6. In Ref. [14] among others it was proved that

$$\frac{I_\nu(x)}{I_\nu(y)} < \left(\frac{x}{y}\right)^\nu, \qquad y > x > 0, \quad \nu > -\frac{1}{2}, \tag{3.17}$$

which improves an inequality given in [15]. Integration of the well-known recurrence relation [16], p. 79]

$$\frac{I'_\nu(x)}{I_\nu(x)} = \frac{\nu}{x} + \frac{I_{\nu+1}(x)}{I_\nu(x)},$$

between x and y gives

$$\frac{I_\nu(y)}{I_\nu(x)}=\left(\frac{y}{x}\right)^\nu \exp\left\{\int_x^y \frac{I_{\nu+1}(t)}{I_\nu(t)}\,dt\right\}. \tag{3.18}$$

From (3.18) and the inequality (2.19) it follows

$$\frac{I_\nu(y)}{I_\nu(x)}>\left(\frac{y}{x}\right)^\nu \left(\frac{j_{\nu,1}^2+y^2}{j_{\nu,1}^2+x^2}\right)^{j_{\nu,1}^2/4(\nu+1)}$$

or

$$\frac{I_\nu(x)}{I_\nu(y)}<\left(\frac{x}{y}\right)^\nu \left(\frac{j_{\nu,1}^2+x^2}{j_{\nu,1}^2+y^2}\right)^{j_{\nu,1}^2/4(\nu+1)}, \qquad y>x>0, \quad \nu>-1. \tag{3.19}$$

This inequality is an improvement of (3.17) not only with respect to the bound but also with respect to the validity of ν. Finally from (3.19) for $x\to 0$, we obtain the inequality

$$\Gamma(\nu+1)\left(\frac{2}{y}\right)^\nu I_\nu(y)>\left(1+\frac{y^2}{j_{\nu,1}^2}\right)^{j_{\nu,1}^2/4(\nu+1)}, \qquad y>0, \quad \nu>-1 \tag{3.20}$$

which is more stringent than the inequality

$$\Gamma(\nu+1)\left(\frac{2}{y}\right)^\nu I_\nu(y)>1, \qquad y>0, \quad \nu>-\frac{1}{2}$$

given by Y. L. Luke in [12, p. 29].

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