Normal generalized selfadjoint operators in Krein spaces

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Abstract

A full description is given of n-selfadjoint normal operators in Krein spaces of finite defect, as well as in general Krein spaces. Both real and complex Krein spaces are considered. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

This note is a follow up on the papers [1,2]. Let J be a selfadjoint operator on a (complex or real, finite or infinite dimensional) Hilbert space H such that J^2 = I. Consider the sesquilinear form [.,.] induced by J:

[x, y] = ⟨Jx, y⟩, \quad x, y ∈ H,

where ⟨., .⟩ stands for the inner product in H. The corresponding quadratic form [x, x] is indefinite (unless J = I or J = −I). The space H, together with the sesquilinear form [., .] generated by some J as above, is called a Krein space.

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Let \( n \) be a fixed positive integer. A (linear bounded) operator \( A \) on the Krein space \( H \) is called \( n \)-selfadjoint if

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} A^k A^{n-k} = 0
\]

(\( A^0 \) and \( A^0 \) are interpreted as the identity operator \( I \)), where \( A^* \) stands for the Krein space adjoint: \([Ax, y] = [x, A'y]\) for all \( x, y \in H \). (Everywhere from now on the adjoint is understood in the Krein space sense.)

Some information on the structure of Krein space \( n \)-selfadjoint operators is found in [2]. In this note we describe \( n \)-selfadjoint operators \( A \), in the case \( A \) is assumed, in addition, normal. Recall that an operator \( A \) on \( H \) is called normal if \( AA^* = A'A \). In contrast with [1,2], we allow real Krein spaces as well as the complex ones.

We denote by \( P_+ = \frac{1}{2} (I + J) \) the orthogonal (with respect to the usual scalar product \((\cdot, \cdot)\)) projector on the eigenspace of \( J \) corresponding to the eigenvalue 1. The defect \( q \) of the Krein space is defined by \( q = \min (\text{rank } P_+, \text{rank } (I - P_+)) \). Thus, \( q \) is either a nonnegative integer or infinity.

The following proposition is useful:

**Proposition 1.1.** If \( A \) is \( n \)-selfadjoint, then \( A - \lambda I \) is \( m \)-selfadjoint for every real \( \lambda \) and every integer \( m \geq n \).

The proof goes through as in the complex Hilbert case (see [1]). \( \mathbb{R} \) and \( \mathbb{C} \) denote the fields of the real numbers and of the complex numbers, respectively.

### 2. Main results

We start with two simple lemmas.

**Lemma 2.1.** Let \( H \) be a Krein space, and let \( A \) be a normal operator on \( H \). Denote \( S = \frac{1}{2} (A - A^*) \). Then \( A \) is \( n \)-selfadjoint if and only if \( S^n = 0 \).

For the proof, simply compute, using \( A^* A = AA^* \):

\[
S^n = 2^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} A^k A^{n-k}.
\]

**Lemma 2.2.** Let \( H \) be a Krein space of finite defect \( q \) (Pontryagin space). If a nilpotent operator \( T \) on \( H \) is either selfadjoint or skew-adjoint, i.e., \( T^* = \pm T \), then \( T^{2q+1} = 0 \); moreover, \( T^{2q} = 0 \) if the smallest nonnegative integer \( n \) such that \( T^n = 0 \) is even.
Proof. Since $H$ has defect $q$, we have

$$J = \pm (I - P),$$

(2.1)

where $P$ has rank $q$. Without loss of generality, we assume that the plus sign holds in Eq. (2.1); otherwise replace $J$ by $-J$. Let $m$ denote $n$ if $n$ is even and $n + 1$ if $n$ is odd, where $n$ is the smallest nonnegative integer such that $T^n = 0$. Denote $Y = T^{m/2}$ and let $Y'$ be the $\langle \cdot, \cdot \rangle$-adjoint of $Y$. We have:

$$0 = JT^m = \pm JY'Y = \pm Y'JY = \pm (Y'Y - Y'PY).$$

Since $P$ has rank $q$, the rank of $Y'PY$ is at most $q$. Consequently, $Y'Y$ has rank at most $q$, and since $Y'$ is one-to-one on the range of $Y$, the rank of $Y = T^{m/2}$ is at most $q$. It follows that, as $T$ is nilpotent, $T^{(m/2)+q} = 0$. So $\frac{1}{2}m + q \geq n$. Thus, if $m = n$, then $2q \geq n$, and if $m = n + 1$, then $2q + 1 \geq n$. \qed

We now state one of the main results of this note.

Theorem 2.3. Let $H$ be a Krein space of defect $q \leq \infty$. An operator $A$ on $H$ is normal and $n$-selfadjoint if and only if $A$ has the form $A = T + N$, where the operators $T$ and $N$ are such that $T = T^*$, $N = -N^*$, $TN = NT$, and $N^m = 0$, where $m = \min(2q + 1, n)$.

Proof. If $A$ is normal and $n$-selfadjoint, let $T = \frac{1}{2}(A + A^*)$ and $N = \frac{1}{2}(A - A^*)$, and use Lemmas 2.1 and 2.2 to verify the required properties of $T$ and $N$. Conversely, if $A = T + N$, where $T$ and $N$ have the properties described in the theorem, then clearly $A$ is normal. Now, since $\frac{1}{2}(A - A^*) = N$, by Lemma 2.1 $A$ is $n$-selfadjoint as well. \qed

A particular case of Theorem 2.3 (for 2-selfadjoint operators in complex Krein spaces) is presented in [2], Theorem 4.1(i). We use this opportunity to point out that the equality $NN^* = 0$ is missing in the formulation of Theorem 4.1(i) in [2].

When $q = 0$, Theorem 2.3 says that every normal $n$-selfadjoint operator on a Hilbert space is in fact selfadjoint. This is an immediate consequence of the fact (see [3–5]) that $n$-selfadjoint operators in Hilbert spaces have real spectrum.

Denote by $\mathcal{S}_n$ the class of normal $n$-selfadjoint operators.

Theorem 2.4. Assume the Krein space $H$ has finite defect $q$. Then

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \subseteq \cdots \subseteq \mathcal{S}_{2q+1} = \mathcal{S}_p$$

for every $p \geq 2q + 1$. If, in addition, the (possibly infinite) dimension of $H$ exceeds $2q$, then the classes $\mathcal{S}_1, \ldots, \mathcal{S}_{2q+1}$ are all distinct in the complex case, and $\mathcal{S}_2 \neq \mathcal{S}_3, \mathcal{S}_4 \neq \mathcal{S}_5, \ldots, \mathcal{S}_{2q} \neq \mathcal{S}_{2q+1}$ in the real case.
Proof. The first part of Theorem 2.4 follows from Proposition 1.1 and Theorem 2.3. For the second part, we use induction on \( q \). It will be convenient to consider the complex case first. Start with the basis of induction, i.e., \( q = 1 \). It suffices to assume that \( H = \mathbb{C}^3 \) with the standard Hilbert space structure, and

\[
J = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\]

Let

\[
B_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{bmatrix}, \quad B_2 = \begin{bmatrix}
0 & 0 & i \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\] (2.2)

It turns out that \( B_1 \) and \( B_2 \) are normal, and \( B_1 \) is 3-selfadjoint but not 2-selfadjoint, whereas \( B_2 \) is 2-selfadjoint but not selfadjoint. We pass now to the general \( q \), still assuming the complex scalars. Let \( \dim H \geq 2q + 1 \). Using induction on \( q \) we may assume that we have proved already that the classes \( \mathcal{S}_1, \ldots, \mathcal{S}_{2q-1} \) are all distinct. It remains therefore to demonstrate that the classes \( \mathcal{S}_{2q-1}, \mathcal{S}_{2q}, \mathcal{S}_{2q+1} \) are distinct. To this end, we may restrict ourselves to the case when \( H = \mathbb{C}^{2q+1} \), with the standard Hilbert space structure on \( \mathbb{C}^{2q+1} \), and \( J = [\alpha_{jk}]_{2q+1}^{2q+1} \) is defined by \( \alpha_{jk} = 1 \) if \( j + k = 2q + 2 \) and \( \alpha_{jk} = 0 \) otherwise. Define the \((2q+1) \times (2q+1)\) matrix \( N_{2q+1} \) by the property that the \((j,k)\) entry of \( N_{2q+1} \) is equal to \( i = \sqrt{-1} \) if \( k - j = 1 \), and zero otherwise. Then \( N_{2q+1} = -N_{2q+1}^* \), and since \( N_{2q+1}^{2q+1} = 0 \neq N_{2q+1}^{2q+1} \), Theorem 2.3 shows that \( N_{2q+1} \in \mathcal{S}_{2q+1} \setminus \mathcal{S}_{2q} \). To show that \( \mathcal{S}_{2q} \neq \mathcal{S}_{2q+1} \), we consider the Krein space structure defined on \( \mathbb{C}^{2q+1} \) by \( J = [\alpha_{jk}]_{2q+1}^{2q+1} \), where \( \alpha_{jk} = 1 \) if \( j + k = 2q + 1 \) and \( \alpha_{jk} = 0 \) otherwise, then the matrix \( N_{2q+1} \) belongs to \( \mathcal{S}_{2q} \setminus \mathcal{S}_{2q+1} \).

Consider now the real case, and assume first \( q = 1 \). The matrix \( B_1 \) of Eq. (2.2) shows that \( \mathcal{S}_3 \neq \mathcal{S}_2 \) if \( \dim H \geq 3 \). Using the induction on \( q \), it remains to prove that \( \mathcal{S}_{2q} \neq \mathcal{S}_{2q+1} \) if \( \dim H \geq 2q + 1 \). It suffices to prove this for the case when \( H = \mathbb{R}^{2q+1} \) and \( J = [\alpha_{jk}]_{2q+1}^{2q+1} \), where \( \alpha_{jk} = 1 \) if \( j + k = 2q + 2 \) and \( \alpha_{jk} = 0 \) otherwise. Define a \((2q+1) \times (2q+1)\) matrix \( B \) by

\[
B = \begin{bmatrix}
0_{q \times q} & I_{q \times q} & 0_{q \times q} \\
0_{q \times q} & 0_{q \times q} & -I_{q \times q} \\
0_{1 \times q} & 0_{1 \times q} & 0_{1 \times q}
\end{bmatrix},
\]

where \( 0_{r \times k} \) and \( I_{r \times k} \) stand for the \( j \times k \) zero and identity matrix, respectively. One verifies that \( B^* = -B \), and since \( B^{2q} \neq 0 \sim B^{2q+1} \), it follows that \( B \in \mathcal{S}_{2q+1} \setminus \mathcal{S}_{2q} \). \( \square \)
The fact that $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ are distinct if $q = 1$ and the dimension of $H$ is at least three (in the complex case), was observed in Theorem 4.1 of [2].

The question whether $\mathcal{S}_{2r-1} = \mathcal{S}_{2r}$ for real Krein spaces is more involved.

**Theorem 2.5.** Let $H$ be a real Krein space with defect $q$. Then, for a positive integer $r$, the equality $\mathcal{S}_{2r-1} = \mathcal{S}_{2r}$ holds if and only if $q < 2r$.

For the proof of Theorem 2.5, we need a canonical form for nilpotent skew-adjoint operators in finite dimensional real Krein spaces. We denote by $K_m(0)$ the (upper triangular) $m \times m$ nilpotent Jordan block. Denote also

$$F_j = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & (-1)^{j-2} & \ldots & 0 & 0 \\ (-1)^{j-1} & 0 & \ldots & 0 & 0 \end{bmatrix},$$

so $F_j$ is a $j \times j$ matrix which is symmetric if $j$ is odd and skew-symmetric if $j$ is even. $X^T$ will denote the transpose of a matrix $X$.

**Lemma 2.6.** Let $J$ be an invertible real symmetric $n \times n$ matrix, and let $N$ be a nilpotent real matrix such that $N^* = -N$, where the adjoint is taken with respect to the Krein space structure in $\mathbb{R}^n$ determined by $J$. Then there exists a real invertible matrix $S$ such that $S^{-1}NS$ and $S^TJS$ have the following forms:

$$S^{-1}NS = \bigoplus_{j=1}^{p} K_{2n_j+1}(0) \oplus \bigoplus_{j=1}^{k} (K_{n_j+1}(0) \oplus -(K_{n_j+1}(0))^T), \quad (2.3)$$

$$S^TJS = \bigoplus_{j=1}^{p} \kappa_j F_{2n_j+1} \oplus \bigoplus_{j=1}^{k} \left[ \begin{array}{cc} 0 & I_{n_j} \\ I_{n_j} & 0 \end{array} \right]. \quad (2.4)$$

In Eqs. (2.3) and (2.4), $n_1, \ldots, n_p$ are nonnegative integers, $n_{p+1}, \ldots, n_{p+s}$ are even integers, and $\kappa_1, \ldots, \kappa_p$ are signs $\pm 1$.

Of course, the cases when $p = 0$ (i.e., the blocks with $K_{2n_j+1}(0)$ and $F_{2n_j+1}$ are absent) or $s = 0$ are not excluded in Eqs. (2.3) and (2.4).

Lemma 2.6 follows immediately from the canonical forms of pairs of real symmetric or skew-symmetric matrices (see [6]); in the form presented here, Lemma 2.6 can be found in [7], for example.

**Proof of Theorem 2.5.** In the proof we will use repeatedly the following fact: If $H$ is a real (or complex) Krein space with finite defect $q$, then no $J$-neutral subspace is of dimension greater than $q$. (Recall that a subspace $\mathcal{M}$ of $H$ is
called $J$-neutral if $[x, y] = 0$ for every $x, y \in \mathcal{H}$.) For the proof see, for example, Lemma 1.2 of [5], or Theorem 1.5 of [3]; the latter reference concerns finite dimensional spaces only.

We consider first the case when $H$ is finite dimensional. Assume first $q < 2r$. By Lemma 2.6, a skew-adjoint nilpotent operator $N$ on $H$ cannot have Jordan blocks of size $2r \times 2r$, since this would violate the condition $q < 2r$. Therefore, the every such $N$, if $N^{2r} = 0$, then also $N^{2r-1} = 0$. By Theorem 2.3, we have $\mathcal{S}_{2r} \subseteq \mathcal{S}_{2r-1}$, and the equality of these sets follows. Conversely, if $q \geq 2r$, then by Lemma 2.6 there exists a skew-adjoint nilpotent operator $N$ on $H$ such that $N^{2r} = 0 \neq N^{2r-1}$. By Theorem 2.3, $N \in \mathcal{S}_{2r} \setminus \mathcal{S}_{2r-1}$.

Now let $H$ be infinite dimensional. If $q \geq 2r$, then there exists a $4r$-dimensional subspace $H_0$ of $H$ on which $J$ is given by

\[
J = \begin{bmatrix}
0 & I_{2r} \\
I_{2r} & 0
\end{bmatrix},
\]

with respect to some basis in $H_0$. Then the operator $N$ given by $K_{2r}(0) \oplus - (K_{2r}(0))^\top$ on $H_0$ and by zero on the $J$-orthogonal complement of $H_0$ has the properties that $N$ is skew-adjoint and $N^{2r} = 0 \neq N^{2r-1}$. Thus, $\mathcal{S}_{2r} \neq \mathcal{S}_{2r-1}$.

It remains to consider the case when $H$ is infinite dimensional and $q < 2r$. In view of Theorem 2.3, we have to show that if $N^* = -N$ and $N^m = 0$, where $m = \min(2q + 1, 2r)$, then also $N^{\min(2q + 1, 2r - 1)} = 0$. If $2q + 1 < 2r$ this is obvious. So let $2q + 1 \geq 2r$. Arguing by contradiction, assume there exists an operator $N$ on $H$ with the properties $N^{2r} = 0 \neq N^{2r-1}$ and $N^* = -N$. Select $x \in H$ such that $N^{2r-1}x \neq 0$. Then clearly the vectors $x, Nx, \ldots, N^{2r-1}x$ are linearly independent. Choose $y \in H$ such that $[y, N^{2r-1}x] = 1$. It is easy to see that $y$ exists because $N^{2r-1}x \neq 0$, say $y = (1/(N^{2r-1}x, N^{2r-1}x)) J N^{2r-1}x$. Since $N$ is skew-adjoint, we have $[N^{2r-1}x, y] = -[y, N^{2r-1}x] = -1$, in particular, $N^{2r-1}y \neq 0$. Next, we show that the vectors

\[
x, Nx, \ldots, N^{2r-1}x, y, Ny, \ldots, N^{2r-1}y
\]

are linearly independent. Indeed, suppose that some $N^p y, (p = 0, \ldots, 2r - 1)$ is a linear combination of $x, Nx, \ldots, N^{2r-1}x, N^{p+1}y, \ldots, N^{2r-1}y$. Applying $N^{2r-1-p}$ to this linear combination, we see that $N^{2r-1}y$ is a linear combination of $x, Nx, \ldots, N^{2r-1}x$:

\[
N^{2r-1}y = \sum_{j=0}^{2r-1} a_j N^j x.
\]

Applying $N$ to both sides of Eq. (2.6) we obtain a contradiction with the linear independence of $x, Nx, \ldots, N^{2r-1}x$, unless $a_{2r-1}$ is the only nonzero coefficient in Eq. (2.6). But then, using the skew-adjointness of $N$. 
\[ -1 = [N^{2r-1}y, x] = [x_{2r-1}N^{2r-1}x, x] = x_{2r-1}[N^{2r-1}x, x] = -x_{2r-1}[x, N^{2r-1}x], \] (2.7)

a contradiction with the symmetry of the bilinear form \([\cdot, \cdot]\). Thus, Eq. (2.5) are linear independent, and the 4r-dimensional subspace \(H_1\) spanned by the vectors Eq. (2.5) is obviously \(N\)-invariant. The subspace \(H_1\) also turns out to be \(J\)-regular, i.e., zero is the only vector in \(H_1\) which is \(J\)-orthogonal to every vector in \(H_1\). Indeed, let \(w = \sum_{j=0}^{3r-1} \alpha_j N^j x + \sum_{k=1}^{2r-1} \beta_k N^k y\), where at least one of the scalars \(\alpha_u, \beta_r\) is nonzero, be such that \([w, z] = 0\) for every \(z \in H_1\). Using this equality with \(z = N^s x\) and \(z = N^s y\) for an appropriate \(s\), we obtain that at least one of the following three sets of conditions is valid, in correspondence with the possible relations \(u = v, u < v\) or \(u > v\):

\[ x_u N^{2r-1} x + \beta_r N^{2r-1} y \perp x, y; \] (2.8)

\[ x_u N^{2r-1} x \perp x, y; \quad x_u \neq 0; \] (2.9)

\[ \beta_r N^{2r-1} y \perp x, y; \quad \beta_r \neq 0. \] (2.10)

(The orthogonality here is understood with respect to \([\cdot, \cdot]\)). However, each one of Eqs. (2.8)–(2.10) results in a contradiction, by using the equalities \([N^{2r-1}x, x] = [N^{2r-1}y, y] = 0\) (which in turn are consequence of the skew-adjointness of \(N\)). Once we have shown that \(H_1\) is \(N\)-invariant and \(J\)-regular, Lemma 2.6 is applicable to the restrictions of \(N\) and \(J\) to \(H_1\). Since the restriction of \(N\) to \(H_1\) has a Jordan block of size \(2r \times 2r\), Lemma 2.6 shows that there exists a \(2r\)-dimensional \(J\)-neutral subspace of \(H_1\). This contradicts the inequality \(q < 2r\).

\[ \square \]

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