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Inclusion hyperspaces and capacities on Tychonoff spaces: Functors and monads ☆

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ABSTRACT

The inclusion hyperspace functor, the capacity functor and monads for these functors have been extended from the category of compact Hausdorff spaces to the category of Tychonoff spaces. Properties of spaces and maps of inclusion hyperspaces and capacities (non-additive measures) on Tychonoff spaces are investigated.

0. Introduction

The category of compact Hausdorff topological spaces is probably the most convenient topological category for a categorical topologist. A situation is usual when some results are first obtained for compacta and then extended with much effort to a wider class of spaces and maps, see e.g. factorization theorems for inverse limits [13]. Many classical construction on topological spaces lead to covariant functors in the category of compacta, and categorical methods proved to be efficient tools to study hyperspaces, spaces of measures, symmetric products etc. [17]. We can mention the hyperspace functor exp [15], the inclusion hyperspace functor G [8], the probability measure functor P [5], and the capacity functor M which was recently introduced by Zarichnyi and Nykyforchyn [18] to study non-additive regular measures on compacta.

Functors exp, P, G, M have rather good properties. The functors exp and P belong to a defined by Ščepin class of normal functors, while G and M satisfy all requirements of normality but preservation of preimages, hence are only weakly normal. They are functorial parts of monads [15,18].

Unfortunately the functors exp and G lose most of their nice properties when they are extended from the category of compacta to the category of Tychonoff spaces. Moreover, a meaningful extension usually is not unique. An interested reader is referred, e.g. to [1], where *four* extensions to the category of Tychonoff spaces of the probability measure functor P are discussed, and two of them are investigated in detail.

The aim of this paper is extend the inclusion hyperspace functor, the capacity functor and monads for these functors from the category of compacta to the category of Tychonoff spaces, and to study properties of these extensions. We will use "fine tuning" of standard definitions of hyperspaces and inclusion hyperspaces to "save" as much topological and categorical properties valid for the compact case as possible.

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1. Preliminaries

In the sequel a *compactum* is a compact Hausdorff topological space. The *unit segment* $I = [0; 1]$ is considered as a subspace of the real line $\mathbb{R}$ with the natural topology. We say that a function $\varphi : X \to I$ *separates* subsets $A, B \subset X$ if $\varphi|_A \equiv 1$, $\varphi|_B \equiv 0$. If such φ exists for A and B and is continuous, then we call these sets *completely separated*. We write $A \underset{\text{op}}{\subset} X$ or $A \underset{\text{cl}}{\subset} X$ if A is respectively an open or a closed subset of a space X. The set of all continuous functions from a space X to a space Y is denoted by $C(X, Y)$.

See [7] for definitions of category, functor, natural transformation, monad (triple), morphism of monads. For a category $\mathcal{C}$ we denote the class of its objects by $\mathrm{Ob}\,\mathcal{C}$. The category of Tychonoff spaces $\mathcal{T}\mathrm{ych}$ consists of all Tychonoff (= completely regular) spaces and continuous maps between them. The *category of compacta* $\mathcal{C}\mathrm{omp}$ is a full subcategory of $\mathcal{T}\mathrm{ych}$ and contains all compacta and their continuous maps. We say that a functor F_1 in $\mathcal{T}\mathrm{ych}$ or in $\mathcal{C}\mathrm{omp}$ is a *subfunctor* of a functor F_2 in the same category if there is a natural transformation $F_1 \to F_2$ with all components being embeddings. Similarly a monad $\mathbb{F}_1$ is a *submonad* of a monad $\mathbb{F}_2$ if there is a morphism of monads $\mathbb{F}_1 \to \mathbb{F}_2$ such that all its components are embeddings.

From now on we denote the set of all non-empty closed subsets of a topological space X by $\exp X$, though sometimes this notation is used for the set of all *compact* non-empty subsets, and the two meaning can even coexist in one text [6]. A lot of topologies on $\exp X$ can be found in literature. The *upper topology* τ_u is generated by the base which consist of all sets $\{F \in \exp X \mid F \subset U\}$, where U is open in X. The *lower topology* τ_l has the subbase $\{\{F \in \exp X \mid F \cap X \neq \varnothing\} \mid U \underset{\text{op}}{\subset} X\}$. The *Vietoris topology* τ_v is the least topology that contains both the upper and the lower topologies. It is *de facto* the default topology on $\exp X$, to the great extent due to an important fact that, for a compact Hausdorff space X, the space $\exp X$ with the Vietoris topology is compact and Hausdorff. It $f : X \to Y$ is a continuous map of compacta, then the map $\exp f : \exp X \to \exp Y$, which sends each non-empty closed subset F of X to its image $f(Y)$, is continuous. Thus we obtain the *hyperspace functor* $\exp : \mathcal{C}\mathrm{omp} \to \mathcal{C}\mathrm{omp}$.

A non-empty closed with respect to the Vietoris topology subset $\mathcal{F} \subset \exp X$ is called an *inclusion hyperspace* if $A \subset B \in \exp X$, $A \in \mathcal{F}$ imply $B \in \mathcal{F}$. The set GX of all inclusion hyperspaces on the space X is closed in $\exp^2 X$, hence is a compactum with the induced topology if X is a compactum. This topology can also be determined by a subbase which consists of all sets of the form

$$U^+ = \{\mathcal{F} \in GX \mid \text{there is } F \in \mathcal{F}, F \subset U\},$$

$$U^- = \{\mathcal{F} \in GX \mid F \cap U \neq \varnothing \text{ for all } F \in \mathcal{F}\},$$

with U open in X. If the map $Gf : GX \to GY$ for a continuous map $f : X \to Y$ of compacta is defined as $Gf(G) = \{B \underset{\text{cl}}{\subset} Y \mid B \supset f(A) \text{ for some } A \in \mathcal{F}\}$, $\mathcal{F} \in GX$, then G is the *inclusion hyperspace functor* in $\mathcal{C}\mathrm{omp}$.

We follow a terminology of [18] and call a function $c : \exp X \cup \{\varnothing\} \to I$ a *capacity* on a compactum X if the three following properties hold for all closed subsets F, G of X:

(1) $c(\varnothing) = 0$, $c(X) = 1$;
(2) if $F \subset G$, then $c(F) \leqslant c(G)$ (monotonicity);
(3) if $c(F) < a$, then there exists an open set $U \supset F$ such that for any $G \subset U$ we have $c(G) < a$ (upper semicontinuity).

The set of all capacities on a compactum X is denoted by MX. It was shown in [18] that a compact Hausdorff topology is determined on MX with a subbase which consists of all sets of the form

$$O_-(F, a) = \bigl\{c \in MX \bigm| c(F) < a\bigr\},$$

where $F \underset{\text{cl}}{\subset} X$, $a \in \mathbb{R}$, and

$$O_+(U, a) = \bigl\{c \in MX \bigm| c(U) > a\bigr\} = \bigl\{c \in MX \bigm| \text{there exists a compactum } F \subset U,\ c(F) > a\bigr\},$$

where $U \underset{\text{op}}{\subset} X$, $a \in \mathbb{R}$. The same topology can be defined as weak* topology, i.e. the weakest topology on MX such that for each continuous function $\varphi : X \to [0; +\infty)$ the correspondence which sends each $c \in MX$ to the *Choquet integral* [3] of φ w.r.t. c

$$\int_X \varphi(x)\, dc(x) = \int_0^{+\infty} c\bigl(\bigl\{x \in X \bigm| \varphi(x) \geqslant a\bigr\}\bigr)\, da$$

is continuous. If $f : X \to Y$ is a continuous map of compacta, then the map $Mf : MX \to MY$ is defined as follows: $Mf(c)(F) = c(f^{-1}(F))$, for $c \in MX$ and $F \underset{\text{cl}}{\subset} Y$. This map is continuous, and we obtain the *capacity functor* M in the category of compacta.

A monad $\mathbb{F}$ in a category $\mathcal{C}$ is a triple (F, η_F, μ_F), with $F : \mathcal{C} \to \mathcal{C}$ a functor, $\eta_F : \mathbf{1}_{\mathcal{C}} \to F$ and $\mu_F : F^2 \to F$ natural transformations, such that $\mu_F X \circ \eta_F FX = \mu_F X \circ F\eta_F X = \mathbf{1}_{FX}$, $\mu_F X \circ F\mu_F X = \mu_F X \circ \mu_F FX$ for all objects X of $\mathcal{C}$. Then F, η_F, μ_F are called respectively the *functorial part*, the *unit* and the *multiplication* of $\mathbf{F}$. For the *inclusion hyperspace monad* $\mathbb{G} = (G, \eta_G, \mu_G)$ the components of the unit and the multiplication are defined by the formulae [11]:

$$\eta_G X(x) = \{F \in \exp X \mid F \ni x\}, \quad x \in X,$$

and

$$\mu_G X(\mathbf{F}) = \Big\{F \in \exp X \;\Big|\; F \in \bigcap \mathrm{H} \text{ for some } \mathrm{H} \in \mathbf{F}\Big\}, \quad \mathbf{F} \in G^2 X.$$

In the *capacity monad* $\mathbb{M} = (M, \eta_M, \mu_M)$ [18] the components of the unit and the multiplication are defined as follows:

$$\eta_M(x)(F) = \begin{cases} 1, & x \in F, \\ 0, & x \notin F, \end{cases} \quad x \in X,\ F \underset{\mathrm{cl}}{\subset} X,$$

and

$$\mu_M X(\mathcal{C})(F) = \sup\big\{\alpha \in I \;\big|\; \mathcal{C}\big(\{c \in MX \mid c(F) \geqslant \alpha\}\big) \geqslant \alpha\big\}, \quad \mathcal{C} \in M^2,\ F \underset{\mathrm{cl}}{\subset} X.$$

An internal relation between the inclusion hyperspace monad and the capacity monad is presented in [18,9].

It is well known that the correspondence which sends each Tychonoff space X to its Stone–Čech compactification βX is naturally extended to a functor $\beta : \mathcal{T}\mathrm{ych} \to \mathcal{C}\mathrm{omp}$. For a continuous map $f : X \to Y$ of Tychonoff spaces the map $\beta f : \beta X \to \beta Y$ is the unique continuous extension of f. In fact this functor is *left adjoint* [7] to the inclusion functor U which embeds $\mathcal{C}\mathrm{omp}$ into $\mathcal{T}\mathrm{ych}$. The collection $i = (iX)_{X \in \mathrm{Ob}\,\mathcal{T}\mathrm{ych}}$ of natural embeddings of all Tychonoff spaces into their Stone–Čech compactifications is a unique natural transformation $\mathbf{1}_{\mathcal{T}\mathrm{ych}} \to U\beta$ (a *unit of the adjunction*, cf. [7]).

In this paper "monotonic" always means "isotone".

2. Inclusion hyperspace functor and monad in the category of Tychonoff spaces

First we modify the Vietoris topology on the set $\exp X$ for a Tychonoff space X. Distinct closed sets in X have distinct closures in βX, but the map $e_{\exp} X$ which sends each $F \in \exp X$ to $\mathrm{Cl}_{\beta X} F \in \exp \beta X$ generally is not an embedding when the Vietoris topology are considered on the both spaces, although is continuous. It is easy to prove:

Lemma 2.1. *Let X be a Tychonoff space. Then the unique topology on $\exp X$, such that $e_{\exp} X$ is an embedding into $\exp \beta X$ with the Vietoris topology, is determined by a base which consists of all sets of the form*

$$\langle U_1, \ldots, U_k\rangle = \big\{F \in \exp X \;\big|\; F \text{ is completely separated from } X \setminus (U_1 \cup \cdots \cup U_k),\ F \cap U_i \neq \varnothing,\ i = 1, \ldots, k\big\},$$

with all U_i open in X.

Observe that our use of the notation $\langle \cdots \rangle$ differs from its traditional meaning [15], but agrees with it if X is a compactum. Hence this topology coincides with the Vietoris topology for each compact Hausdorff space X, but may be weaker for noncompact spaces. The topology is not changed when we take a less base which consists only of $\langle U_1, \ldots, U_k\rangle$ for $U_i \underset{\mathrm{op}}{\subset} X$ such that $U_2 \cup \cdots \cup U_k$ is completely separated from $X \setminus U_1$. We can also equivalently determine our topology with a subbase which consists of the sets

$$\langle U\rangle = \{F \in \exp X \mid F \text{ is completely separated from } X \setminus U\}$$

and

$$\langle X, U\rangle = \{F \in \exp X \mid F \cap U \neq \varnothing\}$$

with U running over all open subsets of X.

Observe that the sets of the second type form a subbase of the lower topology τ_l on $\exp X$, while a subbase which consists of the sets of the first form determines a topology that is equal or weaker than the upper topology τ_u on $\exp X$. We call it an *upper separation topology* (not only for Tychonoff spaces) and denote by τ_{us}. Thus the topology introduced in the latter lemma is a lowest upper bound of τ_l and τ_{us}. From now on we *always* consider $\exp X$ with this topology, if otherwise is not specified. We also denote by $\exp_l X$, $\exp_u X$ and $\exp_{us} X$ the set $\exp X$ with the respective topologies.

If $f : X \to Y$ is a continuous map of Tychonoff spaces, then we define the map $\exp f : \exp X \to \exp Y$ by the formula $\exp f(F) = \mathrm{Cl}\, f(F)$. The equality $e_{\exp} Y \circ \exp f = \exp \beta f \circ e_{\exp} X$ implies that $\exp f$ is continuous, and we obtain an extension of the functor exp in $\mathcal{C}\mathrm{omp}$ to $\mathcal{T}\mathrm{ych}$. Unfortunately, the extended functor exp does not preserve embeddings.

Now we consider how to define "valid" inclusion hyperspaces in Tychonoff spaces.

Lemma 2.2. *Let a family $\mathcal{F}$ of non-empty closed sets of a Tychonoff space X is such that $A \subset B \underset{\mathrm{cl}}{\subset} X$, $A \in \mathcal{F}$ imply $B \in \mathcal{F}$. Then the following properties are equivalent*:

(a) *$\mathcal{F}$ is a compact set in $\exp_l X$;*
(b) *for each monotonically decreasing net (F_α) of elements of $\mathcal{F}$ the intersection $\bigcap_\alpha F_\alpha$ also is in $\mathcal{F}$.*

Each such $\mathcal{F}$ is closed in $\exp_{us} X$, hence in $\exp X$. If X is compact, then these conditions are also equivalent to:

(c) *$\mathcal{F}$ is an inclusion hyperspace.*

Proof. Assume (a), and let (F_α) be a monotonically decreasing net of elements of $\mathcal{F}$. If $\bigcap_\alpha F_\alpha \notin \mathcal{F}$, then the collection $\{\langle X, X \setminus F_\alpha \rangle\}$ is an open cover of $\mathcal{F}$ that does not contain a finite subcover, which contradicts the compactness of $\mathcal{F}$ in the lower topology. Thus (a) implies (b).

Let (b) hold, and we have a cover of $\mathcal{F}$ by subbase elements $\langle X, U_\alpha \rangle$, $\alpha \in \mathcal{A}$. If there is no finite subcover, then $\mathcal{F}$ contains all sets of the form $X \setminus (U_{\alpha_1}) \cup \cdots \cup U_{\alpha_k}$, $\alpha_1, \ldots, \alpha_k \in \mathcal{A}$. These sets form a filtered family, which may be considered as a monotonically decreasing net of elements of $\mathcal{F}$. Hence, by the assumption, $\mathcal{F}$ contains their non-empty intersection $B = X \setminus \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ that does not intersect any of U_α. This contradiction shows that each open cover of $\mathcal{F}$ by subbase elements contains a finite subcover, and by Alexander Lemma $\mathcal{F}$ is compact, i.e. (a) is valid.

Let $\mathcal{F}$ satisfy (b), and let C be a point of closure of $\mathcal{F}$ in $\exp_{us} X$. Then for each neighborhood $U \supset C$ there is $F \in \mathcal{F}$ such that F is completely separated from $X \setminus U$, therefore $\mathrm{Cl}\, U \in \mathcal{F}$. The set $\mathcal{U}$ of all closures $\mathrm{Cl}\, U$, with U a neighborhood of C, is filtered. Therefore $\bigcap \mathcal{U} = C \in \mathcal{F}$, hence $\mathcal{F}$ is closed in $\exp_{us} X$. If X is a compactum, then $\mathcal{F}$ satisfies the definition of inclusion hyperspace, i.e. (c) is true.

It is also obvious that an inclusion hyperspace on a compactum satisfies (b). □

Therefore we call a collection $\mathcal{F}$ of non-empty closed sets of a Tychonoff space X a *compact inclusion hyperspace* in X if $A \subset B \underset{\mathrm{cl}}{\subset} X$, $A \in \mathcal{F}$ imply $B \in \mathcal{F}$, and $\mathcal{F}$ is compact in the lower topology on $\exp X$. Note that the lower topology is non-Hausdorff for non-degenerate X. The set of all compact inclusion hyperspaces in X will be denoted by $\check{G}X$.

Let G^*X be the set of all inclusion hyperspaces $\mathcal{G}$ in βX with the property: if $A, B \underset{\mathrm{cl}}{\subset} \beta X$, $A \cap X = B \cap X$, then $A \in \mathcal{G} \iff B \in \mathcal{G}$. Observe that each such $\mathcal{G}$ does not contain subsets of $\beta X \setminus X$.

The latter lemma implies:

Proposition 2.3. *A collection $\mathcal{F} \subset \exp X$ is a compact inclusion hyperspace if and only if it is equal to $\{G \cap X \mid G \in \mathcal{G}\}$ for a unique $\mathcal{G} \in G^*X$.*

We denote the map $\check{G}X \to G\beta X$ which sends each $\mathcal{F} \in \check{G}X$ to the respective $\mathcal{G}$ by $e_G X$. It is easy to see that $e_G X(\mathcal{F})$ is equal to $\{G \in \exp \beta X \mid G \cap X \in \mathcal{F}\}$.

We define a Tychonoff topology on $\check{G}X$ by the requirement that $e_G X$ is an embedding into $G\beta X$. An obvious inclusion $G\beta f(G^*X) \subset G^*Y$ for a continuous map $f : X \to Y$ allows to define a continuous map $\check{G}f : \check{G}X \to \check{G}Y$ as a restriction of the map $G\beta f$, i.e. by the equality $G\beta f \circ e_G X = e_G Y \circ \check{G}f$. Of course, $\check{G}f(\mathcal{F}) = \{G \underset{\mathrm{cl}}{\subset} Y \mid G \supset f(F) \text{ for some } F \in \mathcal{F}\}$ for $\mathcal{F} \in \check{G}X$. A functor $\check{G}$ in the category of Tychonoff spaces is obtained. Its definition implies that $e_G = (e_G X)_{X \in \mathrm{Ob}\, \mathcal{T}\mathrm{ych}}$ is a natural transformation $\check{G} \to UG\beta$, with all components being embeddings, therefore $\check{G}$ is a subfunctor of $UG\beta$. Note also that $e_G X = \check{G}iX$ for all Tychonoff spaces X.

Due to the form of the standard subbase of $G\beta X$, we obtain:

Proposition 2.4. *The topology on $\check{G}X$ can be determined by a subbase which consists of all sets of the form*

$$U^+ = \{\mathcal{F} \in \check{G}X \mid \text{there is } F \in \mathcal{F},\ F \text{ is completely separated from } X \setminus U\},$$
$$U^- = \{\mathcal{F} \in \check{G}X \mid F \cap U \neq \varnothing \text{ for all } F \in \mathcal{F}\},$$

with U open in X.

Observe that this interpretation of U^+, U^- for Tychonoff spaces agrees with the standard one for compact Hausdorff spaces.

As it was said before, the functor $\exp : \mathcal{T}\mathrm{ych} \to \mathcal{T}\mathrm{ych}$ does not preserve embeddings, thus we cannot regard $\exp\exp X$ as a subspace of $\exp\exp \beta X$, although $\exp X$ is a subspace of $\exp \beta X$. We can only say that image under exp of the embedding $\exp X \to \exp \beta X$ is continuous. Therefore a straightforward attempt to embed $\check{G}X$ into $\exp^2 X$ fails, while $\check{G}X$ is embedded into $\exp^2 \beta X$.

Now we will show that the topology on $\check{G}X$ is the weak topology with respect to a collection of maps into the unit interval.

Lemma 2.5. *Let a map $\varphi : X \to I$ be continuous. Then the map $\psi : \exp X \to I$ which sends each non-empty closed subset $F \subset X$ to $\sup_{x\in F} \varphi(x)$ (or $\inf_{x\in F} \varphi(x)$) is continuous.*

Proof. We prove for sup, the other case is analogous. Let $\sup_{x\in F} \varphi(x) = \beta < \alpha$, $\alpha, \beta \in I$. The set $U = \varphi^{-1}([0; \frac{\alpha+\beta}{2}))$ is open, and F is completely separated from $X \setminus U$, hence $F \in \langle U\rangle$. If $G \in \exp X$, $G \in \langle U\rangle$, then $\sup_{x\in F} \varphi(x) \leqslant \frac{\alpha+\beta}{2} < \alpha$ as well, and the preimage of the set $[0; \alpha)$ under the map ψ is open.

Now let $\sup_{x\in F} \varphi(x) = \beta > \alpha$, $\alpha, \beta \in I$. There exists a point $x \in F$ such that $\varphi(x) > \frac{\alpha+\beta}{2}$, hence F intersects the open set $U = \varphi^{-1}((\frac{\alpha+\beta}{2}; 1])$. Then $\langle X, U\rangle \ni F$, and $G \in \exp X$, $G \in \langle X, U\rangle$ implies $\sup_{x\in G} \varphi(x) \geqslant \frac{\alpha+\beta}{2} > \alpha$. Therefore the preimage $\psi^{-1}(\alpha; 1]$ is open as well, which implies the continuity of ψ. □

Lemma 2.6. *Let a function $\psi : \exp X \to I$ be continuous and monotonic. Then φ attains its minimal value on each compact inclusion hyperspace $\mathcal{F} \in \check{G}X$.*

Proof. If ψ is continuous and monotonic, then it is lower semicontinuous with respect to the lower topology. Then the image of the compact set $\mathcal{F}$ under ψ is compact in the topology $\{I \cap (a, +\infty) \mid a \in \mathbb{R}\}$ on I, therefore $\psi(\mathcal{F})$ contains a least element. □

Proposition 2.7. *The topology on $\check{G}X$ is the weakest among topologies such that for each continuous function $\varphi : X \to I$ the map m_φ which sends each $\mathcal{F} \in \check{G}X$ to $\min\{\sup_F \varphi \mid F \in \mathcal{F}\}$ is continuous. If $\psi : \exp X \to I$ is a continuous monotonic map, then the map which sends each $\mathcal{F} \in \check{G}X$ to $\min\{\psi(F) \mid F \in \mathcal{F}\}$ is continuous w.r.t. this topology.*

Proof. Let $\psi : \exp X \to I$ be a continuous monotonic map, and $\min\{\psi(F) \mid F \in \mathcal{F}\} < \alpha$, then there is $F \in \mathcal{F}$ such that $\psi(F) < \alpha$. Due to continuity there is a neighborhood $\langle U_1, \dots, U_k\rangle \ni F$ such that $\psi(G) < \alpha$ for all $G \in \langle U_1, \dots, U_k\rangle$. For φ is monotonic, the inequality $\psi(G) < \alpha$ is valid for all $G \in \langle U_1 \cup \dots \cup U_k\rangle$. Therefore $\min\{\psi(G) \mid G \in \mathcal{G}\} < \alpha$ for all $\mathcal{G} \in (U_1 \cup \dots \cup U_k)^+$, and the latter open set contains $\mathcal{F}$.

If $\min\{\psi(F) \mid F \in \mathcal{F}\} > \alpha$, then $\psi(F) > \alpha$ for all $F \in \mathcal{F}$. The function ψ is continuous, hence each $F \in \mathcal{F}$ is in a basic neighborhood $\langle U_0, U_1, \dots, U_k\rangle$ in $\exp X$ such that for all G in this neighborhood the inequality $\psi(G) > \alpha$ holds. We can assume that $U_1 \cup U_2 \cup \dots \cup U_k$ is completely separated from $X \setminus U_0$, then $\psi(G) > \alpha$ also for all $G \in \langle X, U_1, U_2, \dots, U_k\rangle$. The latter set is an open neighborhood of F in the lower topology. The set $\mathcal{F}$ is compact in $\exp_l X$, therefore we can choose a finite subcover $\langle U_1^1, \dots, U_{k_1}^1, \dots, \langle U_1^n, \dots, U_{k_n}^n$ of $\mathcal{F}$ such that $G \in \langle U_1^l, \dots, U_{k_l}^l\rangle$, $1 \leqslant l \leqslant n$, implies $\psi(G) > \alpha$. Then $\mathcal{F}$ is in an open neighborhood

$$\mathcal{U} = \bigcap \bigl\{ \bigl(U_{j_1}^1 \cup U_{j_2}^2 \cup \dots \cup U_{j_n}^n\bigr)^- \bigm| 1 \leqslant j_1 \leqslant k_1,\ 2 \leqslant j_2 \leqslant k_2,\ \dots,\ n \leqslant j_n \leqslant k_n \bigr\}.$$

Each element G of any compact inclusion hyperspace $\mathcal{G} \in \mathcal{U}$ intersects all $U_1^l, \dots, U_{k_l}^l$ for at least one $l \in \{1, \dots, n\}$, therefore $\min\{\psi(G) \mid G \in \mathcal{G}\} > \alpha$ for all $\mathcal{G} \in \mathcal{U}$. Thus $\min\{\psi(F) \mid F \in \mathcal{F}\}$ is continuous w.r.t. $\mathcal{F} \in \check{G}X$.

Due to Lemma 2.5 it implies that the map $m : \check{G}X \to I^{C(X,I)}$, $m(\mathcal{F}) = (m_\varphi(\mathcal{F}))_{\varphi \in C(X,I)}$ for $\mathcal{F} \in \check{G}X$, is continuous.

Now let $\mathcal{F} \in U^+$ for $U \underset{\text{op}}{\subset} X$, i.e. there is $F \in \mathcal{F}$ and a continuous function $\varphi : X \to I$ such that $\varphi|_F \equiv 0$, $\varphi|_{X\setminus U} = 1$. Then $m_\varphi(\mathcal{F}) < 1/2$, and for any $\mathcal{G} \in \check{G}X$ the inequality $m_\varphi(\mathcal{G}) < 1/2$ implies $\mathcal{G} \in U^+$.

If $\mathcal{F} \in U^-$, $U \underset{\text{op}}{\subset} X$, then due to the compactness of $\mathcal{F}$ we can choose $V \underset{\text{op}}{\subset} X$ such that $\mathcal{F} \in V^-$, and there is a continuous map $\varphi : X \to I$ such that $\varphi|_V = 1$, $\varphi|_{X\setminus U} = 0$. Then $m_\varphi(\mathcal{F}) = 1 > 1/2$, and for each $\mathcal{G} \in \check{G}X$ the inequality $m_\varphi(\mathcal{G}) > 1/2$ implies $\mathcal{G} \in U^-$. Therefore the inverse to m is continuous on $m(\check{G}X)$, thus the map $m : \check{G}X \to I^{C(X,I)}$ is an embedding, which completes the proof. □

Remark 2.8. It is obvious that the topology on $\check{G}X$ can be equivalently defined as the weak topology w.r.t. the collection of maps $m^\varphi : \check{G}X \to I$, $m^\varphi(\mathcal{F}) = \max\{\inf_F \varphi \mid F \in \mathcal{F}\}$, for all $\varphi \in C(X, I)$.

Further we will need the subspace

$$\hat{G}X = \{\mathcal{F} \in \check{G}X \mid \text{for all } F \in \mathcal{F} \text{ there is a compactum } K \subset F,\ K \in \mathcal{F}\} \subset \check{G}X.$$

It is easy to see that its image under $e_G X : \check{G}X \hookrightarrow G\beta X$ is the set

$$G_* X = \bigl\{\mathcal{G} \in G^* X \bigm| \text{for all } G \in \mathcal{G} \text{ there is a compactum } K \subset G \cap X,\ K \in \mathcal{G}\bigr\},$$

and $\check{G}f(\hat{G}X) \subset \hat{G}Y$ for each continuous map $f : X \to Y$ of Tychonoff spaces. Thus we obtain a subfunctor $\hat{G}$ of the functor $\check{G} : \mathcal{T}\text{ych} \to \mathcal{T}\text{ych}$.

Lemma 2.9. *Let X be a Tychonoff space. Then $\mu_G\beta X\circ\check{G}e_GX(\check{G}^2X)\subset e_GX(\check{G}X)$.*

The composition in the above inclusion is legal because $\check{G}\beta X=G\beta X$.

Proof. Let $\mathbf{F}\in\check{G}^2X$, $\mathcal{F}=\mu_G\beta X\circ\check{G}e_GX(\mathbf{F})$, and $F,G\underset{\mathrm{cl}}{\subset}\beta X$ are such that $F\cap X=G\cap X$. Assume $F\in\mathcal{F}$, then there is $\mathrm{H}\in\mathbf{F}$ such that $F\in\mathcal{G}$ for all $\mathcal{G}\in\mathrm{Cl}_{G\beta X}\,e_GX(\mathrm{H})$, therefore for all $\mathcal{G}\in e_GX(\mathrm{H})$. It is equivalent to $F\cap X\in\mathcal{H}$ for all $\mathcal{H}\in\mathrm{H}\subset\check{G}X$, which in particular implies that $F\cap X\neq\varnothing$. By the assumption, $G\cap X\in\mathcal{H}$ for all $\mathcal{H}\in\mathrm{H}$ as well, hence $G\in\mathcal{G}$ for all $\mathcal{G}\in e_GX(\mathrm{H})$. The set of all $\mathcal{H}\in\check{G}X$ such that $\mathcal{H}\ni A$ is closed for any $A\in\exp X$, thus $G\in\mathcal{G}$ for all $\mathcal{G}\in\mathrm{Cl}_{G\beta X}\,e_GX(\mathrm{H})$. We infer that $G\in\mathcal{F}$, and $\mathcal{F}\in\check{G}X$. □

For e_GX is an embedding, we define $\check{\mu}_GX$ as a map $\check{G}^2X\to\check{G}X$ such that $e_GX\circ\check{\mu}_GX=\mu_G\beta X\circ\check{G}e_GX$. This map is unique and continuous. Following the latter proof, we can see that

$$\check{\mu}_G(\mathbf{F})=\left\{F\in\exp X\;\middle|\;F\in\bigcap\mathrm{H}\text{ for some }\mathrm{H}\in\mathbf{F}\right\},\quad\mathbf{F}\in\check{G}^2X,$$

i.e. the formula is the same as in $\mathcal{C}$omp.

For the inclusion $\eta_G\beta X\circ iX(X)\subset e_GX(\check{G}X)$ is also true, there is a unique map $\check{\eta}_GX:X\to\check{G}X$ such that $e_GX\circ\eta_GX=\eta_G\beta X\circ iX$, namely $\check{\eta}_GX(x)=\{F\in\exp X\mid F\ni x\}$ for each $x\in X$, and this map is continuous. It is straightforward to prove that the collections $\check{\eta}_G=(\check{\eta}_GX)_{X\in\mathrm{Ob}\,\mathcal{T}\mathrm{ych}}$ and $\check{\mu}_G=(\check{\mu}_GX)_{X\in\mathrm{Ob}\,\mathcal{T}\mathrm{ych}}$ are natural transformations respectively $\mathbf{1}_{\mathcal{T}\mathrm{ych}}\to\check{G}$ and $\check{G}^2\to\check{G}$.

Theorem 2.10. *The triple $\check{\mathbb{G}}=(\check{G},\check{\eta}_G,\check{\mu}_G)$ is a monad in $\mathcal{T}$ych.*

Proof. Let X be a Tychonoff space and iX its embedding into βX. Then:

$$e_GX\circ\check{\mu}X\circ\check{\eta}\check{G}X=\mu\beta X\circ\check{G}e_GX\circ\check{\eta}\check{G}X=\mu\beta X\circ\eta G\beta X\circ e_GX=\mathbf{1}_{G\beta X}\circ e_GX=e_GX,$$

thus $\check{\mu}_GX\circ\check{G}\check{\eta}X=\check{\mu}_GX\circ\check{\eta}_G\check{G}X=\mathbf{1}_{\check{G}X}$, similarly we obtain the equalities $\check{\mu}_GX\circ\check{G}\check{\eta}_GX=\mathbf{1}_{\check{G}X}$ and $\check{\mu}_GX\circ\check{G}\check{\mu}_GX=\check{\mu}_GX\circ\check{\mu}_G\check{G}X$. □

For $\check{G}X$, $\check{\eta}_GX$, $\check{\mu}_GX$ coincide with GX, η_GX, μ_GX for any compactum X, the monad $\check{\mathbb{G}}$ is an extension of the monad $\mathbb{G}$ in $\mathcal{C}$omp to $\mathcal{T}$ych.

3. Functional representation of the capacity monad in the category of compacta

In the sequel X is a compactum, c is a capacity on X and $\varphi:X\to\mathbb{R}$ is a continuous function. We define the *Sugeno integral* of φ with respect to c by the formula [10,14]:

$$\int_X^{\vee}\varphi(x)\wedge dc(x)=\sup\bigl\{c\bigl(\{x\in X\mid\varphi(x)\geqslant\alpha\}\bigr)\wedge\alpha\;\big|\;\alpha\in I\bigr\}.$$

The following theorem was recently obtained (in an equivalent form) by Radul [12] under more restrictive conditions, namely restrictions of normalizedness and non-expandability were also imposed. Therefore for the readers convenience we provide a formulation and a short proof of a version more suitable for our needs.

Theorem 3.1. *Let X be a compactum, c a capacity on X. Then the functional $i:C(X,I)\to I$, $i(\varphi)=\int_X^{\vee}\varphi(x)\wedge dc(x)$ for $\varphi\in C(X,I)$, has the following properties*:

(1) *for all $\varphi,\psi\in C(X,I)$ the inequality $\varphi\leqslant\psi$ (i.e. $\varphi(x)\leqslant\psi(x)$ for all $x\in X$) implies $i(\varphi)\leqslant i(\psi)$ (i is* monotonic);
(2) *i satisfies the equalities $i(\alpha\wedge\varphi)=\alpha\wedge i(\varphi)$, $i(\alpha\vee\varphi)=\alpha\vee i(\varphi)$ for any $\alpha\in I$, $\varphi\in C(X,I)$.*

Conversely, any functional $i:C(X,I)\to I$ satisfying (1), (2) *has the form $i(\varphi)=\int_X^{\vee}\varphi(x)\wedge dc(x)$ for a uniquely determined capacity $c\in MX$.*

In the two following lemmata $i:C(X,I)\to I$ is a functional that satisfies (1), (2).

Lemma 3.2. *If $\alpha\in I$ and continuous functions $\varphi,\psi:X\to I$ are such that $\{x\in X\mid\varphi(x)\geqslant\alpha\}\subset\{x\in X\mid\psi(x)\geqslant\alpha\}$ and $i(\varphi)\geqslant\alpha$, then $i(\psi)\geqslant\alpha$.*

Proof. For $\alpha = 0$ the statement is trivial. Otherwise assume $i(\varphi) \geqslant \alpha$. Let $0 \leqslant \beta < \alpha$. For φ, ψ are continuous, there is $\gamma \in (\beta; \alpha)$ such that the closed sets $F = \psi^{-1}([0;\beta])$ and $G = \varphi^{-1}([\gamma, 1])$ have an empty intersection. Then, by Brouwer–Tietze–Urysohn Theorem, there is a continuous function $\theta : X \to [\beta; \gamma]$ such that $\theta|_F \equiv \beta$, $\theta|_G \equiv \gamma$. Then we define a function $f : X \to I$ as follows:

$$f(x) = \begin{cases} \psi(x), & x \in F, \\ \theta(x), & x \notin F \cup G, \\ \varphi(x), & x \in G. \end{cases}$$

Then $\gamma \vee f = \gamma \vee \varphi$, thus

$$\gamma \vee i(f) = i(\gamma \vee f) = i(\gamma \vee \varphi) = \gamma \vee i(\varphi) = \alpha,$$

and $i(f) = \alpha$. Taking into account $\beta \wedge f = \beta \wedge \psi$, we obtain

$$\beta = \beta \wedge i(f) = i(\beta \wedge f) = i(\beta \wedge \psi) = \beta \wedge i(\psi),$$

thus $i(\psi) \geqslant \beta$ for all $\beta < \alpha$. It implies $i(\psi) \geqslant \alpha$. □

Obviously if $\{x \in X \mid \varphi(x) \geqslant \alpha\} = \{x \in X \mid \psi(x) \geqslant \alpha\}$, then $i(\varphi) \geqslant \alpha$ if and only if $i(\psi) \geqslant \alpha$.

Lemma 3.3. *For each closed set $F \subset X$ and $\beta \in I$ the equality*

$$\inf\{i(\varphi) \mid \varphi \geqslant \alpha \wedge \chi_F\} = \alpha \wedge \inf\{i(\psi) \mid \varphi \geqslant \chi_F\}$$

is valid.

Proof. It is sufficient to observe that for all $0 \leqslant \beta < \alpha$ the sets $\{\beta \wedge \varphi \mid \varphi \geqslant \alpha \wedge \chi_F\}$ and $\{\beta \wedge \psi \mid \psi \geqslant \chi_F\}$ coincide, therefore by the previous lemma:

$$\beta \wedge \inf\{i(\varphi) \mid \varphi \geqslant \alpha \wedge \chi_F\} = \beta \wedge \inf\{i(\psi) \mid \psi \geqslant \chi_F\} = \beta \wedge \alpha \wedge \inf\{i(\psi) \mid \psi \geqslant \chi_F\}.$$

For the both expressions $\inf\{i(\varphi) \mid \varphi \geqslant \alpha \wedge \chi_F\}$ and $\alpha \wedge \inf\{i(\psi) \mid \psi \geqslant \chi_F\}$ do not exceed α, they are equal. □

Proof of the theorem. It is obvious that Sugeno integral w.r.t. a capacity satisfies (1), (2). If i is Sugeno integral w.r.t. some capacity c, then the equality $c(F) = \inf\{i(\psi) \mid \psi \geqslant \chi_F\}$ must hold for all $F \underset{\text{cl}}{\subset} X$. To prove the converse, we assume that $i : C(X, I) \to I$ satisfies (1), (2) and use the latter formula to define a set function c. It is obvious that the first two conditions of the definition of capacity hold for c. To show upper semicontinuity, assume that $c(F) < \alpha$ for some $F \underset{\text{cl}}{\subset} X$, $\alpha \in I$. Then there is a continuous function $\varphi : X \to I$ such that $\varphi \geqslant \chi_F$, $i(\varphi) < \alpha$. Let $i(\varphi) < \beta < \alpha$, then

$$i(\varphi) = \beta \wedge i(\varphi) = i(\beta \wedge \varphi) \geqslant \beta \wedge c(\{x \in X \mid \varphi(x) \geqslant \beta\}),$$

which implies $c(\{x \in X \mid \varphi(x) \geqslant \beta\}) < \beta < \alpha$. The set $U = \{x \in X \mid \varphi(x) > \beta\}$ is an open neighborhood of F such that $c(G) < \alpha$ for all $G \underset{\text{cl}}{\subset} X$, $G \subset U$. Thus c is upper semicontinuous and therefore it is a capacity.

The two previous lemmata imply that for any $\varphi \in C(X, I)$ we have

$$i(\varphi) = \sup\{\alpha \in I \mid i(\varphi) \geqslant \alpha\} = \sup\{\alpha \in I \mid c(\{x \in X \mid \varphi(x) \geqslant \alpha\}) \geqslant \alpha\}$$
$$= \sup\{\alpha \wedge c(\{x \in X \mid \varphi(x) \geqslant \alpha\}) \mid \alpha \in I\} = \int_X^{\vee} \varphi(x) \wedge dc(x). \qquad \square$$

Lemma 3.4. *Let $\varphi : X \to I$ be a continuous function. Then the map $\delta_\varphi : MX \to I$ which sends each capacity c to $\int_X^{\vee} \varphi(x) \wedge dc(x)$ is continuous.*

Proof. Observe that

$$\delta_\varphi^{-1}([0;\alpha)) = O_-(\varphi^{-1}([0;\alpha]), \alpha), \qquad \delta_\varphi^{-1}((\alpha;1]) = O_+(\varphi^{-1}((\alpha;1]), \alpha)$$

for all $\alpha \in I$. □

Corollary 3.5. *The map $X \to I^{C(X,I)}$ which sends each capacity c on X to $(\delta_\varphi(c))_{\varphi \in C(X,I)}$ is an embedding.*

Recall that its image consists of all monotonic functionals from $C(X, I)$ to I which satisfy (1), (2). Therefore from now on we identify each capacity and the respective functional. By the latter statement the topology on MX can be equivalently defined as weak* topology using Sugeno integral instead of Choquet integral. We also write $c(\varphi)$ for $\int_X^{\vee} \varphi(x) \wedge dc(x)$.

The following observation is a trivial “continuous” version of [10, Theorem 6.5].

Proposition 3.6. *Let $C \in MX$ and $\varphi \in C(X, I)$. Then $\mu_M X(C)(\varphi) = C(\delta_\varphi)$.*

Proof. Indeed, the both sides are greater or equal than $\alpha \in I$ if and only if $C\{c \in MX \mid c(\varphi) \geqslant \alpha\} \geqslant \alpha$. □

It is also easy to see that $\eta_M X(x)(\varphi) = \varphi(x)$ for all $x \in X$, $\varphi \in C(X, I)$. Thus we have obtained a description of the capacity monad $\mathbb{M}$ in terms of functionals which is a complete analogue of the description of the probability monad $\mathbb{P}$ [5,15]. Now we can easily reprove the continuity of $\eta_M X$ and $\mu_M X$, as well as the fact that $\mathbb{M} = (M, \eta_M, \mu_M)$ is a monad.

4. Extensions of the capacity functor and the capacity monad to the category of Tychonoff spaces

We will extend the definition of capacity to Tychonoff spaces. A function $c : \exp X \cup \{\varnothing\} \to I$ is called a *regular capacity* on a Tychonoff space X if it is monotonic, satisfies $c(\varnothing) = 0$, $c(X) = 1$ and the following property of *upper semicontinuity* or *outer regularity*: if $F \underset{\mathrm{cl}}{\subset} X$ and $c'(F) < \alpha$, $\alpha \in I$, then there is an open set $U \supset F$ in X such that F and $X \setminus U$ are completely separated, and $c'(G) < \alpha$ for all $G \subset U$, $G \underset{\mathrm{cl}}{\subset} X$.

This definition implies that each closed set F is contained in some zero-set Z such that $c(F) = c(Z)$.

Each capacity c on any compact space Y satisfies also the property which is called *τ-smoothness* for additive measures and have two slightly different formulations [1,16]. Below we show that they are equivalent for Tychonoff spaces.

Lemma 4.1. *Let X be a Tychonoff space and $m : \exp X \cup \{\varnothing\} \to I$ a monotonic function. Then the two following statements are equivalent:*

(a) *for each monotonically decreasing net (F_α) of closed sets in X and a closed set $G \subset X$, such that $\bigcap_\alpha F_\alpha \subset G$, the inequality $\inf_\alpha c(F_\alpha) \leqslant c(G)$ is valid;*
(b) *for each monotonically decreasing net (Z_α) of zero-sets in X and a closed set $G \subset X$, such that $\bigcap_\alpha Z_\alpha \subset G$, the inequality $\inf_\alpha c(Z_\alpha) \leqslant c(G)$ is valid.*

Proof. It is obvious that (a) implies (b). Let (b) hold, and let a net (F_α) and a set G satisfy the conditions of (a). We denote the set of all pairs (F_α, a) such that $a \in X \setminus F_\alpha$ by A, and let Γ be the set of all non-empty finite subsets of A. The space X is Tychonoff, hence for each pair $(F_\alpha, a) \in A$ there is a zero-set $Z_{\alpha,a} \supset F_\alpha$ such that $Z_{\alpha,a} \not\ni a$. For $\gamma = \{(\alpha_1, a_1), \ldots, (\alpha_k, a_k)\} \in \Gamma$ we put $Z_\gamma = Z_{\alpha_1,a_1} \cap \cdots \cap Z_{\alpha_k,a_k}$. If Γ is ordered by inclusion, then $(Z_\gamma)_{\gamma\in\Gamma}$ is a monotonically decreasing net such that $\bigcap_{\gamma\in\Gamma} Z_\gamma = \bigcap_\alpha F_\alpha \subset G$, thus $\inf_\alpha c(F_\alpha) \leqslant \inf_{\gamma\in\Gamma} Z_\gamma \leqslant c(G)$, and (a) is valid. □

We call a function $c : \exp X \to I$ a *τ-smooth capacity* if it is monotonic, satisfies $c(\varnothing) = 0$, $c(X) = 1$ and any of the two given above equivalent properties of τ-smoothness. It is obvious that each τ-smooth capacity is a regular capacity, but the converse is false. E.g. the function $c : \exp \mathbb{N} \cup \{\varnothing\} \to I$ which is defined by the formulae $c(\varnothing) = 0$, $c(F) = 1$ as $F \subset \mathbb{N}$, $F \neq \varnothing$, is a regular capacity that is not τ-smooth. For compacta the two classes coincide.

From now all capacities are τ-smooth, if otherwise is not specified.

Now we show that capacities on a Tychonoff space X can be naturally identified with capacities with a certain property on the Stone–Čech compactification βX.

Lemma 4.2. *Let c be a capacity on βX. Then the following statements are equivalent:*

(1) *for each closed sets $F, G \subset \beta X$ such that $F \cap X \subset G$, the inequality $c(F) \leqslant c(G)$ is valid;*
(2) *for each monotonically decreasing net (φ_γ) of continuous functions $\beta X \to I$ and a continuous function $\psi : \beta X \to I$ such that $\inf_\gamma \varphi_\gamma(x) \leqslant \psi(x)$ for all $x \in X$, the inequality $\inf_\gamma c(\varphi_\gamma) \leqslant c(\psi)$ is valid.*

Proof. (1) $\Longrightarrow$ (2). Let $c(\psi) < \alpha$, $\alpha \in I$, then $c(Z_0) < \alpha$ for the closed set $Z_0 = \{x \in \beta X \mid \psi(x) \geqslant \alpha\}$. The intersection Z of the closed sets $Z_\gamma = \{x \in \beta X \mid \varphi_\gamma(x) \geqslant \alpha\}$ satisfies the inclusion $Z \cap X \subset Z_0$, hence by (1): $c(Z) \leqslant c(Z_0)$. Due to τ-smoothness of c we obtain $\inf_\gamma c(Z_\gamma) \leqslant c(Z)$. Therefore there exists an index γ such that $c(\{x \in \beta X \mid \varphi_\gamma(x) \geqslant \alpha\}) < \alpha$, thus $c(\varphi_\gamma) < \alpha$, and $\inf_\gamma c(\varphi_\gamma) < \alpha$, which implies the required inequality.

(2) $\Longrightarrow$ (1). Let a continuous function $\psi : \beta X \to I$ be such that $\psi|_G = 1$. Denote the set of all continuous functions $\varphi : \beta X \to I$ such that $\varphi|_F \equiv 1$ by $\mathcal{F}$. We consider the order on $\mathcal{F}$ which is reverse to natural: $\varphi \prec \varphi'$ if $\varphi \geqslant \varphi'$, then the collection $\mathcal{F}$ can be regarded as a monotonically decreasing net such that $(\varphi(x))_{\varphi\in\mathcal{F}}$ converges to 1 for all $x \in X \cap G$, and to 0 for all $x \in X \setminus G$. Therefore $\inf_{\varphi\in\mathcal{F}} \varphi(x) \leqslant \psi(x)$ for all $x \in X$, hence, by the assumption: $\inf_{\varphi\in\mathcal{F}} c(\varphi) \leqslant c(\psi)$. Thus

$$\inf\bigl\{c(\varphi) \bigm| \varphi : \beta X \to I \text{ is continuous}, \varphi|_F \equiv 1\bigr\} \leqslant \inf\bigl\{c(\psi) \bigm| \psi : \beta X \to I \text{ is continuous}, \psi|_G \equiv 1\bigr\},$$

i.e. $c(F) \leqslant c(G)$. □

We define the set of all $c \in M\beta X$ that satisfy (1) $\Longleftrightarrow$ (2) by $M^* X$.

Condition (1) implies that, if closed sets $F, G \subset \beta X$ are such that $F \cap X = G \cap X$, then $c(F) = c(G)$. Therefore we can define a set function $\check{c} : \exp X \cup \{\varnothing\} \to I$ as follows: if $A \underset{\text{cl}}{\subset} X$, then $\check{c}(A) = c(F)$ for any set $F \underset{\text{cl}}{\subset} \beta X$ such that $F \cap X = A$. Obviously $\check{c}(A) = \inf\{c(\psi) \mid \psi \in C(\beta X, I),\ \psi \geqslant \chi_A\}$.

The following observation, although almost obvious, is a crucial point in our exposition.

Proposition 4.3. *A set function $c' : \exp X \cup \{\varnothing\} \to I$ is equal to $\check{c}$ for some $c \in M^*X$ if and only if c' is a τ-smooth capacity on X.*

Therefore we define the set of all capacities on X by $\check{M}X$ and identify it with the subset $M^*X \subset M\beta X$. We obtain an injective map $e_M X : \check{M}X \to M\beta X$, and from now on we assume that a topology on $\check{M}X$ is such that $e_M X$ is an embedding. Thus $\check{M}X$ for a Tychonoff X is Tychonoff as well.

If c is a capacity on X and $\varphi : X \to I$ is a continuous function, we define the Sugeno integral of φ w.r.t. c by the usual formula:

$$c(\varphi) = \int_X^{\vee} \varphi(x) \wedge dc(x) = \sup\bigl\{\alpha \wedge c\bigl\{x \in X \bigm| \varphi(x) \geqslant \alpha\bigr\} \bigm| \alpha \in I\bigr\}.$$

For any continuous function $\varphi : X \to I$ we denote by $\beta\varphi$ its Stone–Čech compactification, i.e. its unique continuous extension to a function $\beta X \to I$.

Proposition 4.4. *Let $c \in M^*X$ and $\check{c}$ is defined as above. Then for any continuous function $\varphi : X \to I$ we have $\check{c}(\varphi) = c(\beta\varphi)$.*

Proof. It is sufficient to observe that

$$\check{c}\bigl(\bigl\{x \in X \bigm| \varphi(x) \geqslant \alpha\bigr\}\bigr) = c\bigl(\bigl\{x \in \beta X \bigm| \beta\varphi(x) \geqslant \alpha\bigr\}\bigr). \qquad \square$$

Thus the topology on $\check{M}X$ can be equivalently defined as the weak*-topology using Sugeno integral. It also immediately implies that the following theorem is valid.

Theorem 4.5. *Let X be a Tychonoff space, c a capacity on X. Then the functional $i : C(X, I) \to I$, $i(\varphi) = \int_X^{\vee} \varphi(x) \wedge dc(x)$ for $\varphi \in C(X, I)$, has the following properties*:

(1) *for all $\varphi, \psi \in C(X, I)$ the inequality $\varphi \leqslant \psi$ (i.e. $\varphi(x) \leqslant \psi(x)$ for all $x \in X$) implies $i(\varphi) \leqslant i(\psi)$ (i is* monotonic);
(2) *i satisfies the equalities $i(\alpha \wedge \varphi) = \alpha \wedge i(\varphi)$, $i(\alpha \vee \varphi) = \alpha \vee i(\varphi)$ for any $\alpha \in I$, $\varphi \in C(X, I)$;*
(3) *for each monotonically decreasing net (φ_α) of continuous functions $X \to I$ and a continuous function $\psi : X \to I$ such that $\inf_\alpha \varphi_\alpha(x) \leqslant \psi(x)$ for all $x \in X$, the inequality $\inf_\alpha i(\varphi_\alpha) \leqslant i(\psi)$ is valid.*

Conversely, any functional $i : C(X, I) \to I$ satisfying (1)–(3) *has the form $i(\varphi) = \int_X^{\vee} \varphi(x) \wedge dc(x)$ for a uniquely determined capacity $c \in \check{M}X$.*

Condition (3) is superfluous for a compact space X, but cannot be omitted for noncompact spaces. E.g. the functional, which sends each $\varphi \in C(\mathbb{R}, I)$ to $\sup \varphi$, has properties (1), (2), but fails to satisfy (3).

The following statement is an immediate corollary of an analogous theorem for the compact case.

Proposition 4.6. *The topology on $\check{M}X$ can be equivalently determined by a subbase which consists of all sets of the form*

$$O_+(U, \alpha) = \bigl\{c \in \check{M}X \bigm| \text{there is } F \underset{\text{cl}}{\subset} X,\ F \text{ is completely separated from } X \setminus U,\ c(F) > \alpha\bigr\}$$

for all open $U \subset X$, $\alpha \in I$, and of the form

$$O_-(F, \alpha) = \bigl\{c \in \check{M}X \bigm| c(F) < \alpha\bigr\}$$

for all closed $F \subset X$, $\alpha \in I$.

Like the compact case, for a continuous map $f : X \to Y$ of Tychonoff spaces we define a map $\check{M}f : \check{M}X \to \check{M}Y$ by the two following equivalent formulae: $\check{M}f(c)(F) = c(f^{-1}(F))$, with $c \in \check{M}X$, $F \underset{\text{cl}}{\subset} Y$ (if set functions are used), or $\check{M}f(c)(\varphi) = c(\varphi \circ f)$ for $c \in \check{M}X$, $\varphi \in C(X, I)$ (if we regard capacities as functionals). The latter representation implies the continuity of $\check{M}f$, and we obtain a functor $\check{M}$ in the category $\mathcal{T}$ych of Tychonoff spaces that is an extension of the capacity functor M in $\mathcal{C}$omp.

The map $e_M X : \check{M}X \to M\beta X$ coincides with $\check{M}iX$, where iX is the embedding $X \hookrightarrow \beta X$ (we identify $\check{M}\beta X$ and $M\beta X$), and the collection $e_M = (e_M X)_{X \in \mathrm{Ob}\,\mathcal{T}\mathrm{ych}}$ is a natural transformation from the functor $\check{M}$ to the functor $UM\beta$, with $U : \mathcal{C}\mathrm{omp} \to \mathcal{T}\mathrm{ych}$ being the inclusion functor. Observe that $\eta_M\beta X(X) \subset M^*X = e_M X(\check{M}X)$, therefore there is a continuous restriction $\check{\eta}_M X = \eta_M \beta X|_X : X \to \check{M}X$ which is a component of a natural transformation $e_M : \mathbf{1}_{\mathcal{T}\mathrm{ych}} \to \check{M}$. For all $x \in X \in \mathrm{Ob}\,\mathcal{T}\mathrm{ych}$, $F \underset{\mathrm{cl}}{\subset} X$ the value $\check{\eta}_G X(x)(F)$ is equal to 1 if $x \in F$, otherwise is equal to 0.

Lemma 4.7. *Let X be a Tychonoff space. Then $\mu_M\beta X \circ \check{M}e_M X(\check{M}^2 X) \subset e_M X(\check{M}X)$.*

Proof. Let $C \in \check{M}^2 X$, and $F, G \underset{\mathrm{cl}}{\subset} \beta X$ are such that $F \cap X \subset G$. Then for all $c \in M^*X$ we have $c(F) \leqslant c(G)$, thus for each $\alpha \in I$:

$$\big\{c \in \check{M}X \bigm| e_M X(c)(F) \geqslant \alpha\big\} \subset \big\{c \in \check{M}X \bigm| e_M X(c)(G) \geqslant \alpha\big\},$$

hence

$$\begin{aligned}\check{M}e_M X(C)\big(\big\{c \in M\beta X \bigm| c(F) \geqslant \alpha\big\}\big) &= C\big(e_M X^{-1}\big(\big\{c \in M\beta X \bigm| c(F) \geqslant \alpha\big\}\big)\big) \leqslant C\big(e_M X^{-1}\big(\big\{c \in M\beta X \bigm| c(G) \geqslant \alpha\big\}\big)\big) \\ &= \check{M}e_M X(C)\big(\big\{c \in M\beta X \bigm| c(G) \geqslant \alpha\big\}\big),\end{aligned}$$

thus

$$\begin{aligned}\mu_M\beta X \circ \check{M}e_M X(C)(F) &= \sup\big\{\alpha \wedge \check{M}e_M X(C)\big(\big\{c \in M\beta X \bigm| c(F) \geqslant \alpha\big\}\big)\big\} \\ &\leqslant \sup\big\{\alpha \wedge \check{M}e_M X(C)\big(\big\{c \in M\beta X \bigm| c(G) \geqslant \alpha\big\}\big)\big\} = \mu_M\beta X \circ \check{M}e_M X(C)(G),\end{aligned}$$

which means that $\mu_M\beta X \circ \check{M}e_M X(C) \in M^*X = e_M X(\check{M}X)$. □

For $e_M X : \check{M}X \to M\beta X$ is an embedding, there is a unique map $\check{\mu}_M X : \check{M}^2 X \to \check{M}X$ such that $\mu_M\beta X \circ Me_M X = e_M X \circ \check{\mu}_M X$, and this map is continuous. It is straightforward to verify that the collection $\check{\mu}_M = (\check{\mu}_M X)_{X \in \mathrm{Ob}\,\mathcal{T}\mathrm{ych}}$ is a natural transformation $\check{M}^2 \to \check{M}$, and $\check{\mu}_M X$ can be defined directly, without involving Stone–Čech compactifications, by the usual formulae:

$$\check{\mu}_M X(C)(F) = \sup\big\{\alpha \wedge C\big(\big\{c \in \check{M}X \bigm| c(F) \geqslant \alpha\big\}\big)\big\}, \quad C \in \check{M}^2 X,\ F \underset{\mathrm{cl}}{\subset} X,$$

or

$$\check{\mu}_M X(C)(\varphi) = C(\delta_\varphi), \quad \varphi \in C(X, I), \quad \text{where } \delta_\varphi(c) = c(\varphi) \text{ for all } c \in \check{M}X.$$

Theorem 4.8. *The triple $\check{\mathbb{M}} = (\check{M}, \check{\eta}_M, \check{\mu}_M)$ is a monad in $\mathcal{T}\mathrm{ych}$.*

Proof is a complete analogue of the proof of Proposition 2.10.

This monad is an extension of the monad $\mathbb{M} = (M, \eta_M, \mu_M)$ in $\mathcal{C}\mathrm{omp}$ in the sense that $\check{M}X = MX$, $\check{\eta}_M X = \eta_M X$ and $\check{\mu}_M X = \mu_M X$ for each compactum X.

Proposition 4.9. *Let for each compact inclusion hyperspace $\mathcal{F}$ on a Tychonoff space X the set function $i_G^M X(\mathcal{F}) : \exp X \cup \{\varnothing\} \to I$ be defined by the formula*

$$i_G^M X(\mathcal{F})(A) = \begin{cases} 1, & A \in \mathcal{F}, \\ 0, & A \notin \mathcal{F}, \end{cases} \quad A \underset{\mathrm{cl}}{\subset} X.$$

Then $i_G^K X$ is an embedding $\check{G}X \hookrightarrow \check{M}X$, and the collection $i_G^K = (i_G^K X)_{X \in \mathrm{Ob}\,\mathcal{T}\mathrm{ych}}$ is a morphism of monads $\check{\mathbb{G}} \to \check{\mathbb{M}}$.

Thus the monad $\check{\mathbb{G}}$ is a submonad of the monad $\check{\mathbb{M}}$.

Now let

$$M_*X = \big\{c \in M\beta X \bigm| c(A) = \sup\big\{c(F) \bigm| F \subset A \cap X \text{ is compact}\big\} \text{ for all } A \underset{\mathrm{cl}}{\subset} \beta X\big\}.$$

It is easy to see that $M_*X \subset M^*X$. As a corollary we obtain

Proposition 4.10. *A set function $c' : \exp X \cup \{\varnothing\} \to I$ is equal to $\check{c}$ for some $c \in M_*X$ if and only if c' is a τ-smooth capacity on X and satisfies the condition $c'(A) = \sup\{c'(F) \mid F \subset A$ is compact$\}$ for all $A \underset{\mathrm{cl}}{\subset} X$* (inner compact regularity).

If a set function satisfies (1)–(4), we call it a *Radon capacity*. The set of all Radon capacities on X is denoted by $\hat{M}X$ and regarded as a subspace of $\check{M}X$. An obvious inclusion $M\beta f(M_*X) \subset M_*Y$ for a continuous map $f : X \to Y$ of Tychonoff spaces implies $\check{M}f(\hat{M}X) \subset \hat{M}Y$. Therefore we denote the restriction of $\check{M}f$ to a mapping $\hat{M}X \to \hat{M}Y$ by $\hat{M}f$ and obtain a subfunctor $\hat{M}$ of the functor $\check{M}$.

Question 4.11. What are necessary and sufficient conditions for a functional $i : C(X, I) \to I$ to have the form $i(\varphi) = \int_X^{\vee} \varphi(x) \wedge dc(x)$ for a some capacity $c \in \hat{M}X$?

Here is a *necessary condition*: for each monotonically *increasing* net (φ_α) of continuous functions $X \to I$ and a continuous function $\psi : X \to I$ such that $\sup_\alpha \varphi_\alpha(x) \geqslant \psi(x)$ for all $x \in X$, the inequality $\sup_\alpha i(\varphi_\alpha) \geqslant i(\psi)$ is valid.

The problem of existence of a restriction of $\check{\mu}_M X$ to a map $\hat{M}^2X \to \hat{M}X$ is still unsolved and is connected with a similar question for inclusion hyperspaces by the following

Proposition 4.12. *Let X be a Tychonoff space. If $\check{\mu}_M X(\hat{M}^2X) \subset \hat{M}X$, then $\check{\mu}_G X(\hat{G}^2X) \subset \hat{G}X$.*

Proof. We will consider equivalent inclusions $\mu_M\beta X(M_*^2X) \subset M_*X$ and $\mu_G\beta X(G_*^2X) \subset G_*X$. The latter one means that, for each set $A \underset{\mathrm{cl}}{\subset} X$ and compact set $\mathcal{G} \subset G\beta X$ such that each element F of any inclusion hyperspace $\mathcal{B} \in \mathcal{G}$ contains a compactum $K \in \mathcal{B}$, $K \subset X$, there is a compact set $H \subset A$, $H \in \bigcap \mathcal{G}$.

Assume that $\mu_G\beta X(G_*^2X) \not\subset G_*X$, then there are $A \underset{\mathrm{cl}}{\subset} X$ and a compact set $\mathcal{G} \subset G_*X$ such that all inclusion hyperspaces in $\mathcal{G}$ contain subsets of A, but there are no compact subsets of A in $\bigcap \mathcal{G}$. For each $\mathcal{B} \in \mathcal{G}$ let a capacity $c_\mathcal{B}$ be defined as follows:

$$c_\mathcal{B}(F) = \begin{cases} 1, & F \in \mathcal{B}, \\ 0, & F \notin \mathcal{B}, \end{cases} \quad F \underset{\mathrm{cl}}{\subset} \beta X.$$

It is obvious that $c_\mathcal{B} \in M_*X$, and the correspondence $\mathcal{B} \mapsto c_\mathcal{B}$ is continuous, thus the set $\mathrm{B} = \{c_\mathcal{B} \mid \mathcal{B} \in \mathcal{G}\} \subset M\beta X$ is compact. Therefore the capacity $C \in M^2\beta X$, defined as

$$C(\mathcal{F}) = \begin{cases} 1, & \mathcal{F} \supset \mathrm{B}, \\ 0, & \mathcal{F} \not\supset \mathrm{B}, \end{cases} \quad \mathcal{F} \underset{\mathrm{cl}}{\subset} M\beta X,$$

is in $M_*(M_*X)$. Then $\mu_M\beta X(C)(\mathrm{Cl}_{\beta X} A) = 1$, but there is no compact subset $K \subset A$ such that $c_\mathcal{B}(K) \neq 0$ for all $\mathcal{B} \in \mathcal{G}$, therefore $\mu_M\beta X(C)(K) = 0$ for all compact $K \subset A = \mathrm{Cl}_{\beta X} A \cap X$, and $\mu_M\beta X(C) \notin M_*X$. □

It is still unknown to the authors:

Question 4.13. Does the converse implication hold? Do all locally compact Hausdorff or (complete) metrizable spaces satisfy the condition of the previous statement?

5. Topological properties of the functors $\check{G}$, $\hat{G}$, $\check{M}$ and $\hat{M}$

Recall that a continuous map of topological spaces is *proper* if the preimage of each compact set under it is compact. A *perfect map* is a closed continuous map such that the preimage of each point is compact. Any perfect map is proper [4].

From now on all maps in this section are considered continuous, and all spaces are Tychonoff if otherwise not specified.

Remark 5.1. We have already seen that properties of the functors $\check{M}$ and $\hat{M}$ are "parallel" to properties of the functors $\check{G}$ and $\hat{G}$. Therefore in this section we present only formulations and proofs of statements for $\check{M}$ and $\hat{M}$. **All** of them are valid also for $\check{G}$ and $\hat{G}$, and it is an easy exercise to simplify the proofs for capacities to obtain proofs for compact inclusion hyperspaces.

Proposition 5.2. *Functors $\check{M}$ and $\hat{M}$ preserves the class of injective maps.*

Proof. Let $f : X \to Y$ be injective. If $c, c' \in \check{M}X$ and $A \underset{\mathrm{cl}}{\subset} X$ are such that $c(A) \neq c'(A)$, then $B = \mathrm{Cl}\, f(A) \in \underset{\mathrm{cl}}{\subset} Y$, and $\check{M}f(c)(B) = c(f^{-1}(B)) = c(A) \neq c'(A) = c'(f^{-1}(B)) = \check{M}f(c)(B)$, hence $\check{M}f(c) \neq \check{M}f(c')$, and $\check{M}f$ is injective, as well as its restriction $\hat{M}f$. □

Proposition 5.3. *Functors $\check{M}$ and $\hat{M}$ preserve the class of closed embeddings.*

Proof. Let a map $f: X \to Y$ be a closed embedding (thus a perfect map), then for the Stone–Čech compactification $\beta f: \beta X \to \beta Y$ the inclusion $\beta f(\beta X \setminus X) \subset \beta Y \setminus Y$ is valid [4]. We know that $M\beta X(M^*X) \subset M^*Y$, $M\beta X(M_*X) \subset M_*Y$.

Let $c \in M\beta X \setminus M^*X$, then there are $F, G \underset{\text{cl}}{\subset} M\beta X$ such that $F \cap X \subset G$, but $c(F) > c(G)$. Then $f(F)$ and $f(G)$ are closed in βY, and $f(F) \setminus f(G) \subset f(\beta X \setminus X) \subset \beta Y \setminus Y$.

The sets $F' = f^{-1}(f(F))$ and $G' = f^{-1}(f(G))$ are closed in βX and satisfy $F' \cap X = F \cap X$, $G' \cap X = G \cap X$, thus $c(F') = M\beta f(c)(f(F)) > c(G') = M\beta f(c)(f(G))$, which implies $M\beta f(c) \notin M^*Y$. Thus $(M\beta f)^{-1}(M^*Y) = M^*X$, and the restriction $M\beta f|_{M^*X}: M^*X \to M^*Y$ is perfect, therefore closed. It is obvious that this restriction is injective, thus is an embedding. For the maps $M\beta f|_{M^*X}$ and $\check{M} f$ are homeomorphic, the same holds for the latter map.

Now let $c \in M\beta X \setminus M_*X$, i.e. there is $F \underset{\text{cl}}{\subset} \beta X$ such that $c(F) > \sup\{c(K) \mid K \subset F \cap X \text{ is compact}\}$. The compact set $F' = \beta f(F)$ is closed in $\mathrm{Cl}\, f(X) \subset \beta Y$. Observe that $F = (\beta f)^{-1}(F')$ and obtain:

$$\begin{aligned}\sup\{M\beta f(c)(L) \mid L \subset F' \cap Y \text{ is compact}\} &= \sup\{c((\beta f)^{-1}(L)) \mid L \subset F' \cap Y \text{ is compact}\} \\ &\leqslant \sup\{c(K) \mid K \subset F \cap X \text{ is compact}\} < c(F) = M\beta f(c)(F'),\end{aligned}$$

and $M\beta f(c) \notin M_*X$. The rest of the proof is analogous to the previous case. □

It allows for a closed subspace $X_0 \subset X$ to identify $\check{M}X_0$ and $\hat{M}X_0$ with the images of the map $\check{M}i$ and $\hat{M}i$, with $i: X_0 \hookrightarrow\!\!\to X$ being the embedding.

We say that a functor F in $\mathcal{T}$ych *preserves intersections (of closed sets)* if for any space X and a family $(i_\alpha: X_\alpha \hookrightarrow X)$ of (closed) embeddings the equality $\bigcap_\alpha FX_\alpha = FX_0$ holds, i.e. $\bigcap_\alpha Fi_\alpha(X_\alpha) = Fi_0(X_0)$, where i_0 is the embedding of $X_0 = \bigcap_\alpha X_\alpha$ into X. This notion is usually used for functors which preserve (closed) embeddings, therefore we verify that:

Proposition 5.4. *Functors $\check{M}$ and $\hat{M}$ preserve intersections of closed sets.*

Proof. Let $c \in \check{M}X$ and closed subspaces $X_\alpha \subset X$, $\alpha \in A$, are such that $c \in \check{M}X_\alpha$ for all $\alpha \in A$. Let 2^A_f be the set of all non-empty finite subsets of A. It is a directed poset when ordered by inclusion. For all $F \underset{\text{cl}}{\subset} X$ and $\{\alpha_1, \dots, \alpha_k\} \in 2^A_f$ we have $c(F) = c(F \cap X_{\alpha_1} \cap \dots \cap X_{\alpha_k})$. The monotonically decreasing net $(F \cap X_{\alpha_1} \cap \dots \cap X_{\alpha_k})_{\{\alpha_1,\dots,\alpha_k\} \in 2^A_f}$ converges to $F \cap X_0$, with $X_0 = \bigcap_{\alpha \in A} X_\alpha$. Thus $c(F) = c(F \cap X_0)$, which implies $c \in \check{M}X_0$.

The statement for $\hat{M}$ is obtained as a corollary due to the following observation: if $X_0 \subset X$ is a closed subspace, then $\hat{M}X_0 = \hat{M}X \cap \check{M}X_0$. □

Therefore for each element $c \in \check{M}X$ there is a least closed subspace $X_0 \subset X$ such that $c \in \check{M}X_0$. It is called the *support* of c and denoted $\operatorname{supp} c$.

It is unknown to the author whether the functor $\check{M}$ preserve finite or countable intersections.

Proposition 5.5. *Functor $\hat{M}$ preserves countable intersections.*

Proof. Let $c \in \hat{M}X$ belong to all $\hat{M}X_n$ for a sequence of subspaces $X_n \subset X$, $n = 1, 2, \dots$. If $A \underset{\text{cl}}{\subset} F$, $\varepsilon > 0$, then there is a compactum $K_1 \subset A \cap X_1$ such that $c(K_1) > c(A) - \varepsilon/2$. Then choose a compactum $K_2 \subset K_1 \cap X_2$ such that $c(K_2) > c(K_1) - \varepsilon/4$, …, a compactum $K_n \subset K_{n-1} \cap X_n$ such that $c(K_n) > c(K_{n-1}) - \varepsilon/2^n$, etc. The intersection $K = \bigcap_{n=1}^\infty K_n$ is a compact subset of $A \cap X_0$, $X_0 = \bigcap_{n=1}^\infty X_n$, and $c(K) > C(A) - \varepsilon$. Thus $\sup\{c(K) \mid K \subset A \cap X_0 \text{ is compact}\} = c(A)$ for all $A \underset{\text{cl}}{\subset} X$, i.e. $c \in \hat{M}X_0$. □

It is easy to show that $\check{M}$ and $\hat{M}$ do not preserve uncountable intersections.

We say that a functor F in $\mathcal{T}$ych (or in $\mathcal{C}$omp) *preserves preimages* if for each continuous map $f: X \to Y$ and a closed subspace $Y_0 \subset Y$ the inclusion $Ff(b) \in FY_0$ for $b \in FX$ implies $b \in F(f^{-1}(Y_0))$, or, more formally, $Ff(b) \in Fj(FY_0)$ implies $b \in Fi(F(f^{-1}(Y_0)))$, where $i: f^{-1}(Y_0) \hookrightarrow X$ and $j: Y_0 \hookrightarrow Y$ are the embeddings.

Proposition 5.6. *Functors $\check{M}$ and $\hat{M}$ do not preserve preimages.*

It is sufficient to recall that the capacity functor $M: \mathcal{C}\text{omp} \to \mathcal{C}\text{omp}$, being the restriction of the two functors in question, does not preserve preimages [18].

Proposition 5.7. *Let $f: X \to Y$ is a continuous map such that $f(X)$ is dense in Y. Then $\check{M}f(\check{M}X)$ is dense in $\check{M}Y$, and $\hat{M}f(\hat{M}X)$ is dense in $\hat{M}Y$.*

Proof. Let $M_\omega X$ be the set of all capacities on X with finite support, i.e.

$$M_\omega X = \bigcup\{MK \mid K \subset X \text{ is finite}\}.$$

Then $M_\omega X \subset \hat{M}X \subset \check{M}X$, $\check{M}f(M_\omega X) = M_\omega(f(X))$, and the latter set is dense in both $\hat{M}Y$ and $\check{M}Y$. □

6. Subgraphs of capacities on Tychonoff space and fuzzy integrals

In [18] for each capacity c on a compactum X its *subgraph* was defined as follows:

$$\operatorname{sub} c = \{(F, \alpha) \in \exp X \times I \mid \alpha \leqslant c(F)\}.$$

Given the subgraph $\operatorname{sub} c$, each capacity c is uniquely restored: $c(F) = \max\{\alpha \in I \mid (F, \alpha) \in \operatorname{sub} c\}$ for each $F \in \exp X$.

Moreover, the map sub is an embedding $MX \hookrightarrow \exp(\exp X \times I)$. Its image consists of all sets $S \subset \exp X \times I$ such that [18] the following conditions are satisfied for all closed non-empty subsets F, G of X and all $\alpha, \beta \in I$:

(1) if $(F, \alpha) \in S$, $\alpha \geqslant \beta$, then $(F, \beta) \in S$;
(2) if $(F, \alpha), (G, \beta) \in S$, then $(F \cup G, \alpha \vee \beta) \in S$;
(3) $S \supset \exp X \times \{0\} \cup \{X\} \times I$;
(4) S is closed.

The topology on the subspace $\operatorname{sub}(MX) \subset \exp(\exp X \times I)$ can be equivalently determined by the subbase which consists of all sets of the form

$$V_+(U, \alpha) = \{S \in \operatorname{sub}(MX) \mid \text{there is } (F, \beta) \in S,\ F \subset U,\ \beta > \alpha\}$$

for all open $U \subset X$, $\alpha \in I$, and of the form

$$V_-(F, \alpha) = \{S \in \operatorname{sub}(MX) \mid \beta < \alpha \text{ for all } (F, \beta) \in S\}$$

for all closed $F \subset X$, $\alpha \in I$.

Let the subgraph of a τ-smooth capacity c on a Tychonoff space X be defined by the same formula at the beginning of the section. Consider the intersection $\operatorname{sub} c \cap (\exp X \times \{\alpha\})$. It is equal to $S_\alpha(c) \times \{\alpha\}$, with $S_\alpha(c) = \{F \in \exp X \mid c(F) \geqslant \alpha\}$. The latter set is called the *α-section* [18] of the capacity c and is a compact inclusion hyperspace for each $\alpha > 0$. Of course, $S_0(c) = \exp X$ is not compact if X is not compact. If $0 \leqslant \alpha < \beta \leqslant 1$, then $S_\alpha(c) \supset S_\beta$, and $S_\beta(c) = \bigcup_{0 \leqslant \alpha < \beta} S_\alpha(c)$.

We present necessary and sufficient conditions for a set $S \subset \exp X \times I$ to be the subgraph of some capacity $c \in \hat{M}X$.

Proposition 6.1. *Let X be a Tychonoff space. A set $S \subset \exp X \times I$ is a subgraph of a τ-smooth capacity on X if and only if the following conditions are satisfied for all closed non-empty subsets F, G of X and all $\alpha, \beta \in I$:*

(1) *if $(F, \alpha) \in S$, $\alpha \geqslant \beta$, then $(F, \beta) \in S$;*
(2) *if $(F, \alpha), (G, \beta) \in S$, then $(F \cup G, \alpha \vee \beta) \in S$;*
(3) $S \supset \exp X \times \{0\} \cup \{X\} \times I$;
(4) $S \cap (\exp X \times [\gamma; 1])$ *is compact in* $\exp_l X \times I$ *for all* $\gamma \in (0; 1]$.

Such S is closed in $\exp X \times I$.

Proof. Let $c \in \check{M}X$ and $S = \operatorname{sub} c$. It is easy to see that S satisfies (1)–(3). To show that $S \cap (\exp X \times [\gamma; 1])$ is compact, assume that it is covered by subbase elements $U_i^- \times (a_i; b_i)$, $U_i \underset{\text{op}}{\subset} X$, $i \in \mathcal{I}$. For any $\alpha \in [\gamma; 1]$ the intersection $S \cap (\exp \times \{\alpha\}) = S_\alpha(c) \times \{\alpha\}$ is compact and covered by $U_i^- \times (a_i; b_i)$ for those $i \in \mathcal{I}$ that $(a_i, b_i) \ni \alpha$. Therefore there is a finite subcover $U_{i_1}^-, \ldots, U_{i_k}^-$ of $S_\alpha(c)$, $\max\{a_{i_1}, \ldots, a_{i_k}\} < \alpha < \min\{b_{i_1}, \ldots, b_{i_k}\}$. When $a \nearrow \alpha$, the compact set $S_a(c)$ decreases to $S_\alpha(c)$, thus there is $a \in (\max\{a_{i_1}, \ldots, a_{i_k}\}; \alpha)$ such that $S_a(c) \subset U_{i_1}^- \cup \cdots \cup U_{i_k}^-$. If we denote $b = \min\{b_{i_1}, \ldots, b_{i_k}\}$, we obtain that for each $\alpha \in [\gamma; 1]$ there is an interval $(a, b) \ni \alpha$ such that $S \cap (\exp X \times (a, b))$ is covered by a finite number of sets $U_i^- \times (a_i, b_i)$. For $[\gamma, 1]$ is compact, we infer that there is a finite subcover of the whole set $S \cap (\exp X \times [\gamma; 1])$, thus (4) holds.

Now let a set $S \subset \exp X \times I$ satisfy (1)–(4), and let $S_\alpha = \operatorname{pr}_1(S \cap (\exp X \times I))$ for all $\alpha \in I$. By (1) $S_\alpha \supset S_\beta$ whenever $a < \beta$. Assume $S_\beta \neq \bigcap_{0<\alpha<\beta} S_\alpha$ for some $\beta \in (0; 1]$, i.e. there is $F \in \exp X$ such that $F \in S_\alpha$ for all $\alpha \in (0; \beta)$, but $F \notin S_\beta$. Then the sets $(X \setminus F)^- \times I$ and $\exp X \times [0; \alpha)$, with $\alpha \in (0; \beta)$, form an open cover of the set $S \cap (\exp X \times [\beta/2; 1])$ for which there is no finite subcover, which contradicts to compactness. Thus $S_\beta = \bigcap_{0<\alpha<\beta} S_\alpha$. It implies that for $(F, \beta) \notin S$, i.e. $F \notin S_\beta$, there is $\alpha \in (0; \beta)$ such that $F \notin S_\alpha$. The set S_α is a compact inclusion hyperspace, thus is closed in $\exp X$. Then $(\exp X \setminus S_\alpha) \times (\alpha; 1]$ is an open neighborhood of (F, β) which does not intersect S, hence S is closed in $\exp X \times I$.

For each $F \in \exp X$ we put $c(F) = \max\{\alpha \mid (F, \alpha) \in S\}$. It is straightforward to verify that c is a τ-smooth capacity such that $\operatorname{sub} c = S$. □

Proposition 6.2. *Let $\psi : \exp X \times I \to I$ be a continuous function such that*:

(1) *ψ in antitone in the first argument and isotone in the second one*;
(2) *$\psi(F, \alpha)$ uniformly converges to 0 as $\alpha \to 0$.*

Then the correspondence $\Psi : c \mapsto \max\{\psi(F, c(F)) \mid F \in \exp X\}$ is a well defined continuous function $\check{M}X \to I$.

Proof. Let $S = \operatorname{sub} c$. Observe that Ψ can be equivalently defined as $\Psi(c) = \max\{\psi(F, \alpha) \mid (F, \alpha) \in S\}$. The function $\psi : \exp_l X \times I \to I$ is upper semicontinuous, and $\psi(S)$ is either $\{0\}$ or equal to $\psi(S \cap (\exp X \times [\gamma; 1]))$ for some $\gamma \in (0; 1)$. Hence $\psi(S)$ is a compact subset of I, therefore contains a greatest element, and use of "max" in the definition of Ψ is legal.

Assume that $\Psi(c) < b$ for some $b \in I$. We take some $a \in (\Psi(c); b)$. There exists $\gamma \in I$ such that $\psi(F, \alpha) < a$ for all $\alpha \in [0; \gamma)$, $F \in \exp X$. If $(F, \alpha) \in S$, $\alpha \geqslant \gamma$, then there is a neighborhood $\mathcal{V} = \langle U_0, U_1, \ldots, U_k\rangle \times (u, v) \ni (F, \alpha)$ such that $U_1 \cup \cdots \cup U_k$ is completely separated from $X \setminus U_0$, and $\psi(G, \beta) < a$ for all $(G, \beta) \in \mathcal{V}$. The inequality $\psi(G, \beta) < a$ holds also for all $(G, \beta) \in \langle X, U_1, \ldots, U_k\rangle \times [0, v)$. Thus we obtain a cover of $S \cap (\exp X \times [\gamma; 1])$ by open sets in $\exp_l X \times I$, and there is a finite subcover by sets $\langle X, U^l_1, \ldots, U^l_{k_l}\rangle \times [0, v_l)$, $1 \leqslant l \leqslant n$. We may assume $0 < v_1 \leqslant v_2 \leqslant \cdots \leqslant v_n > 1$. It is routine but straightforward to verify that c is in an open neighborhood

$$\mathcal{U} = \bigcap \bigl\{ O_-\bigl(X \setminus \bigl(U^{m+1}_{j_{m+1}} \cup \cdots \cup U^n_{j_n}\bigr), v_m\bigr) \bigm| 1 \leqslant m < n,\ 1 \leqslant j_{m+1} \leqslant k_{m+1}, \ldots, 1 \leqslant j_n \leqslant k_n \bigr\},$$

and for each capacity $c' \in \mathcal{U}$ the set $\operatorname{sub} c' \cap [\gamma; 1]$ is also covered by the sets

$$\bigl\langle X, U^l_1, \ldots, U^l_{k_l}\bigr\rangle \times [0, v_l), \quad 1 \leqslant l \leqslant n,$$

therefore

$$\Psi\bigl(c'\bigr) \leqslant \max\bigl\{a, \max\bigl\{\psi(F, \alpha) \bigm| (F, \alpha) \in S,\ \alpha \geqslant \gamma\bigr\}\bigr\} = a < b.$$

Hence Ψ is upper semicontinuous. To prove lower semicontinuity, assume that $\Psi(c) > b$ for some $b \in I$. Then there is $F \in \exp X$ such that $\psi(F, c(F)) > b$. By continuity there are open neighborhood $U \supset F$ and $\gamma \in (0; c(F))$ such that F is completely separated from $X \setminus U$, and for all $G \in \exp X$, G completely separated from $X \setminus U$, $\alpha \in I$, $\alpha > \gamma$ the inequality $\psi(G, \alpha) > b$ is valid. Then $c \in O_+(U, \alpha)$, and for all $c' \in O_+(U, \alpha)$ we have $\Psi(c') > b$. □

The reason to consider such form of Ψ is that not only Sugeno integral can be represented this way (for $\psi(F, \alpha) = \inf\{\varphi(x) \mid x \in F\} \wedge \alpha$), but a whole class of fuzzy integrals obtained by replacement of "$\wedge$" by an another "pseudomultiplication" $\odot : I \times I \to I$ [2], e.g. by usual multiplication or the operation $h(a, b) = a + b - ab$. The latter statement provides the continuity of a fuzzy integral with respect to a capacity on a Tychonoff space, provided "$\odot$" is continuous, isotone in the both variables and uniformly converges to 0 as the second argument tends to 0 (which is not the case for the h given above).

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