Theoretical Computer Science 412 (2011) 1941–1957

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Correctness of linear logic proof structures is *NL*-complete☆

Paulin Jacobé de Naurois *, Virgile Mogbil

Laboratoire d'Informatique de l'Université Paris Nord (LIPN), UMR CNRS 7030, Institut Galilée - Université Paris 13, 99, avenue Jean-Baptiste Clément, 93430 Villetaneuse, France

ARTICLE INFO

Keywords:
Complexity classes
Correctness criteria
Linear logic
Nondeterministic logspace

ABSTRACT

We provide new correctness criteria for all fragments (multiplicative, exponential, additive) of linear logic. We use these criteria for proving that deciding the correctness of a linear logic proof structure is *NL*-complete.

0. Introduction

The *proof nets* [5,3] of linear logic (LL) are a parallel syntax for logical proofs without all the bureaucracy of sequent calculus. They are a non-sequential graph-theoretic representation of proofs, where the order in which some rules are used in a sequent calculus derivation, when irrelevant, is neglected. The unit-free multiplicative proof nets are inductively defined from sequent calculus rules of unit-free multiplicative linear logic (MLL). The MLL *proof structures* are freely built on the same syntax as proof nets, without any reference to a sequent calculus derivation. The same holds for MELL and MALL proof nets and proof structures with respect to MELL and MALL sequent calculus.

In LL, we are mainly interested in the following decision problems: deciding the provability of a given formula, which gives the expressiveness of the logic; deciding if two given proofs reduce to the same normal form, i.e., the cut-elimination problem which corresponds to program equivalence using the Curry–Howard isomorphism; and deciding the correctness of a given proof structure, i.e., whether it comes from a sequent calculus derivation. For this last decision problem, one uses a *correctness criterion* to distinguish proof nets among proof structures. We recall the following main results [13,15] and we complete (in bold) the correctness cases:

fragment		decision problem		
	units	provability	cut-elimination	correctness
MLL	no	*NP*-complete	*P*-complete	***NL*-complete**
MELL	yes	open	non-elementary	
MALL	no	*PSPACE*-complete	*coNP*-complete	

Correctness is equivalent to provability for unit only MLL because proof nets are formulae syntactic trees. However, it is not so obvious for the propositional case as one can observe following the long story of correctness criteria.

- The long-trip criterion [5] is based on travels, and was the first one.
- The acyclic-connected criterion [3] is a condition that is based on switchings, i.e., the choice of one premise for each $\parr$ connective. The condition is that all the associated graphs are trees. A naive implementation of the acyclic-connected criterion uses exponential time.
- The contractibility criterion [2] is done in quadratic time by repeating two graph rewriting rules until one obtains a simple node.

☆ Work supported by project NOCoST (ANR, JC05_43380).
* Corresponding author.
E-mail addresses: denaurois@lipn.univ-paris13.fr (P. Jacobé de Naurois), Virgile.Mogbil@lipn.univ-paris13.fr (V. Mogbil).

0304-3975/$ – see front matter © 2011 Published by Elsevier B.V.
doi:10.1016/j.tcs.2010.12.020

- The graph parsing criterion [14] is a strategy for contractibility which is implemented in linear time as a sort of unification [7].
- The dominator tree criterion [17,18] is a linear-time correctness criterion for essential nets, to which proof structure correctness reduces in linear time.
- The ribbon criterion [16] is a topological condition requiring homeomorphism to the disk.

For other fragments of linear logic, some of these criteria apply or are extended as for MELL[1] [2,8] or MALL [6,4,9].

A feature of these criteria is that they successively lower the complexity of sequential, deterministic algorithms deciding correctness for MLL. Switching from proof structures to paired graphs, that is, undirected graphs with a distinguished set of edges, we give a new correctness criterion for MLL and more generally for MELL. This new correctness criterion gives, for the first time, a lower bound for the correctness decision problem for both MLL (MLL-CORR) and MELL (MELL-CORR). This lower bound yields an exact characterization of the complexity of this problem, and induces naturally efficient parallel and randomized algorithms for it. The classical inclusion $NL \subseteq P$ induces a deterministic polynomial-time version of our algorithm; note, however, that there is little hope for this to be linear time.

Our new criterion also induces an *NL* algorithm for the correctness problem for MALL proof structures (MALL-CORR) as defined in [9], thus establishing as well the *NL*-hardness of this problem.

The paper is organized as follows. We recall preliminary definitions and results in linear logic and complexity theory in Section 1. Section 2 is devoted to the exposition of a new correctness criterion for unit-free MLL and MELL with units (Theorem 2.6). Proposition 2.7 establishes the *NL*-hardness of MLL-CORR and Proposition 2.9 the *NL*-membership of MELL-CORR. Section 3 is devoted to the proof of the *NL*-membership of MALL-CORR. The *NL*-completeness of MLL-CORR, MELL-CORR and MALL-CORR is established in Theorem 3.20.

1. Background

1.1. MLL *and proof structures*

Italic capitals A, B stand for MLL formulae, which are given by the following grammar, where $\otimes$ and $\parr$ are duals for the negation $^\perp$, accordingly to the De Morgan laws:

$$\text{MLL}: \quad F ::= A \mid A^\perp \mid F \otimes F \mid F \parr F$$

Greek capitals Γ, Δ stand for sequents, which are multisets of formulae, so that exchange is implicit. The MLL sequent calculus is given by the following rules:

$$\frac{}{\vdash A, A^\perp}\ (ax) \qquad \frac{\vdash \Gamma, C \quad \vdash \Delta, C^\perp}{\vdash \Gamma, \Delta}\ (cut) \qquad \frac{\vdash \Gamma, A \quad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}\ \otimes \qquad \frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr$$

Definition 1.1. An MLL *skeleton* is a directed acyclic graph (DAG) whose edges are labelled with MLL formulae, and whose nodes are labelled, and defined with an arity and coarity as follows:

Node label		Atom		Cut		$\otimes$		$\parr$	
Arity	Edges	0	$\emptyset$	2	$A, A^\perp$	2	A, B	2	A, B
Coarity	Edges	1	A	0	$\emptyset$	1	$A \otimes B$	1	$A \parr B$

We allow edges with a source but no target (i.e., pending or dangling edges); they are called the *conclusions* of the proof structure. The set of conclusions of an MLL skeleton is clearly an MLL sequent. We also denote as *premises* of a node the edges incident to it, and *conclusion* of a node its outgoing edge.
For a given node x of arity 2, its left (respectively right) parent is denoted x^l (respectively x^r).
An *axiom link*, or simply a *link*, on an MLL skeleton $\mathcal{S}$ is a bidirected edge between complementary atoms in $\mathcal{S}$, i.e., atoms labelled with dual literals P and $P^\perp$.
A *linking* on $\mathcal{S}$ is a partitioning of the atom nodes of $\mathcal{S}$ into links, i.e., a set of disjoint links whose union contains every atom of $\mathcal{S}$.

An MLL *proof structure* is $(\mathcal{S}, \lambda)$, where $\mathcal{S}$ is an MLL skeleton and λ a linking on $\mathcal{S}$.

An MLL *proof net* is an MLL proof structure inductively defined as follows.

- (ax): $(\mathcal{S}, \lambda)$, where $\mathcal{S} = (\{A, A^\perp\}, \emptyset)$, $\lambda = \{(A, A^\perp)\}$ is an MLL proof net with conclusions $A, A^\perp$.
- $\parr$: if $(\mathcal{S}, \lambda)$ is an MLL proof net with conclusions Γ, A, B, then $(\mathcal{S}', \lambda)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with a $\parr$-link of premises A and B is an MLL proof net with conclusions $\Gamma, A \parr B$.
- $\otimes$: if $(\mathcal{S}_1, \lambda_1)$ with conclusions Γ, A and $(\mathcal{S}_2, \lambda_2)$ with conclusions Δ, B are disjoint MLL proof nets, $(\mathcal{S}, \lambda_1 \uplus \lambda_2)$, where $\mathcal{S}$ is $\mathcal{S}_1 \uplus \mathcal{S}_2$ extended with a $\otimes$-link of premises A and B is an MLL proof net with conclusions $\Gamma, A \otimes B, \Delta$.

---

[1] As usual M, A and E denote respectively multiplicative, additive and exponential fragments of LL.

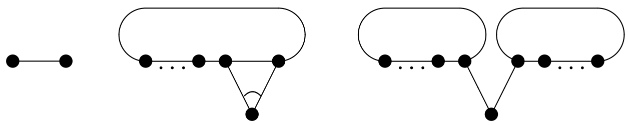

Fig. 1. Paired graph constructors associated to MLL proof nets: *ax* link, ⅋-link and ⊗ (cut) link.

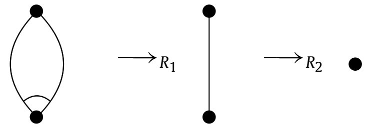

Fig. 2. Contraction rules $\rightarrow_c$.

- (cut): if $(\mathcal{S}_1, \lambda_1)$ with conclusions Γ, A and $(\mathcal{S}_2, \lambda_2)$ with conclusions $\Delta, A^\perp$ are disjoint MLL proof nets, $(\mathcal{S}, \lambda_1 \uplus \lambda_2)$, where $\mathcal{S}$ is $\mathcal{S}_1 \uplus \mathcal{S}_2$ extended with a cut-link of premises A and $A^\perp$ is an MLL proof net with conclusions Γ, Δ.

The inductive definition of MLL proof nets corresponds to a graph-theoretic abstraction of the derivation rules of MLL; any proof net is sequentializable, i.e., corresponds to an MLL derivation: given a proof net P of conclusion Γ, there exists a sequent calculus proof of $\vdash \Gamma$ which infers P.

Definition 1.2. A *paired graph* is an undirected graph $G = (V, E)$ with a set of *pairs* $C(G) \subseteq E \times E$ which are pairwise disjoint couples of edges with the same target, called a *pair node*, and two (possibly distinct) sources called the *premise nodes*.

A *switching* S of G is the choice of an edge for every pair of $C(G)$. With each switching S is associated a subgraph $S(G)$ of G: for every pair of $C(G)$, erase the edges which are not selected by S. When S selects the (abusively speaking) left edge of each pair, $S(G)$ is denoted as $G[\forall \mapsto \backslash\!\cdot\,]$. Also, $G[\forall \mapsto \cdot\!\cdot\,]$ stands for $G \setminus \{e, e' | \; (e, e') \in C(G)\}$.

Remark 1.3. Without loss of generality, we allow tuples of edges, i.e., $C(G) \subseteq \bigcup_{n\in\mathbb{N}} E$. A tuple of edges incident to a node x can be seen as a binary tree rooted at x with all in-going edges being coupled.

Let $\mathcal{S} = (V, E)$ be an MLL skeleton. To $\mathcal{S}$, we associate the paired graph $G_{\mathcal{S}} = (V, E)$, where $C(G_{\mathcal{S}})$ contains the premises of each ⅋-link of $\mathcal{S}$.
To an MLL proof structure $(\mathcal{S}, \lambda)$, we associate the paired graph $G_{(\mathcal{S},\lambda)} = G_{\mathcal{S}} \uplus \lambda$, where $C(G_{(\mathcal{S},\lambda)}) = C(G_{\mathcal{S}})$ (Fig. 1).
For a pair of edges $(v, x), (w, x)$, we adopt the representation of Fig. 1, where the two edges of the pair are joined by an arc.
We define the following graph rewriting rules $\rightarrow_c$ of Fig. 2 on paired graphs where all the nodes are distinct and rule $\rightarrow_{R_2}$ applies only for a non-pair edge. We denote by $G \rightarrow_c^* \bullet$ the fact that G contracts to a single vertex with no edge.

Definition 1.4. An MLL proof structure $(\mathcal{S}, \lambda)$ is *DR-correct* if, for all switchings S of $G_{(\mathcal{S},\lambda)}$, the graph $S(G_{(\mathcal{S},\lambda)})$ is acyclic and connected.

An MLL proof structure $(\mathcal{S}, \lambda)$ is *contractile* if $G_{(\mathcal{S},\lambda)} \rightarrow_c^* \bullet$.

Theorem 1.5 (*[3,2]*)**.** *An* MLL *proof structure* $(\mathcal{S}, \lambda)$ *is an* MLL *proof net iff* $(\mathcal{S}, \lambda)$ *is DR-correct iff* $(\mathcal{S}, \lambda)$ *is contractile.* □

We define the following decision problem MLL-CORR:
GIVEN: An MLL proof structure $(\mathcal{S}, \lambda)$.
PROBLEM: Is $(\mathcal{S}, \lambda)$ an MLL proof net?

1.2. MELL *and proof structures*

The definition of MELL formulae follows that of MLL formulae in Section 1.1, with ! and ? duals for the negation $^\perp$, as well as the neutral elements 1 and $\perp$:

$$\text{MELL}: \quad F ::= A \mid A^\perp \mid F \otimes F \mid F \parr F \mid !F \mid ?F \mid 1 \mid \perp$$

The MELL sequent calculus contains the rules of the MLL sequent calculus, as well as the following rules:

$$\frac{\vdash \Gamma}{\vdash \Gamma, \perp}\ \perp \qquad \frac{}{\vdash 1}\ 1 \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?W \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?C \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?D \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ !P$$

Definition 1.6. MELL *skeletons* are defined similarly to MLL skeletons (Definition 1.1), with the following additional nodes, where the $?W$ node subsumes both $?W$ and $\perp$ MELL sequent calculus rules:

Node label		1		?W		?C		?D		!P	
Arity	Edges	0	$\emptyset$	0	$\emptyset$	2	$?A, ?A$	1	A	1	A
Coarity	Edges	1	1	1	$\perp, ?A$	1	$?A$	1	$?A$	1	$!A$

Definition 1.7. An *exponential box* is an MELL skeleton whose conclusions are all ? formulae but one, its *principal door*, which is conclusion of a $!P$ node. Similarly, a *weakening box* is an MELL skeleton with a distinguished conclusion, its *principal door*, which is the conclusion of a $?W$ node. A *box* is either an exponential or a weakening box.

Definition 1.8. An MELL *boxed structure* $(\mathcal{S}, B)$ is given by a MELL skeleton $\mathcal{S}$ and a set of exponential and weakening boxes $B = \{B_1, \ldots, B_k\}$. Moreover, boxes may nest but may not partially overlap. For a given node in $\mathcal{S}$, its associated box (if there is any) is the smallest box in B that contains it. A unique exponential (respectively weakening) box is associated to each $!P$ node (respectively $?W$ node). The set B of boxes is identified with a box mapping B which, for a given node in $\mathcal{S}$, returns its associated box if there is any, and $\mathcal{S}$ otherwise.

Definition 1.9. An MELL *proof structure* is $(\mathcal{S}, B, \lambda)$, where $(\mathcal{S}, B)$ is a boxed structure and λ is a linking on $\mathcal{S}$.

As for MLL, the set of conclusions of an MELL proof structure is by construction an MELL sequent.

Definition 1.10. An MELL *proof net* is an MELL proof structure defined inductively as follows.

- (ax): $((\{A, A^\perp\}, \emptyset), \emptyset, \{(A, A^\perp)\})$ is an MELL proof net with conclusions $A, A^\perp$.
- $\parr$: if $(\mathcal{S}, B, \lambda)$ is an MELL proof net with conclusions Γ, A, C, then $(\mathcal{S}', B, \lambda)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with a $\parr$-link of premises A and C, is an MELL proof net with conclusions $\Gamma, A \parr C$.
- $\otimes$: if $(\mathcal{S}_1, B_1, \lambda_1)$ with conclusions Γ, A and $(\mathcal{S}_2, B_2, \lambda_2)$ with conclusions Δ, C are disjoint MELL proof nets, $(\mathcal{S}, B_1 \uplus B_2, \lambda_1 \uplus \lambda_2)$, where $\mathcal{S}$ is $\mathcal{S}_1 \uplus \mathcal{S}_2$ extended with a $\otimes$-link of premises A and C, is a MELL proof net with conclusions $\Gamma, A \otimes C, \Delta$.
- (cut): if $(\mathcal{S}_1, B_1, \lambda_1)$ with conclusions Γ, A and $(\mathcal{S}_2, B_2, \lambda_2)$ with conclusions $\Delta, A^\perp$ are disjoint MELL proof nets, $(\mathcal{S}, B_1 \uplus B_2, \lambda_1 \uplus \lambda_2)$, where $\mathcal{S}$ is $\mathcal{S}_1 \uplus \mathcal{S}_2$ extended with a cut-link of premises A and $A^\perp$, is an MELL proof net with conclusions Γ, Δ.
- 1: $((\{1\}, \emptyset), \emptyset, \emptyset)$ is a MELL proof net with conclusion 1.
- ?W: if $(\mathcal{S}, B, \lambda)$ is an MELL proof net with conclusions Γ, then, for any MELL formula A, $(\mathcal{S}', B \uplus \mathcal{S}', \lambda)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with a ?W node with conclusion $?A$ (respectively $\perp$), is a MELL proof net with conclusions $\Gamma, ?A$ (respectively $\Gamma, \perp$).
- ?C: if $(\mathcal{S}, B, \lambda)$ is an MELL proof net with conclusions $\Gamma, ?A, ?A$, then $(\mathcal{S}', B, \lambda)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with a ?C node of premises $?A$ and $?A$, is an MELL proof net with conclusions $\Gamma, ?A$.
- ?D: if $(\mathcal{S}, B, \lambda)$ is an MELL proof net with conclusions Γ, A, then $(\mathcal{S}', B, \lambda)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with a ?D node of premise A, is a MELL proof net with conclusions $\Gamma, ?A$.
- !P: if $(\mathcal{S}, B, \lambda)$ is an MELL proof net with conclusions $?\Gamma, A$, then $(\mathcal{S}', B \uplus \mathcal{S}', \lambda)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with a !P node of premise A, is an MELL proof net with conclusions $?\Gamma, !A$.

As for MLL, MELL proof nets are induced by MELL sequent calculus proofs.

Definition 1.11. Let $(\mathcal{S}, B, \lambda)$ be an MELL proof structure, with boxes $b_1, \ldots, b_n$. Let $b_0 = \mathcal{S}$. We define as follows the family $\mathcal{G}_{(\mathcal{S},B,\lambda)} = \{G^i_{(\mathcal{S},B,\lambda)}\}_{i=0\ldots n}$ of paired graphs.

- $G^i_{(\mathcal{S},B,\lambda)}$ contains a node l for every node l of $\mathcal{S} \setminus \{?W \text{ nodes}\}$ with $B(l) = b_i$.
- $G^i_{(\mathcal{S},B,\lambda)}$ contains an edge (l, l') for $l, l' \in \mathcal{S} \setminus \{?W \text{ nodes}\}$ such that $B(l) = B(l') = b_i$ if and only if
  - there is an edge (l, l') in $\mathcal{S}$, or
  - there is an edge (l', l) in $\mathcal{S}$, or
  - $(l, l') \in \lambda$.
- $C(G^i_{(\mathcal{S},B,\lambda)})$ contains the premises of each $\parr$-link and $?C$ node l of $\mathcal{S}$ with $B(l) = b_i$.
- Assume that b_j is an outermost box included in b_i. A node $\overline{b_j} \in G^i_{(\mathcal{S},B,\lambda)}$ is associated to b_j, and an edge $(\overline{b_j}, l) \in G^i_{(\mathcal{S},B,\lambda)}$ for any node l conclusion of a node in b_j and such that $B(l) = b_i$.

Essentially, $G^i_{(\mathcal{S},B,\lambda)}$ is the paired graph corresponding to the box b_i, where all inner boxes are considered contracted to a single node. Moreover, for the sake of connectivity, the $?W$ node (if there is any) corresponding to b_i is removed.
An MELL proof structure $(\mathcal{S}, B, \lambda)$ with boxes $b_1, \ldots, b_n$ is *DR-correct* if, for all $i \in \{0, \ldots, n\}$, and for all switchings S of $G^i_{(\mathcal{S},B,\lambda)}$, the graph $S(G^i_{(\mathcal{S},B,\lambda)})$ is acyclic and connected.
An MELL proof structure $(\mathcal{S}, B, \lambda)$ with boxes $b_1, \ldots, b_n$ is *contractile* if $\forall i \in \{0, \ldots, n\}$, $G^i_{(\mathcal{S},B,\lambda)} \to^*_c \bullet$.

Theorem 1.12 ([8]). *An* MELL *proof structure* $(\mathcal{S}, B, \lambda)$ *is an* MELL *proof net iff* $(\mathcal{S}, B, \lambda)$ *is DR-correct iff* $(\mathcal{S}, B, \lambda)$ *is contractile.* $\square$

We define the following decision problem MELL-CORR.
GIVEN: An MELL proof structure $(\mathcal{S}, B, \lambda)$.
PROBLEM: Is $(\mathcal{S}, B, \lambda)$ an MELL proof net?

1.3. MALL

We recall (and adapt to our formalism) the notion of MALL proof structures and proof nets defined in [9]. The definition of MALL formulae follows that of MLL formulae in Section 1.1, with the additive connectives $\oplus$ and $\&$, duals under the De Morgan laws:

$$\text{MALL}: \quad F ::= A \mid A^{\perp} \mid F \otimes F \mid F \parr F \mid F \oplus F \mid F \& F$$

The MALL sequent calculus contains the rules of the MLL sequent calculus, as well as the following rules:

$$\frac{\vdash \Gamma, A}{\vdash \Gamma, A \oplus B}\ \oplus_1 \qquad \frac{\vdash \Gamma, B}{\vdash \Gamma, A \oplus B}\ \oplus_2 \qquad \frac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \& B}\ \&$$

Definition 1.13. MALL *skeletons* are defined similarly to MLL skeletons (Definition 1.1), with the following additional nodes:

Node label		$\oplus$		$\&$	
Arity	Edges	2	A, B	2	A, B
Coarity	Edges	1	$A \oplus B$	1	$A \& B$

Definition 1.14. Let $\mathcal{S}$ be an MALL skeleton. An *additive resolution* of $\mathcal{S}$ is any result of deleting one argument subtree of each additive ($\oplus$ or $\&$) node in $\mathcal{S}$. A $\&$*-resolution* of $\mathcal{S}$ is any result of deleting one argument subtree of each $\&$ node in $\mathcal{S}$.

A *linking* on an MALL skeleton $\mathcal{S}$ is a set of disjoint links on $\mathcal{S}$ such that its set of vertices is the set of leaves of an additive resolution of $\mathcal{S}$. Note that, in the case where $\mathcal{S}$ contains no additive node, this definition subsumes Definition 1.1. The additive resolution of $\mathcal{S}$ induced by a linking λ is denoted $\mathcal{S} \downharpoonleft \lambda$.

An additive resolution of $\mathcal{S}$ naturally induces an MLL skeleton, and, for any linking λ, $(\mathcal{S} \downharpoonleft \lambda, \lambda)$ induces an MLL proof structure. Denote by $G_{(\mathcal{S}|\lambda,\lambda)}$ the paired graph associated to it.

An MALL *proof structure* is $(\mathcal{S}, \Theta)$, where $\mathcal{S}$ is an MALL skeleton and Θ is a set of linkings on $\mathcal{S}$. The set of conclusions of an MALL proof structure is a MALL sequent.

Definition 1.15. An MALL *proof net* is an MALL proof structure inductively defined as follows.

- (ax): $((\{A, A^{\perp}\}, \emptyset), \{\{(A, A^{\perp})\}\})$ is an MALL proof net with conclusions $A, A^{\perp}$.
- $\parr$: if $(\mathcal{S}, \Theta)$ is an MALL proof net with conclusions Γ, A, B, then $(\mathcal{S}', \Theta)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with a $\parr$-link of premises A and B, is an MALL proof net with conclusions $\Gamma, A \parr B$.
- $\otimes$: if $(\mathcal{S}_1, \Theta_1)$ with conclusions Γ, A and $(\mathcal{S}_2, \Theta_2)$ with conclusions Δ, B are disjoint MALL proof nets, $(\mathcal{S}, \Theta)$, where $\mathcal{S}$ is $\mathcal{S}_1 \uplus \mathcal{S}_2$ extended with a $\otimes$-link of premises A and B and Θ is $\{\lambda_1 \uplus \lambda_2, \lambda_1 \in \Theta_1, \lambda_2 \in \Theta_2\})$, is an MALL proof net with conclusions $\Gamma, A \otimes B, \Delta$.
- (cut): if $(\mathcal{S}_1, \Theta_1)$ with conclusions Γ, A and $(\mathcal{S}_2, \Theta_2)$ with conclusions $\Delta, A^{\perp}$ are disjoint MALL proof nets, $(\mathcal{S}, \Theta)$, where $\mathcal{S}$ is $\mathcal{S}_1 \uplus \mathcal{S}_2$ extended with a cut-link of premises A and $A^{\perp}$ and Θ is $\{\lambda_1 \uplus \lambda_2, \lambda_1 \in \Theta_1, \lambda_2 \in \Theta_2\})$, is an MALL proof net with conclusions Γ, Δ.
- $\&$: if $(\mathcal{S} \uplus \mathcal{S}_A, \Theta_A)$, where $\mathcal{S}$ (respectively $\mathcal{S}_A$) has conclusions Γ (respectively A) and $(\mathcal{S} \uplus \mathcal{S}_B, \Theta_B)$, where $\mathcal{S}_B$ has conclusion B are MALL proof nets, then $(\mathcal{S} \uplus \mathcal{S}', \Theta_A \uplus \Theta_B)$, where $\mathcal{S}'$ is $\mathcal{S}_A \uplus \mathcal{S}_B$ extended with a $\&$ node of premises A and B, is a MALL proof net with conclusions $\Gamma, A \& B$.
- $\oplus$: for any MALL formula B, if $(\mathcal{S}, \Theta)$ is a MALL proof net with conclusions Γ, A, then $(\mathcal{S}', \Theta)$, where $\mathcal{S}'$ is $\mathcal{S}$ extended with the syntactic tree of B and a $\oplus$ node of premises A and B (respectively B and A), is an MALL proof net with conclusions $\Gamma, A \oplus B$ (respectively $\Gamma, B \oplus A$).

Again, MALL proof nets are induced by MALL sequent calculus proofs.

Definition 1.16. Let $(\mathcal{S}, \Theta)$ be an MALL proof structure.

Let W be a $\&$-resolution of $\mathcal{S}$ and let $\lambda \in \Theta$ be a linking on $\mathcal{S}$. We write $\lambda \sqsubseteq W$ if and only if every vertex of every link in λ is a leaf of W.

Let $\Lambda \subseteq \Theta$ be a set of linkings on $\mathcal{S}$.

Λ is said to *toggle* a $\&$ node $x_{\&}$ (respectively a $\oplus$ node $x_{\oplus}$) of $\mathcal{S}$ if there exist $\lambda_1, \lambda_2 \in \Lambda$ such that $x^l_{\&} \in \mathcal{S} \downharpoonleft \lambda_1$ and $x^r_{\&} \in \mathcal{S} \downharpoonleft \lambda_2$ (respectively $x^l_{\oplus} \in \mathcal{S} \downharpoonleft \lambda_1$ and $x^r_{\oplus} \in \mathcal{S} \downharpoonleft \lambda_2$).

Let $\mathcal{S} \downharpoonleft \Lambda = \bigcup_{\lambda \in \Lambda} \mathcal{S} \downharpoonleft \lambda$, and let $G_{\mathcal{S}|\Lambda} = \bigcup_{\lambda \in \Lambda} G_{(\mathcal{S}|\lambda,\lambda)}$.

Let $x_{\&}$ be a $\&$ node in $\mathcal{S}$ and let a be an atom of $\mathcal{S}$. Let $\{\lambda_1, \lambda_2\} \subseteq \Lambda$. A *jump edge* $(x_{\&}, a)$ is admissible for $\{\lambda_1, \lambda_2\}$ if and only if

1. $x_{\&}$ is the *unique* $\&$ node toggled by $\{\lambda_1, \lambda_2\}$, and
2. there exists a link $l = (a, b) \in \lambda_1 \setminus \lambda_2$.

Let $H_{\mathcal{S}|\Lambda}$ be $G_{\mathcal{S}|\Lambda}$ extended with all admissible jump edges for all $\{\lambda_1, \lambda_2\} \subseteq \Lambda$, and where $C(H_{\mathcal{S}|\Lambda})$ contains the premise – and jump – edges incident to all $\parr/\&$ nodes of $\mathcal{S} \downharpoonleft \Lambda$ (the pair edges are actually tuples as in Remark 1.3). Let G be a paired graph. A *switching cycle* $\mathcal{C}$ in G is a cycle in $S(G)$ for some switching S of G.

Theorem 1.17 (*Correctness Criterion*). *[9]*
An MALL *proof structure* $(\mathcal{S}, \Theta)$ *is an* MALL *proof net iff the following hold.*

1. *(RES): For every* $\&$*-resolution* W *of* $\mathcal{S}$*, there exists a unique* $\lambda \in \Theta$ *such that* $\lambda \sqsubseteq W$*;*
2. *(MLL): For every* $\lambda \in \Theta$*,* $(\mathcal{S} \downharpoonright \lambda, \lambda)$ *is an* MLL *proof net; and*
3. *(TOG): For every* $\Lambda \subseteq \Theta$ *of two or more linkings,* Λ *toggles a* $\&$ *node* $x_\&$ *such that* $x_\&$ *does not belong to any switching cycle of* $H_{\mathcal{S}\downharpoonright\Lambda}$*.* □

We define the following decision problem MALL-corr.
Given: An MALL proof structure $(\mathcal{S}, \Theta)$.
Problem: Is $(\mathcal{S}, \Theta)$ an MALL proof net?

1.4. Complexity classes and related problems

Let us mention several major complexity classes below P, some of which having natural complete problems that we will use in this paper. Let us briefly recall some basic definitions and results.

Definition 1.18. Complexity classes.

- AC^0 (respectively AC^1) is the class of problems solvable by a uniform family of circuits of constant (respectively logarithmic) depth and polynomial size, with NOT gates and AND, OR gates of unbounded fan-in.
- L is the class of problems solvable by a deterministic Turing machine which only uses a logarithmic working space.
- NL (respectively $coNL$) is the class of problems solvable by a non-deterministic Turing machine which only uses a logarithmic working space, such that the following hold.

  1. If the answer is "yes", at least one (respectively all) computation path accepts.
  2. If the answer is "no", all (respectively at least one) computation paths reject.

Theorem 1.19 (*[10,20]*). $NL = coNL$. □

The following inclusion results are also well known.

$$AC^0 \subseteq L \subseteq NL \subseteq AC^1 \subseteq P,$$

where it remains unknown whether any of these inclusions is strict.

It is important to note that our NL-completeness results for MLL-corr, MELL-corr and MALL-corr are under constant-depth (actually AC^0) reductions. From the inclusion above, it should be clear to the reader that the reductions lie indeed in a class small enough to be relevant. For a good exposition of constant-depth reducibility, see [1].

In what follows, we will often use the notion of a *path* in a directed – or undirected – graph. A path is a sequence of vertices such that there is an edge between any two consecutive vertices in the path. A path will be called *elementary* when any node occurs at most once in the path.

Let us now list some graph-theoretic problems that will be used in this paper.

Is Tree (IT): Given an undirected graph $G = (V, E)$, is it a tree?
IT is L-complete under constant-depth reductions [11].

Source–Target Connectivity (STCONN): Given a directed graph $G = (V, E)$ and two vertices s and t, is there a path from s to t in G?
STCONN is NL-complete under constant-depth reductions [12].

Undirected Source–Target Connectivity (USTCONN): Given an undirected graph $G = (V, E)$ and two vertices s and t, do s and t belong to the same connected component of G?
USTCONN is L-complete under constant-depth reductions [19].

Universal Source DAG (SDAG): Given a directed graph $G = (V, E)$, is it acyclic and does there exist a source node s such that there is a path from s to each vertex?

Proposition 1.20. SDAG $\in NL$.

Proof. Given $G = (V, E)$ a directed graph, its acyclicity can be expressed as follows:

$$\forall (x, y) \in V^2 : \ \neg\mathrm{STCONN}(G, x, y) \vee \neg\mathrm{STCONN}(G, y, x).$$

Since $NL = coNL$ (Theorem 1.19) and STCONN $\in NL$, acyclicity is clearly in NL. Checking whether each vertex can be reached from a vertex s can also be done with STCONN subroutines; therefore SDAG is in NL. □

Proposition 1.21. SDAG *is coNL-hard under constant-depth reductions.*

Proof. Let $\mathcal{L}$ be any language in *coNL*. $\mathcal{L}$ is then decided by a non-deterministic Turing machine M in space less than $k\log(n)$ on inputs of size n, for some $k \geq 0$.

As it is usual in complexity, we denote by a configuration of a single-tape Turing machine the tuple (S, T, pos), where S is the current state of the machine, T the content of its tape and pos the position of the scanning head on the tape. The size of a configuration is the size of the non-empty part of its tape.

Let $\mathcal{C}_n$ be the set of configurations of M of size less or equal to $k\log(n)$, and define $T = |\mathcal{C}_n|$. Clearly, $T = \mathcal{O}(n^{2k})$ is an upper bound for the computation time of M on inputs of size n. Without loss of generality, we assume that every configuration of M has at least one outgoing transition, possibly towards itself, and that the result of the computation is given by the state reached by M after exactly T computation steps. A configuration is thus either accepting or rejecting.

Let us consider the following directed graph $G_n = (V_n, E_n)$, where the following hold.

- $V_n = \bigcup_{c\in\mathcal{C}_n, t\in[0,T]}\{(c,t)\} \cup \{c_A\} \cup \{c_R\} \cup \{s\}$.
- For (c,t), $(c', t+1) \in V_n$, $((c', t+1) \to (c,t)) \in E_n$ if and only if $c \to c'$ is a transition of M.
- For $c \in \mathcal{C}_n$, $(c_A \to (c,T)) \in E_n$ iff c is an accepting configuration of M.
- For $c \in \mathcal{C}_n$, $(c_R \to (c,T)) \in E_n$ iff c is a rejecting configuration of M.
- $(s \to c_A) \in E_n$, $(s \to c_R) \in E_n$.

A path $(c_1, t_1) \to \cdots \to (c_k, t_k)$ in G_n follows by construction a sequence $t_1, \ldots, t_k$ that is strictly decreasing. Since there is no edge $(c,t) \to c_A$, $(c,t) \to c_R$ nor $(c,t) \to s$, it is then clear that G_n is acyclic.

Moreover, since every configuration of M has at least one outgoing transition, every vertex (c,t), $t < T$ in G_n has at least one parent node $(c', t+1)$. By induction on t, it follows that every vertex in G_n is reachable from s. Therefore, G_n satisfies SDAG.

Let x be an input of size n to M. An initial configuration $c_x \in \mathcal{C}_n$ of M is naturally associated to this input x. Consider now the directed graph $H_n^x = G_n \cup \{(c_x, 0) \to c_R\}$.

Then, H_n^x satisfies SDAG if and only if $x \in \mathcal{L}$. Indeed, by Definition 1.18, $x \in \mathcal{L}$ if and only if there exists no computation path $c_x \to \cdots \to c_r$ of length T in M, where c_r is a rejecting configuration. By construction of G_n, such a path corresponds to a path $(c_r, T) \to \cdots \to (c_x, 0)$ in G_n. Then $x \in \mathcal{L}$ if and only if there exists no path $c_R \to \cdots \to (c_x, 0)$ in G_n, if and only if H_n^x is acyclic. Since G_n satisfies SDAG, it follows that H_n^x satisfies SDAG if and only if $x \in \mathcal{L}$.

Moreover, it is well known that the configuration graph of a Turing machine can be computed with constant-depth circuits. Computing H_n^x from the configuration graph of M requires only purely local rewriting rules, which can all be performed in parallel. Therefore, H_n^x can also be computed with constant-depth circuits. $\square$

Propositions 1.20 and 1.21, and Theorem 1.19 yield the following result.

Theorem 1.22. SDAG *is NL-complete under constant-depth reductions.* $\square$

2. MLL and MELL

2.1. *New correctness criteria for* MLL *and* MELL

For a given paired graph, the following notion of a dependency graph provides a partial order among its pair nodes corresponding to some valid contraction sequences accordingly to rule R_1 of Fig. 2. Lemmas 2.3 and 2.4 establish that a paired graph G is DR-correct if and only if the graph $G[\forall \mapsto \setminus\!\cdot\,]$ of Definition 1.2 is a tree and its dependency graph satisfies SDAG. This yields a new correctness criterion for MLL-CORR and MELL-CORR given by Theorem 2.6.

Definition 2.1. Let G be a paired graph. The *dependency graph* $D(G)$ of G is the directed graph (V_G, E_G) defined as follows.

- $V_G = \{v \mid v \text{ is a pair node in } G\} \cup \{s\}$.
- Let x be a pair node in G, with premise nodes x_l and x_r. The edge $(s \to x)$ is in E_G if and only if the following hold.
  1. There exists an elementary path $p_x = x_l, \ldots, x_r$ in $G[\forall \mapsto \setminus\!\cdot\,]$.
  2. $x \notin p_x$, and for all pair nodes y in G, $y \notin p_x$.
- Let x be a pair node in G, with premise nodes x_l and x_r, and let $y \neq x$ be another pair node in G. The edge $(y \to x)$ is in E_G if and only if the following hold.
  1. There exists an elementary path $p_x = x_l, \ldots, x_r$ in $G[\forall \mapsto \setminus\!\cdot\,]$.
  2. $x \notin p_x$, and for every elementary path $p_x = x_l, \ldots, x_r$ in $G[\forall \mapsto \setminus\!\cdot\,]$ with $x \notin p_x$, $y \in p_x$.

For examples of MLL proof structures, corresponding paired graphs and their dependency graphs, see Fig. 3.

Lemma 2.2. *Let G and H be paired graphs, with $G \to_c H$. Then, $G[\forall \mapsto \setminus\!\cdot\,] \to_c^* H[\forall \mapsto \setminus\!\cdot\,]$, and $G[\forall \mapsto \setminus\!\cdot\,]$ is a tree if and only if $H[\forall \mapsto \setminus\!\cdot\,]$ is a tree.*

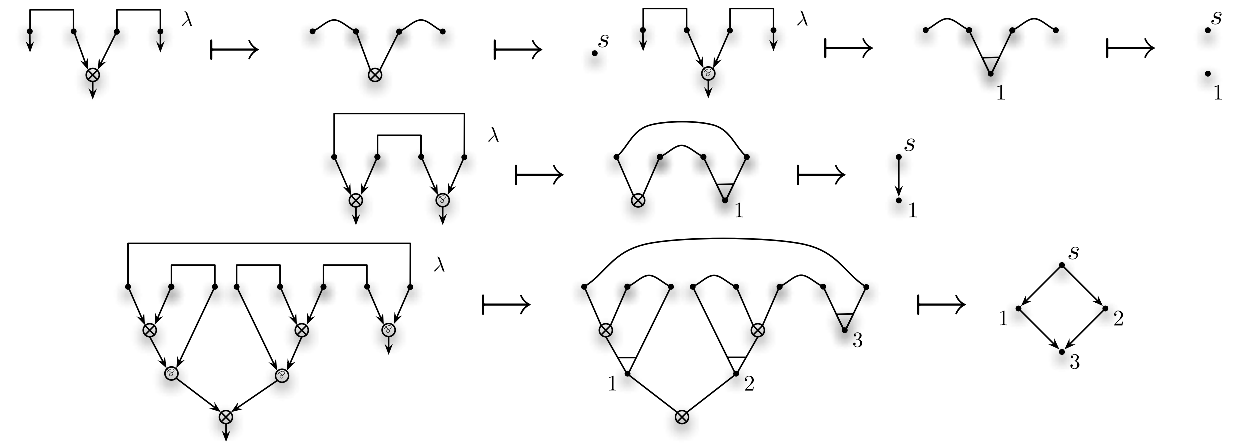

Fig. 3. MLL proof structures, corresponding paired graph and dependency graphs, for the sequents $A^{\perp}, A \otimes B, B^{\perp}$ (correct), $A^{\perp}, A ⅋ B, B^{\perp}$ (incorrect), $A \otimes B, A^{\perp} ⅋ B^{\perp}$ (correct) and $((A \otimes B) ⅋ B^{\perp}) \otimes (C^{\perp} ⅋ (C \otimes D)), D^{\perp} ⅋ A^{\perp}$ (correct).

Proof. If $G \to_{R_1} H$, denote by v the redex pair node in G, with premise w. The reduced pattern in H is the non-pair edge (v, w); therefore $G[\forall \mapsto \backslash\cdot] = H[\forall \mapsto \backslash\cdot]$. If $G \to_{R_2} H$, it is clear that $G[\forall \mapsto \backslash\cdot] \to_{R_2} H[\forall \mapsto \backslash\cdot]$ with the same redex. It is also clear that rule $\to_{R_2}$ preserves connectivity and acyclicity. $\square$

Lemma 2.3. *If $G \to_c^* \bullet$ then $D(G)$ satisfies* SDAG.

Proof. Since $\bullet[\forall \mapsto \backslash\cdot]$ is a tree, by Lemma 2.2 so is $G[\forall \mapsto \backslash\cdot]$. Therefore, for any pair node x with premise nodes x_l and x_r in G, there exists a unique elementary path $p_x = x_l - \cdots - x_r$ in $G[\forall \mapsto \backslash\cdot]$. It follows by construction of $D(G)$ that x has at least one parent node in $D(G)$. Moreover, a path $x \to \cdots \to y$ in $D(G)$ induces by construction an elementary path $x_l - \cdots - y$ in $G[\forall \mapsto \backslash\cdot]$. Therefore, a cycle $x \to \cdots \to y$, $y \to \cdots \to x$ in $D(G)$ induces a cycle $x_l - \cdots - y$, $y_l - \cdots - x$ in $G[\forall \mapsto \backslash\cdot]$. Since $G[\forall \mapsto \backslash\cdot]$ is a tree, $D(G)$ is acyclic. Since every vertex of $D(G)$ but s has at least one parent node and $D(G)$ is acyclic, $D(G)$ satisfies SDAG. $\square$

Lemma 2.4. *Let G be a paired graph such that $G[\forall \mapsto \backslash\cdot]$ is a tree. If the dependency graph $D(G)$ of G satisfies* SDAG *then $G \to_c^* \bullet$.*

Proof. Let $d(v)$, the depth of a pair node $v \in G$, be the length of the longest path from the source s of $D(G)$ to the vertex $v \in D(G)$. Assume that $D(G)$ satisfies SDAG, and let $X^d = \{x \text{ pair node in } G | d(x) = d\}$ and $Y^d = \cup_{d' \leqslant d} X^{d'}$.

By induction on the depth we prove that there exists a sequence of contractions $\mathcal{C}_d$ such that $G \to^{\mathcal{C}_d} G^d$ satisfies the following.

$$\text{Each pair node } y \in G \text{ s.t. } d(y) \leqslant d \text{ is contracted in } G^d. \tag{1}$$

The proof by induction is as follows.

- For $d = 1$, let $x \in X^1$, with premise nodes x_l and x_r. By definition of X^1, there exists an elementary path $p_x = x_l - \cdots - x_r$ in $G[\forall \mapsto \backslash\cdot]$ such that $x \notin p_x$, and for any pair node y in $G[\forall \mapsto \backslash\cdot]$, $y \notin p_x$. The same holds for the path $p_x = x_l - \cdots - x_r$ in G, with respect to any pair node $y \in G$.

  Let $\mathcal{E}_x^1 = \{e \text{ edge of } p_x \mid x \in X^1\}$. The set of contractions $\mathcal{R}_x^1 = \{e \to_c \bullet \mid e \in \mathcal{E}_x^1\}$ contracts the edges of p_x, and let $\mathcal{R}^1 = \cup_{x \in X^1} \mathcal{R}_x^1$. Clearly, $x_l = x_r \neq x$ in the contracted paired graph obtained from G by $\mathcal{R}_x^1$. Since $x \notin p_y$ for any $y \in X^1$, the same holds for the contracted paired graph obtained from G by $\mathcal{R}^1$.

  Let $\mathcal{C}_1$ be the sequence $\mathcal{R}^1$, followed by the set of contraction rules of the pair nodes $x \in X^1$. Define G^1 such that $G \to^{\mathcal{C}_1} G^1$. It is clear that G^1 satisfies (1).
- Assume by induction that there exists a sequence of contractions $\mathcal{C}_d$ such that $G \to^{\mathcal{C}_d} G^d$ satisfies (1).

  Let $x \in X^{d+1}$, with premise nodes x_l and x_r.

  Since $G \to^{\mathcal{C}_d} G^d$ and $G[\forall \mapsto \backslash\cdot]$ is a tree, Lemma 2.2 applies:

$$G[\forall \mapsto \backslash\cdot] \to^{\mathcal{C}'_d} G^d[\forall \mapsto \backslash\cdot], \quad \text{and} \quad G^d[\forall \mapsto \backslash\cdot] \text{ is a tree.} \tag{2}$$

  By definition of X^{d+1}, there exists an elementary path $p_x = x_l - \cdots - x_r$ in $G[\forall \mapsto \backslash\cdot]$ such that $x \notin p_x$ and, for every pair node $y \in G$ of depth $d(y) > d$, $y \notin p_x$.

  Define p_x^d such that $p_x \to^{\mathcal{C}'_d} p_x^d$. By (2), p_x^d is an elementary path in $G^d[\forall \mapsto \backslash\cdot]$ such that $x \notin p_x^d$ and, for every pair node $y \in G^d[\forall \mapsto \backslash\cdot]$ of depth $d(y) > d$, $y \notin p_x^d$. The same holds for p_x^d in G^d, with respect to any pair node $y \in G^d$, since, by induction, for any pair node $y \in G^d$, $d(y) > d$.

  Let $\mathcal{E}_x^{d+1} = \{e \text{ edge of } p_x \mid x \in X^{d+1}\}$. The set of contractions $\mathcal{R}_x^{d+1} = \{e \to_c \bullet \mid e \in \mathcal{E}_x^{d+1}\}$ contracts the edges of p_x^d, and let $\mathcal{R}^{d+1} = \cup_{x \in X^{d+1}} \mathcal{R}_x^{d+1}$. Clearly, $x_l = x_r \neq x$ in the contracted paired graph obtained from G by $\mathcal{R}_x^{d+1}$. Since $x \notin p_y$ for any $y \in X^{d+1}$, the same holds for the contracted paired graph obtained from G by $\mathcal{R}^{d+1}$.

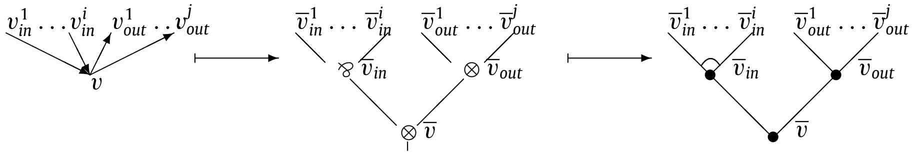

Fig. 4. Construction of $(\mathcal{S}_G, \lambda_G)$ and $G_{(\mathcal{S}_G,\lambda_G)}$.

Let $\mathcal{C}_{d+1}$ be the sequence $\mathcal{C}_d$, followed by $\mathcal{R}_{d+1}$, and followed by the set of contraction rules of the pair nodes $x \in X^{d+1}$. Define G^{d+1} such that $G \to^{\mathcal{C}_{d+1}} G^{d+1}$. G^{d+1} satisfies (1).

Since $D(G)$ satisfies SDAG, the maximal depth $m = max\{d(x)|x \in D(G)\}$ is well defined, and every pair node x of G belongs to X^m. Therefore, $G \to^{\mathcal{C}_m} G^m$ and G^m satisfies (1). Since $G[\forall \mapsto \backslash\!\cdot\,]$ is a tree, by Lemma 2.2 so is $G^m[\forall \mapsto \backslash\!\cdot\,] = G^m$. It follows that $G \to_c^* \bullet$. □

Define a paired graph G to be *DR-connected* if and only if, for any switching S of G, the switched graph $S(G)$ is connected. Lemmas 2.3 and 2.4 yields the following corollary.

Corollary 2.5. *A paired graph G is DR-connected if and only if its dependency graph has a node s from which every node is reachable.*

Proof. An induction on the number of edges shows that G is DR-connected if and only if there exists $G' \subseteq G$ DR-correct with the same set of vertices. By Lemmas 2.3 and 2.4, G' is DR-correct if and only if its dependency graph satisfies SDAG. Since the dependency graph of G' is a subgraph of the dependency graph of G, it follows that G is DR-connected if and only if its dependency graph has a node s from which every node is reachable. □

Lemmas 2.3 and 2.4 and Theorems 1.5 and 1.12 imply the following.

Theorem 2.6 (*Correctness Criteria*). *An* MLL *proof structure* $(\mathcal{S}, \lambda)$ *is an* MLL *proof net if and only if the following hold.*

1. $D(G_{(\mathcal{S},\lambda)})$ *satisfies* SDAG, *and*
2. $G_{(\mathcal{S},\lambda)}[\forall \mapsto \backslash\!\cdot\,]$ *is a tree.*

An MELL *proof structure* $(\mathcal{S}, B, \lambda)$ *with boxes* $b_1, \ldots, b_n$ *is an* MELL *proof net if and only if the following hold.*

1. $\forall i \in \{0, \ldots, n\}$, $D(G^i_{(\mathcal{S},B,\lambda)})$ *satisfies* SDAG, *and*
2. $\forall i \in \{0, \ldots, n\}$, $G^i_{(\mathcal{S},B,\lambda)}[\forall \mapsto \backslash\!\cdot\,]$ *is a tree.* □

2.2. NL-completeness of the criteria for MLL *and* MELL

Proposition 2.7. *MLL-*CORR *is NL-hard under constant-depth reductions.*

Proof. We actually reduce SDAG to MLL-CORR. Let G be a directed graph, and consider the proof structure $(\mathcal{S}_G, \lambda_G)$ defined as follows (see Fig. 4), and let $G_{(\mathcal{S}_G,\lambda_G)}$ be its associated paired graph.

1. To any vertex v of G, we associate a $\otimes$ node $\overline{v}$ with parent nodes $\overline{v}_{in}$ and $\overline{v}_{out}$.
2. If there are $i > 0$ in-going edges to v, $\overline{v}_{in}$ is a $⅋$-link of arity i, with parent nodes $\overline{v}^1_{in}, \ldots, \overline{v}^i_{in}$. $\overline{v}^1_{in}, \ldots, \overline{v}^i_{in}$ are axiom nodes. If v has no in-going edge, $\overline{v}_{in}$ is an axiom node, and λ_G contains a link $(\overline{v}_{in}, \overline{v}^2_{in})$, where $\overline{v}^2_{in}$ is a conclusion of $\mathcal{S}_G$.
3. If there are $j > 0$ outgoing edges from v, $\overline{v}_{out}$ is a $\otimes$-link of arity j, with parent links $\overline{v}^1_{out}, \ldots, \overline{v}^j_{out}$. $\overline{v}^1_{out}, \ldots, \overline{v}^j_{out}$ are axiom nodes. If v has no outgoing edge, $\overline{v}_{out}$ is an axiom node, and λ_G contains a link $(\overline{v}_{out}, \overline{v}^2_{out})$, where $\overline{v}^2_{out}$ is a conclusion of $\mathcal{S}_G$.
4. Let $v \to w$ be an edge of G, and assume that it is the kth outgoing edge from v and the lth in-going edge to w. To $v \to w$ we associate a link $(\overline{v}^k_{out}, \overline{w}^l_{in})$ in λ_G.

It is quite clear that this reduction can be computed by constant-depth circuits. We now claim that $(\mathcal{S}_G, \lambda_G)$ is correct if and only if G satisfies SDAG.

Assume that G contains a cycle. There exists then an elementary path $p = x_1 \to \cdots \to x_l$, with $x_l \to x_1 \in G$. Then, for any edge $x_t \to x_{t+1} \in p$, there exists a switching of the pair node $\overline{x_{t+1}}_{in}$ in $G_{(\mathcal{S}_G,\lambda_G)}$, which connects $\overline{x_t}$ and $\overline{x_{t+1}}$. Similarly for the edge $x_l \to x_1 \in G$. Since p is elementary, these pair nodes are all different; therefore, there exists cyclic switching of $G_{(\mathcal{S}_G,\lambda_G)}$, and $(\mathcal{S}_G, \lambda_G)$ is not correct.

It is clear that, if G is acyclic, it has at least one node of arity 0. Moreover, if G is acyclic and has only one node of arity 0, a proof by induction shows that G satisfies SDAG.

Assume therefore that G is acyclic and has at least two nodes, r and s, of arity 0. Let S' be any switching of $G_{(\mathcal{S}_G,\lambda_G)}$, and assume that there exists an elementary path p from $\overline{r}$ to $\overline{s}$ in S'. Let $p' = \overline{r}, \overline{x_1}, \ldots, \overline{x_k}, \overline{s}$ be the sequence of non pair nodes of p corresponding to vertices of G. p' follows by construction edges of G, accordingly to their orientation or not. Since r and s have arity 0, there exist three nodes $\overline{x_t}, \overline{x_{t+1}}, \overline{x_{t+2}}$ in p' such that $(x_t \to x_{t+1})$ and $(x_{t+2} \to x_{t+1})$ are edges of G. By construction of $G_{(\mathcal{S}_G,\lambda_G)}$, $\overline{x_t}$ and $\overline{x_{t+2}}$ are then premise nodes of the same pair node $\overline{x_{t+1}}_{in}$ in $G_{(\mathcal{S}_G,\lambda_G)}$, which contradicts that p is a path in S'. Therefore, S' is not connected, and $(\mathcal{S}_G, \lambda_G)$ is not correct.

Assume now that G satisfies SDAG, and let $d(v)$, the depth of a vertex v of G, be the length of the longest path from the source s of G to v. Denote by G^d the subgraph of G consisting only in the vertices of depth less than d, and by $G_{(\mathcal{S}_G,\lambda_G)^d}$ the corresponding paired graph. It is easy to see that the rules of Fig. 1 can be turned into an n-ary version, and that $G_{(\mathcal{S}_G,\lambda_G)^{d+1}}$ can be obtained from $G_{(\mathcal{S}_G,\lambda_G)^d}$ by these n-ary rules. By induction on d, it follows that $(\mathcal{S}_G, \lambda_G)$ is correct. $\square$

We denote by *FL* the class of functions computable in logarithmic working space (which is known to be stable under composition). Let DepGRAPH be the function $G \mapsto D(G)$, which associates its dependency graph to a paired graph G.

Lemma 2.8. DepGRAPH $\in$ *FL*.

Proof. The following functions can easily be computed in *FL*.

- $G, x \in G \mapsto (G[\forall \mapsto \cdot\!\backslash\!\cdot]) \setminus \{x\}$.
- $G, x \in G \mapsto (G[\forall \mapsto \because]) \setminus \{x\}$.
- $G, x \in G, y \in G \mapsto (G[\forall \mapsto \cdot\!\backslash\!\cdot]) \setminus \{x, y\}$.

Consider now the following algorithm for DepGRAPH.

INPUT (G)
FOR ALL x pair node in G, with premise nodes x_l and x_r DO
  IF USTCONN$((G[\forall \mapsto \because]) \setminus \{x\}, x_l, x_r)$ THEN OUTPUT $(s \to x) \in D(G)$
FOR ALL (x pair node in G, with premise nodes x_l and x_r, y pair node in G) DO
  IF $\neg$USTCONN$((G[\forall \mapsto \cdot\!\backslash\!\cdot]) \setminus \{x, y\}, x_l, x_r)$
  AND USTCONN$((G[\forall \mapsto \cdot\!\backslash\!\cdot]) \setminus \{x\}, x_l, x_r)$ THEN
    OUTPUT $(y \to \{x\}) \in D(G)$.

- USTCONN$((G[\forall \mapsto \because]) \setminus \{x\}, x_l, x_r)$ tests whether there exists an elementary path $p_x = x_l - \cdots - x_r$ such that $x \notin p_x$ and, for all pair nodes y in G, $y \notin p_x$.
- $\neg$ USTCONN$((G[\forall \mapsto \cdot\!\backslash\!\cdot]) \setminus \{x, y\}, x_l, x_r)$ tests whether any elementary path $p_x = x_l - \cdots - x_r$ such that $x \notin p_x$ contains y.
- USTCONN$((G[\forall \mapsto \cdot\!\backslash\!\cdot]) \setminus \{x\}, x_l, x_r)$ tests whether there exists a path $p_x = x_l - \cdots - x_r$ in G' such that $x \notin p_x$. From the previous point, if such a path p_x exists, $y \in p_x$.

It follows that this algorithm computes DepGRAPH. Since USTCONN $\in L$ [19], this algorithm belongs to FL^L (the class of functions computable in logspace with oracles in L). Since $FL^L = FL$, DepGRAPH $\in FL$. $\square$

Proposition 2.9. *MELL* − corr $\in$ *NL*.

Proof. Let $(\mathcal{S}, B, \lambda)$ be an MELL-proof structure with boxes $b_1, \ldots, b_n$. Each function $(\mathcal{S}, B, \lambda), i \in \{0, \ldots, n\} \mapsto G^i_{(\mathcal{S},B,\lambda)}$ can easily be computed in *FL*. Checking that $G^i_{(\mathcal{S},B,\lambda)}[\forall \mapsto \cdot\!\backslash\!\cdot]$ is a tree is doable in L since IT $\in L$. Checking that $D(G^i_{(\mathcal{S},B,\lambda)})$ satisfies SDAG can be done in *NL*, by composing the function DepGRAPH in *FL* (Lemma 2.8) with an *NL* algorithm for SDAG (Theorem 1.22).

Since the number of paired graphs $G^i_{(\mathcal{S},B,\lambda)}$ is linearly bounded, it suffices to sequentially perform these tasks for $i = 0, \ldots, n$, with a counter i of logarithmic size. $\square$

Note that the previous best algorithms for MELL-corr [14,7] are not likely to be implemented in logarithmic space, since they require on-line modification of the structure they manipulate. The purpose of our criterion of Theorem 2.6 is precisely that it allows a space-efficient implementation, at the cost of nonlinear (actually quadratic) time execution.

For MLL-corr, the linear-time algorithms for essential nets of [17,18] are actually *NL* algorithms. However, they do not yield *NL* algorithms for MLL proof structures, since the reduction they use is not computable in logarithmic space.

3. MALL

This section is devoted to the proof of the *NL*-completeness of MALL-corr. The situation for MALL differs quite a lot from the situation for MLL and MELL in the sense that the size of a sequent and of a corresponding proof structure – or proof net – may be of different order. For MLL and MELL, it is clear that the size of a proof structure is linear in the size of its skeleton. Yet, for MALL, the situation is more complex: while some MALL proof structures and proof nets have size linear in the size of their skeleton (e.g., pure MLL proof structures), others have size exponential in the size of their skeleton. Define the following correct sequents:

$$\Gamma_1 = A_1^\perp \oplus \cdots \oplus A_n^\perp, A_1 \& \cdots \& A_n$$
$$\Gamma_2 = A^\perp \oplus \cdots \oplus A^\perp, A \& \cdots \& A$$
$$\Sigma_1 = A_1^\perp \otimes \cdots \otimes A_n^\perp, A_1 \& A_1, \ldots, A_n \& A_n$$
$$\Sigma_2 = A^\perp \otimes \cdots \otimes A^\perp, A \& A, \ldots, A \& A.$$

For each of these sequents, the size of the corresponding cut-free skeleton is linear in n. The following table shows, for a cut-free MALL skeleton for each of these sequents, its number of additive resolutions, &-resolutions and possible links.

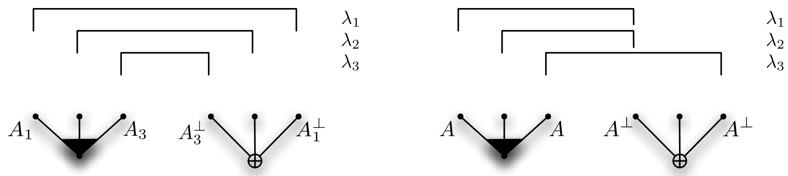

Fig. 5. The MALL proof net on Γ_1, and an example of proof net on Γ_2, with $n = 3$.

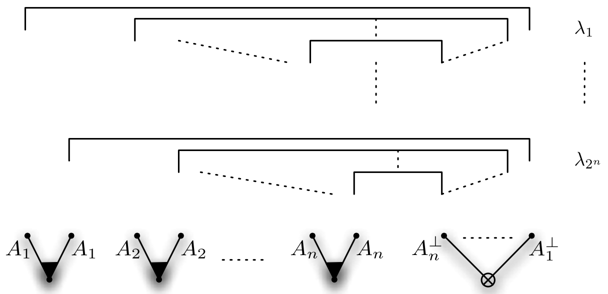

Fig. 6. The MALL proof net (Σ_1, Θ_1) on Σ_1, with $\Theta_1 = \bigcup_{i=1}^{2^n} \lambda_i$.

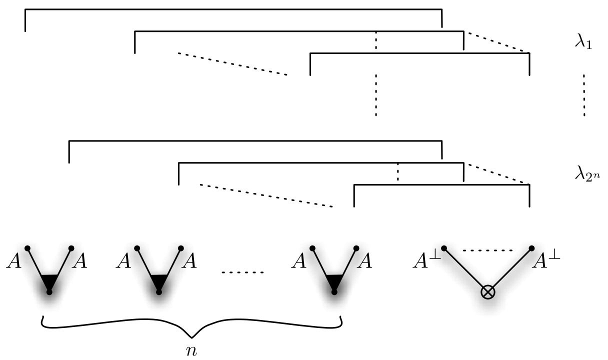

Fig. 7. An example of MALL proof net $(\Sigma_2, \Theta_{n!})$ on Σ_2, with $\Theta_{n!} = \bigcup_{i=1}^{2^n} \lambda_i$. Note that the set Θ_1 of Fig. 6 yields another proof net (Σ_2, Θ_1) on Σ_2, as well as the $n!$ possible combination of choices among the order in which the premises of the $\otimes$ node are linked to the $\&$ nodes.

The last two lines show the number of links in any cut-free MALL proof net, and the number of different cut-free MALL proof nets for each of these sequents.

sequent	Γ_1	Γ_2	Σ_1	Σ_2
# add-resolutions	n^2	n^2	2^n	2^n
# $\&$-resolutions	n	n	2^n	2^n
# links	n	n^2	2^n	$n!2^n$
$\lvert\Theta\rvert$	n	n	2^n	2
# Θ	1	n^2	1	$n!$

This table illustrates how some very simple MALL sequents can yield very large MALL proof nets. These proof nets are exemplified in Figs. 5–7. Here, the reader should keep in mind that the input to the MALL-corr problem is actually an MALL proof structure, of size maybe much larger than the size of the corresponding sequent. Recall from Theorem 1.17 that an MALL proof structure is a positive input to MALL-corr if and only if it satisfies Conditions (MLL), (RES) and (TOG). The *NL*-hardness of MALL-corr follows directly from the *NL*-hardness of MLL-corr (since MLL is a subsystem of MALL). The *NL*-membership of Condition (MLL) follows directly from the *NL*-membership of MELL-corr (and thus of MELL-corr). Therefore, proving the *NL*-membership of MALL-corr requires proving the *NL*-membership of (RES) and (TOG). We exhibit in this section algorithms for checking non-deterministically (RES) and (TOG) in space logarithmic in the size of the proof structure, which, in some cases, is actually polynomial in the size of the sequent.

3.1. Checking (RES)

We recall Condition (RES) of Theorem 1.17. For every $\&$-resolution W of $\mathcal{S}$, there exists a unique $\lambda \in \Theta$ such that $\lambda \sqsubseteq W$.

Let us illustrate the difficulty in checking (RES) on a simple example. Let us consider the proof structure (Σ_1, Θ), where Σ_1 is as above.
$A_1^{\perp} \otimes \cdots \otimes A_n^{\perp}, A_1 \& A_1, \ldots, A_n \& A_n$, and Θ is a subset of Θ_1 of Fig. 6 containing $n^{\lceil \log(n) \rceil}$ linkings. The size of (Σ_1, Θ) is therefore $\mathcal{O}(n^{\lceil \log(n) \rceil})$.

We have seen that the number of $\&$-resolutions of Σ_1 is 2^n. Enumerating (and explicitly describing) all $\&$-resolutions requires at least $\Omega(n)$ space, and is not feasible in space $\mathcal{O}(\log(n^{\lceil \log(n) \rceil})) = \mathcal{O}(\log(n)^2)$. Therefore an *NL* algorithm for (RES) may not proceed by first plainly enumerating all $\&$-resolutions.

The idea of our algorithm is to define a notion of distance of edition on the $\&$-resolutions such that one can pass from any $\&$-resolution to any other $\&$-resolution with intermediate steps of distance at most 1 (Condition L1). Lemma 3.6 shows that (RES) fails if there exists a $\&$-resolution W with $\lambda \sqsubseteq W$ at distance 1 to a $\&$-resolution W' with no $\lambda' \sqsubseteq W'$ (Condition L3). Note however that, as on (Σ_1, Θ), the working space may not be large enough to describe the $\&$-resolutions explicitly: instead, a $\&$-resolution W with $\lambda \sqsubseteq W$ is implicitly described by λ. The difficulty then is to describe a $\&$-resolution W' with no $\lambda' \sqsubseteq W'$. We establish in Lemma 3.9 that (RES) fails if there exists a $\&$-resolution W with $\lambda \sqsubseteq W$ at distance 1 to a $\&$-resolution W' with no $\lambda' \sqsubseteq W'$, where moreover W' can be implicitly described by λ and some $\&$ node (Condition L4). Our algorithm enumerates (in logarithmic space) the λ and the $\&$ nodes in search of such a configuration.

Definition 3.1 (*Condition L1*)**.** Let $(\mathcal{S}, \Theta)$ be an MALL proof structure.
For any $\&$-resolution W of $\mathcal{S}$, let $\mathsf{switch}_W : \{x_\& : \& \text{ node of } \mathcal{S}\} \to \{l, r\}$ be the following function:

$$\mathsf{switch}_W(x_\&) = \begin{cases} l & \text{if } x_\&^l \in W \text{ or } x_\& \notin W \\ r & \text{if } x_\&^r \in W. \end{cases}$$

Let $\mathcal{W}_\mathcal{S}$ be the set of $\&$-resolutions of $\mathcal{S}$.
Let $\mathcal{W}_\Theta = \{W \in \mathcal{W}_\mathcal{S} : \exists \lambda \in \Theta,\ \lambda \sqsubseteq W\}$.
We define the distance Dist on $\mathcal{W}_\mathcal{S}$ by

$$\mathsf{Dist}(W, W') = |\{x_\&\ \& \text{ node of } \mathcal{S} : \mathsf{switch}_W(x_\&) \neq \mathsf{switch}_{W'}(x_\&)\}|.$$

Let $\mathcal{W} \subseteq \mathcal{W}_\mathcal{S}$. We say that $\mathcal{W}$ satisfies Condition L1 if and only if

$$\forall W_0, W_k \in \mathcal{W}\ \exists W_1, \ldots, W_{k-1} \in \mathcal{W} \text{ s.t. } \mathsf{Dist}(W_i, W_{i+1})_{0 \leq i < k \leq 1}.$$

Lemma 3.2. *$\mathcal{W}_\mathcal{S}$ satisfies condition L1.*

Proof. By induction on the skeleton $\mathcal{S}$. □

Definition 3.3 (*Condition L2*)**.** Let $(\mathcal{S}, \Theta)$ be an MALL proof structure.

$(\mathcal{S}, \Theta)$ is said to satisfy Condition L2 if and only if, $\forall y_\oplus$ $\oplus$-nodes in $\mathcal{S}$, $\forall \lambda_1, \lambda_2 \in \Theta$ that toggle $y_\oplus$, there exists a $\&$ node $x_\&$ also toggled by $\{\lambda_1, \lambda_2\}$.

Lemma 3.4. *If $(\mathcal{S}, \Theta)$ is an* MALL *proof net, then it satisfies Condition L2.*

Proof. By induction on $(\mathcal{S}, \Theta)$, along Definition 1.15. The only critical case is that of a $\&$ rule:
if $(\mathcal{S} \uplus \mathcal{S}_A, \Theta_A)$, where $\mathcal{S}$ (respectively $\mathcal{S}_A$) has conclusions Γ (respectively A) and $(\mathcal{S} \uplus \mathcal{S}_B, \Theta_B)$, where $\mathcal{S}_B$ has conclusion B are MALL proof nets, then $(\mathcal{S} \uplus \mathcal{S}', \Theta_A \uplus \Theta_B)$, where $\mathcal{S}'$ is $\mathcal{S}_A \uplus \mathcal{S}_B$ extended with a $\&$-node of premises A and B, is an MALL proof net with conclusions $\Gamma, A \& B$.

Two cases arise:

1. Assume there exist a $\oplus$ node $y_\oplus \in \mathcal{S}$, $\lambda \in \Theta_A$, and $\lambda' \in \Theta_A$ such that λ, λ' toggle $y_\oplus$. Then the induction hypothesis on $(\mathcal{S} \uplus \mathcal{S}_A, \Theta_A)$ ensures that there exists a $\&$ node $x_\& \in \mathcal{S} \uplus \mathcal{S}_A$ also toggled by λ, λ'. Similarly for $\lambda \in \Theta_B, \lambda' \in \Theta_B$.
2. Assume there exist a $\oplus$ node $y_\oplus \in \mathcal{S}$, $\lambda \in \Theta_A$, and $\lambda' \in \Theta_B$ such that λ, λ' toggle $y_\oplus$. Then the $\&$ node of premises A and B in $\mathcal{S}'$ is also toggled by λ, λ'. □

Definition 3.5 (*Condition L3*)**.** Let $(\mathcal{S}, \Theta)$ be an MALL proof structure.
Let $\lambda \in \Theta$, and define $\mathcal{S} \downharpoonleft_\& \lambda = \{W \in \mathcal{W}_\mathcal{S} : \lambda \sqsubseteq W\}$.
Let $x_\&$ be a $\&$ node in $\mathcal{S}$.

$(\lambda, x_\&)$ are said to satisfy Condition L3 in $(\mathcal{S}, \Theta)$ if and only if

$$\exists W_+^\lambda \in \mathcal{S} \downharpoonleft_\& \lambda, W_-^\lambda \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta \ \text{ s.t. } \mathsf{Dist}(W_+^\lambda, W_-^\lambda) = 1 \text{ and } \mathsf{switch}_{W_+^\lambda}(x_\&) \neq \mathsf{switch}_{W_-^\lambda}(x_\&).$$

Lemma 3.6. *Assume that $(\mathcal{S}, \Theta)$ is an* MALL *proof structure. Then, $(\mathcal{S}, \Theta)$ satisfies (RES) of* Theorem 1.17 *if and only if the following hold.*

1. $\forall\lambda, \lambda' \in \Theta,\ \ \lambda \neq \lambda' \Rightarrow \mathcal{S} \downharpoonleft \lambda \neq \mathcal{S} \downharpoonleft \lambda'$, *and*
2. $\forall\lambda \in \Theta,\ \forall x_{\&}$ $\&$ *node in* $\mathcal{S}$, $(\lambda, x_{\&})$ *does not satisfy L3 in* $(\mathcal{S}, \Theta)$.

Proof. 1. Let $W \in \mathcal{W}_\Theta$ and let $\lambda \in \Theta$ s.t. $\lambda \sqsubseteq W$. By induction on W, if there exists $\lambda' \neq \lambda$ s.t. $\lambda' \sqsubseteq W$, then $\mathcal{S} \downharpoonleft \lambda = \mathcal{S} \downharpoonleft \lambda'$. It follows that (1) above is equivalent to the unicity, for any $\&$-resolution W of $\mathcal{S}$, of a $\lambda \in \Theta$ such that $\lambda \sqsubseteq W$.

2. Assume that there exists a $\&$-resolution W of $\mathcal{S}$ s.t. $\forall\lambda \in \Theta, \lambda \not\sqsubseteq W$. Then, $\mathcal{W}_\Theta \subsetneq \mathcal{W}_\mathcal{S}$. Assume that $\Theta \neq \emptyset$; then, $\mathcal{W}_\Theta \neq \emptyset$. Therefore, there exist $W_+ \in \mathcal{W}_\Theta$ and $W_- \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta$. By Lemma 3.4, there then exist $W_1, \ldots, W_k \in \mathcal{W}$ s.t. $\mathsf{Dist}(W_+, W_1) \leq 1$, $\mathsf{Dist}(W_i, W_{i+1})_{0\leq i<k} \leq 1$, and $\mathsf{Dist}(W_k, W_-) \leq 1$. Since any of the W_i belongs either to $\mathcal{W}_\Theta$ or to $\mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta$, there exist $W'_+, W'_- \in \{W_+, W_1, \ldots, W_k, W_-\}$ such that $\mathsf{Dist}(W'_+, W'_-) = 1$, $W'_+ \in \mathcal{W}_\Theta$ and $W'_- \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta$. Let $\lambda \in \Theta$ such that $\lambda \sqsubseteq W'_+$, and let $x_{\&}$ be the $\&$ node such that $\mathsf{switch}_{W'_+}(x_{\&}) \neq \mathsf{switch}_{W'_-}(x_{\&})$. Clearly, $(\lambda, x_{\&})$ satisfy Condition L3.
Conversely, if there exist $\lambda \in \Theta$ and $x_{\&}$ a $\&$ node in $\mathcal{S}$ such that $(\lambda, x_{\&})$ satisfies Condition L3 in $(\mathcal{S}, \Theta)$, then there exists a $\&$-resolution W of $\mathcal{S}$ s.t. $\forall\lambda \in \Theta, \lambda \not\sqsubseteq W$. It follows that (2) above is equivalent to the existence, for any $\&$-resolution W of $\mathcal{S}$, of a $\lambda \in \Theta$ such that $\lambda \sqsubseteq W$. $\square$

Definition 3.7 (*Condition L4*)**.** Let $(\mathcal{S}, \Theta)$ be an MALL proof structure.
Let $x_{\&}$ be a $\&$ node in $\mathcal{S}$. Define

$$\mathcal{W}^l_{x_{\&}} = \{W \in \mathcal{W}_\mathcal{S} \text{ s.t. } \forall x'_{\&} \text{ s.t. there exists a path } x'_{\&} \to \cdots \to x^l_{\&}, \mathsf{switch}_W(x'_{\&}) = l\}$$
$$\mathcal{W}^r_{x_{\&}} = \{W \in \mathcal{W}_\mathcal{S} \text{ s.t. } \forall x'_{\&} \text{ s.t. there exists a path } x'_{\&} \to \cdots \to x^r_{\&}, \mathsf{switch}_W(x'_{\&}) = l\}$$

Let $\lambda \in \Theta$, and define

$$\mathsf{Mirror}(\lambda, x_{\&}) = \{W \in \mathcal{W}_\mathcal{S} \text{ s.t. } \exists W' \in \mathcal{S} \downharpoonleft_{\&} \lambda \cap \mathcal{W}^l_{x_{\&}} \cap \mathcal{W}^r_{x_{\&}} : \mathsf{Dist}(W, W') = 1 \text{ and } \mathsf{switch}_W(x_{\&}) \neq \mathsf{switch}_{W'}(x_{\&})\}.$$

$(\lambda, x_{\&})$ are said to satisfy Condition L4 in $(\mathcal{S}, \Theta)$ if and only if

$$\forall\lambda' \in \Theta, \forall W \in \mathsf{Mirror}(\lambda, x_{\&}), \lambda' \not\sqsubseteq W.$$

Lemma 3.8. *Assume that* $(\mathcal{S}, \Theta)$ *is an* MALL *proof structure satisfying Condition L2. Let* $\lambda \in \Theta$ *and* $x_{\&}$ *be a* $\&$ *node in* $\mathcal{S}$ *such that the following hold.*

1. $(\lambda, x_{\&})$ *satisfies Condition L3 in* $(\mathcal{S}, \Theta)$, *and*
2. $\forall y_\oplus$ $\oplus$ *node in* $\mathcal{S} \downharpoonleft \lambda$, $\forall\lambda' \in \Theta$ *such that* λ, λ' *toggle* $y_\oplus$, $x_{\&}$ *is not toggled by* λ, λ'.

Then, $(\lambda, x_{\&})$ *satisfies Condition L4 in* $(\mathcal{S}, \Theta)$.

Proof. Let $y_\oplus$ be a $\oplus$ node in $\mathcal{S} \downharpoonleft \lambda$. Without loss of generality, assume that $y^l_\oplus \in \mathcal{S} \downharpoonleft \lambda$ and $x^l_{\&} \in \mathcal{S} \downharpoonleft \lambda$. Assume that $(\lambda, x_{\&})$ satisfies Condition L3 in $(\mathcal{S}, \Theta)$:

$$\exists W^\lambda_+ \in \mathcal{S} \downharpoonleft_{\&} \lambda, W^\lambda_- \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta \text{ s.t. } \mathsf{Dist}(W^\lambda_+, W^\lambda_-) = 1 \text{ and } \mathsf{switch}_{W^\lambda_+}(x_{\&}) \neq \mathsf{switch}_{W^\lambda_-}(x_{\&}).$$

Let $\theta_\lambda = \{\lambda_i \in \Theta :\ \lambda_i \sqsubseteq W_i \in \mathsf{Mirror}(\lambda, x_{\&})\}$.
Assume by contradiction that $\theta_\lambda \neq \emptyset$.

Let us show by contradiction that, for all $\lambda' \in \theta_\lambda$, $y^r_\oplus \notin \mathcal{S} \downharpoonleft \lambda'$. Assume that $\exists\lambda' \in \theta_\lambda, y^r_\oplus \in \mathcal{S} \downharpoonleft \lambda'$. Then λ, λ' toggle $y_\oplus$. By Condition L2, there exists a $\&$ node $x'_{\&} \neq x_{\&}$ also toggled by λ, λ'. Assume without loss of generality that ${x'_{\&}}^l \in \mathcal{S} \downharpoonleft \lambda$ and ${x'_{\&}}^r \in \mathcal{S} \downharpoonleft \lambda'$.

Since ${x'_{\&}}^l \in \mathcal{S} \downharpoonleft \lambda$, for all $W \in \mathsf{Mirror}(\lambda, x_{\&})$, $\mathsf{switch}_W(x'_{\&}) = l$. Since ${x'_{\&}}^r \in \mathcal{S} \downharpoonleft \lambda'$, for any $W' \in \mathsf{Mirror}(\lambda, x_{\&})$ s.t. $\lambda' \sqsubseteq W'$, $\mathsf{switch}_{W'}(x'_{\&}) = r$: contradiction.

Therefore, for all $\lambda' \in \theta_\lambda$, $y^r_\oplus \notin \mathcal{S} \downharpoonleft \lambda'$.

Let $\lambda' \in \theta_\lambda$ and let $x'_{\&}$ (respectively $y'_\oplus$) be any $\&$ node (respectively $\oplus$ node) such that there exists no path $x'_{\&} \to \cdots \to x_{\&}$ (respectively $y'_\oplus \to \cdots \to x_{\&}$). Then, by induction on $\mathcal{S}$,

$$x'_{\&} \in \mathcal{S} \downharpoonleft \lambda \Rightarrow x'_{\&} \in \mathcal{S} \downharpoonleft \lambda', y'_\oplus \in \mathcal{S} \downharpoonleft \lambda \Rightarrow y'_\oplus \in \mathcal{S} \downharpoonleft \lambda',$$
$${x'_{\&}}^l \in \mathcal{S} \downharpoonleft \lambda \Rightarrow {x'_{\&}}^l \in \mathcal{S} \downharpoonleft \lambda', {y'_\oplus}^l \in \mathcal{S} \downharpoonleft \lambda \Rightarrow {y'_\oplus}^l \in \mathcal{S} \downharpoonleft \lambda',$$
$${x'_{\&}}^r \in \mathcal{S} \downharpoonleft \lambda \Rightarrow {x'_{\&}}^r \in \mathcal{S} \downharpoonleft \lambda', {y'_\oplus}^r \in \mathcal{S} \downharpoonleft \lambda \Rightarrow {y'_\oplus}^r \in \mathcal{S} \downharpoonleft \lambda'.$$

It follows that $\lambda' \sqsubseteq W^\lambda_-$: contradiction. $\square$

Lemma 3.9. *Assume* $(\mathcal{S}, \Theta)$ *is an* MALL *proof structure satisfying L2. Let* $\lambda \in \Theta$ *and let* $x_{\&}$ *be a* $\&$ *node in* $\mathcal{S}$ *such that*

1. $(\lambda, x_{\&})$ *satisfy Condition L3 in* $(\mathcal{S}, \Theta)$, *and*
2. $\exists y_\oplus$ $\oplus$ *node in* $\mathcal{S} \downharpoonleft \lambda$, *and* $\lambda' \in \Theta$ *such that* λ, λ' *toggle both* $y_\oplus$ *and* $x_{\&}$.

Then, there exists $x'_{\&}$ $\&$ *node in* $\mathcal{S}$ *such that* $(\lambda', x'_{\&})$ *satisfies Condition L4 in* $(\mathcal{S}, \Theta)$.

Proof. By induction on the maximal number of $\&$ and $\oplus$ nodes traversed along a path $x \to \cdots \to x_\&$ or $x \to \cdots \to y_\oplus$ in $\mathcal{S}$. Since $\mathcal{S}$ is acyclic, this number is well defined. Assume that $(\lambda, x_\&)$ satisfies Condition L3 in $(\mathcal{S}, \Theta)$:

$$\exists W_+^\lambda \in \mathcal{S}\downharpoonright_\& \lambda, W_-^\lambda \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta \text{ s.t. } \mathsf{Dist}(W_+^\lambda, W_-^\lambda) = 1 \text{ and } \mathsf{switch}_{W_+^\lambda}(x_\&) \neq \mathsf{switch}_{W_-^\lambda}(x_\&).$$

Without loss of generality, assume that $y_\oplus^l \in \mathcal{S}\downharpoonright\lambda$ and $x_\&^l \in \mathcal{S}\downharpoonright\lambda$.

Let $\theta_\lambda = \{\lambda_i \in \Theta : \lambda_i \sqsubseteq W_i \in \mathsf{Mirror}(\lambda, x_\&)\}$. If there is no $\&$ or $\oplus$ node along any path $x \to \cdots \to x_\&$ or $x \to \cdots \to y_\oplus$, $\theta_\lambda = \emptyset$. If $\theta_\lambda = \emptyset$, $(\lambda, x_\&)$ satisfies Condition L4 in $(\mathcal{S}, \Theta)$. Assume in the following that $\theta_\lambda \neq \emptyset$.

1. Let $y'_\oplus$ be a $\oplus$ node in $\mathcal{S}\downharpoonright\lambda$ such that there exists no path $y'_\oplus \to \cdots \to y_\oplus$ and no path $y'_\oplus \to \cdots \to x_\&$.
   Let us show by contradiction that $y'_\oplus$ is toggled by no (λ, λ_i), $\lambda_i \in \theta_\lambda$.
   Assume that $y'_\oplus$ is toggled by (λ, λ_i), $\lambda_i \in \theta_\lambda$, and, without loss of generality, that ${y'_\oplus}^l \in \mathcal{S}\downharpoonright\lambda$, ${y'_\oplus}^r \in \mathcal{S}\downharpoonright\lambda_i$. Then, by Condition L2, there exists a $\&$ node $x'_\& \in \mathcal{S}\downharpoonright\lambda \cap \mathcal{S}\downharpoonright\lambda_i$ toggled by (λ, λ_i), and, without loss of generality, ${x'_\&}^l \in \mathcal{S}\downharpoonright\lambda$ and ${x'_\&}^r \in \mathcal{S}\downharpoonright\lambda_i$. Let W'_i be any $\&$-resolution such that $\lambda_i \sqsubseteq W'_i$: $\forall W \in \mathcal{S}\downharpoonright_\& \lambda \cap \mathcal{W}^l_{x_\&} \cap \mathcal{W}^r_{x_\&}$ $x^l_\& \in W$, ${x'_\&}^l \in W$, $x^r_\& \in W'_i$, ${x'_\&}^r \in W'_i$, and $\mathsf{Dist}(W, W') \geq 1$. Therefore, W'_i cannot possibly be in $\mathsf{Mirror}(\lambda, x_\&)$, which contradicts the hypothesis that $y'_\oplus$ is toggled by (λ, λ_i), $\lambda_i \in \theta_\lambda$.
2. By Condition L3, $\forall \lambda_i \in \theta_\lambda$, $\exists (x_i, y_i) \in \lambda_i : x_i \notin W_-^\lambda$. Let us show that, $\forall (x_i, y_i) \in \lambda_i \in \theta_\lambda$, $x_i \notin W_-^\lambda$, there exists a path $x_i \to \cdots \to y^r_\oplus$ or a path $x_i \to \cdots \to x^r_\&$.
   Assume that there exists no such path. For any $\oplus$ node $y'_\oplus$ such that there exists a path $x_i \to \cdots \to y'_\oplus$, there exists no path $y'_\oplus \to \cdots \to y_\oplus$ and no path $y'_\oplus \to \cdots \to x_\&$. By (1) above, $y'_\oplus$ is toggled by no (λ, λ_i), $\lambda_i \in \theta_\lambda$. Moreover, for any $\&$ node $x'_\&$ such that there exists a path $x_i \to \cdots \to x'_\&$, there exists no path $x'_\& \to \cdots \to x_\&$. By definition of θ_λ, $x'_\&$ is then toggled by no (λ, λ_i), $\lambda_i \in \theta_\lambda$, and $x_i \in \mathcal{S}\downharpoonright\lambda$. Therefore, $\forall W' \in \mathcal{S}\downharpoonright_\& \lambda$, $x_i \in W'$. By Condition L3, there exists $W_+^\lambda \in \mathcal{S}\downharpoonright_\& \lambda$ s.t. $\mathsf{Dist}(W_+^\lambda, W_-^\lambda) = 1$ and $\mathsf{switch}_{W_+^\lambda}(x_\&) \neq \mathsf{switch}_{W_-^\lambda}(x_\&)$. Since $x_i \in W_+^\lambda$, and since there exists no path $x_i \to \cdots \to x_\&$, it follows that $x_i \in W_-^\lambda$: contradiction.
3. By hypothesis, $W_+^\lambda \in \mathcal{S}\downharpoonright_\& \lambda$, and $\mathsf{switch}_{W_+^\lambda}(x_\&) = l$. Since $\mathsf{Dist}(W_+^\lambda, W_-^\lambda) = 1$ and $\mathsf{switch}_{W_-^\lambda}(x_\&) = r$, it follows that $W_+^\lambda \in \mathcal{W}^l_{x_\&} \cap \mathcal{W}^r_{x_\&}$, and therefore $W_-^\lambda \in \mathsf{Mirror}(\lambda, x_\&)$.
4. It is clear that $\mathcal{S}\downharpoonright_\& \lambda$, $\mathcal{W}^l_{x_\&}$ and $\mathcal{W}^r_{x_\&}$ satisfy condition L1. Therefore, so does $\mathsf{Mirror}(\lambda, x_\&)$. Since $W_-^\lambda \in \mathsf{Mirror}(\lambda, x_\&)$ and $\theta_\lambda \neq \emptyset$, there exist $W_+^{\lambda_i}, W_-^{\lambda_i} \in \mathsf{Mirror}(\lambda, x_\&)$, $\lambda_i \in \theta_\lambda$ such that $\lambda_i \sqsubseteq W_+^{\lambda_i}$, $W_-^{\lambda_i} \in \mathcal{W}_\mathcal{S} \setminus \mathcal{W}_\Theta$ and $\mathsf{Dist}(W_+^{\lambda_i}, W_-^{\lambda_i}) = 1$. Let $x'_\&$ be the unique $\&$ node in $\mathcal{S}$ such that $\mathsf{switch}_{W_+^{\lambda_i}}(x'_\&) \neq \mathsf{switch}_{W_-^{\lambda_i}}(x'_\&)$. By (2) above, there exists a path $x'_\& \to \cdots \to y_\oplus$. If there exists a $\oplus$ node $y'_\oplus$ in $\mathcal{S}\downharpoonright\lambda_i$ and $\lambda_j \in \theta_\lambda$ such that λ_i, λ_j toggle both $x'_\&$ and $y'_\oplus$, by (1) above, there exists a path $y'_\oplus \to \cdots \to y_\oplus$ or a path $y'_\oplus \to \cdots \to x_\&$. Therefore we can apply the induction hypothesis to conclude that $(\lambda', x'_\&)$ satisfies Condition L4 in $(\mathcal{S}, \Theta)$. $\square$

Proposition 3.10. *Assume that $(\mathcal{S}, \Theta)$ is an* MALL *proof structure. Then, $(\mathcal{S}, \Theta)$ satisfies (RES) of Theorem 1.17 if and only if the following hold.*

1. $\forall \lambda, \lambda' \in \Theta$, $\lambda \neq \lambda' \Leftrightarrow \mathcal{S}\downharpoonright\lambda \neq \mathcal{S}\downharpoonright\lambda'$,
2. $(\mathcal{S}, \Theta)$ *satisfies Condition L2, and*
3. $\forall \lambda \in \Theta$, $\forall x_\&$ *$\&$ node in $\mathcal{S}$, $(\lambda, x_\&)$ does not satisfy L4 in $(\mathcal{S}, \Theta)$.*

Proof. Apply Lemmas 3.6, 3.8 and 3.9. $\square$

A consequence of Proposition 3.10 is an *NL* algorithm deciding whether a given MALL proof structure satisfies (RES). Indeed, by Proposition 2.9, (1) can be checked in *NL*, and Conditions L2 and L4 can easily be checked in *NL* by parsing the set of linkings and the skeleton.

3.2. Checking (TOG)

We recall Condition (TOG) of Theorem 1.17.

For every $\Lambda \subseteq \Theta$ of two or more linkings, Λ toggles a $\&$ node $x_\&$ such that $x_\&$ does not belong to any switching cycle of $H_{\mathcal{S}\downharpoonright\Lambda}$.

Checking Condition (TOG) in non-deterministic logarithmic space involves two difficulties, which we address in this section.

1. The number of sets $\Lambda \subseteq \Theta$ of two or more linkings is exponential in the size of Θ, i.e., exponential in the size of the input in the worst case. Consider for instance the sequent $\Gamma = A\&\cdots\&A, A^\perp$ of Fig. 8: a proof net (Γ, Θ) contains n linkings, each linking containing a single link. The number of sets $\Lambda \subseteq \Theta$ of two or more linkings is then $2^n - n - 1$. Clearly, there is no possibility to enumerate all the sets $\Lambda \subseteq \Theta$ of two or more linkings in logarithmic space.[2] Lemma 3.12 below shows that it is actually enough to consider only a quadratic number of well-chosen such sets of linkings.

[2] It is mentioned in [9] that it suffices to check (TOG) merely for *saturated* sets Λ of linkings only, namely, such that any strictly larger subset of Θ toggles more $\&$ nodes than Λ. Note however that the saturated sets of linkings are also exponentially many, and cannot be enumerated in logspace.

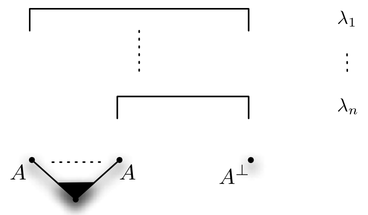

Fig. 8. A proof net (Γ, Θ), with $\Theta = \bigcup_{i=1}^{n} \lambda_i$.

2. Given a set $\Lambda \subseteq \Theta$ of two or more linkings and a $\&$ node $x_\&$ toggled by Λ, it remains to be checked whether $x_\&$ belongs to a switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$. In the worst case, the number of switched graphs of $H_{\mathcal{S}\lfloor\Lambda}$ to be investigated may be also exponential in the size of the input. Moreover, it is unclear whether $H_{\mathcal{S}\lfloor\Lambda}$ enjoys properties such as DR-correctness that allow space-efficient algorithms. Lemma 3.17 below shows that the switching cycles of $H_{\mathcal{S}\lfloor\Lambda}$ are actually the switching cycles of a graph $I_{\mathcal{S}\lfloor\Lambda}$ which, in turns, enjoys the property of being DR-connected.

The two points above are necessary stepping stones towards an *NL* algorithm for condition (TOG) exhibited in Proposition 3.18.

Definition 3.11. Let $\{\lambda_1, \lambda_2\} \subseteq \Theta$; we define $\Theta_{\lambda_1,\lambda_2} = \{\lambda \in \Theta : \mathcal{S} \lfloor \lambda_1 \cap \mathcal{S} \lfloor \lambda_2 \subseteq \mathcal{S} \lfloor \lambda\}$.

Lemma 3.12. *Let $(\mathcal{S}, \Theta)$ be an* MALL *proof structure satisfying (RES).*

$(\mathcal{S}, \Theta)$ satisfies (TOG) if and only if, for all $\{\lambda_1, \lambda_2\} \subseteq \Theta$, there exists a $\&$ node $x_\&$ toggled by λ_1, λ_2 such that $x_\&$ does not belong to any switching cycle of $H_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$.

Proof. In a first step, we show by induction on $\mathcal{S} \setminus (\mathcal{S} \lfloor \lambda_1 \cap \mathcal{S} \lfloor \lambda_2)$ that, for all $\Lambda \subseteq \Theta_{\lambda_1,\lambda_2}$ with at least two linkings, Λ toggles a $\&$ node $x'_\&$ such that $x'_\&$ does not belong to any switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$.

Let $\lambda_1, \lambda_2 \in \Theta$, let $x_\&$ be a $\&$-node toggled by $\{\lambda_1, \lambda_2\}$ and let $\Lambda \subseteq \Theta_{\lambda_1,\lambda_2}$. Then, $H_{\mathcal{S}\lfloor\Lambda} \subseteq H_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$, and the switching cycles of $H_{\mathcal{S}\lfloor\Lambda}$ are switching cycles of $H_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$.

1. If Λ toggles $x_\&$, then $x_\&$ belongs to no switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$ (otherwise it would belong to a switching cycle of $H_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$).
2. Assume that Λ does not toggle $x_\&$. Then, $(\mathcal{S} \lfloor \lambda_1 \cap \mathcal{S} \lfloor \lambda_2) \subsetneq \bigcap_{\lambda\in\Lambda} \mathcal{S} \lfloor \lambda$.

   Let W^l_Λ be the $\&$-resolution of $\mathcal{S}$ defined as follows:

$$\bigcap_{\lambda\in\Lambda} \mathcal{S} \lfloor \lambda \subseteq W^l_1, \text{ and}$$
$$\forall \& \text{ node } x'_\& \in \mathcal{S}, x'_\& \notin \bigcap_{\lambda\in\Lambda} \mathcal{S} \lfloor \lambda \Rightarrow x'^r_\& \text{ is erased in } W^l_1,$$

   and let W^r_Λ be defined as follows:

$$\bigcap_{\lambda\in\Lambda} \mathcal{S} \lfloor \lambda \subseteq W^r_1, \text{ and}$$
$$\forall \& \text{ node } x'_\& \in \mathcal{S}, x'_\& \notin \bigcap_{\lambda\in\Lambda} \mathcal{S} \lfloor \lambda \Rightarrow x'^l_\& \text{ is erased in } W^r_1.$$

   By Condition (RES), there exist $\lambda^l, \lambda^r \in \Theta$ s.t. $\lambda^l \sqsubseteq W^l_\Lambda$ and $\lambda^r \sqsubseteq W^r_\Lambda$. Then, clearly, $\Lambda \subseteq \Theta_{\lambda^l,\lambda^r} \subsetneq \Theta_{\lambda_1,\lambda_2}$. Since $|\Theta_{\lambda^l,\lambda^r}| > 2$, by Condition (RES), $\Theta_{\lambda^l,\lambda^r}$ toggles a $\&$ node $x'_\& \neq x_\&$. By construction, $x'_\&$ is also toggled by Λ. The induction hypothesis on $\Theta_{\lambda^l,\lambda^r}$ and the arguments of (1) above yield that $x'_\&$ belongs to no switching cycle of $H_{\mathcal{S}\lfloor\Lambda}$.

The second step is to show that there exist $\lambda_1, \lambda_2 \in \Theta$ s.t. $\Theta = \Theta_{\lambda_1,\lambda_2}$. Consider W_l the $\&$-resolution of $\mathcal{S}$, where all right premises of $\&$ nodes are erased, and W_r the one where all left premises of $\&$ nodes are erased. By Condition (RES), there exist $\lambda_1, \lambda_2 \in \Theta$ such that $\lambda_1 \sqsubseteq W_l$ and $\lambda_2 \sqsubseteq W_r$. It is clear that, for all $\lambda \in \Theta$, $\mathcal{S} \lfloor \lambda_1 \cap \mathcal{S} \lfloor \lambda_2 \subseteq \mathcal{S} \lfloor \lambda$. Therefore, $\Theta \subseteq \Theta_{\lambda_1,\lambda_2}$. □

Definition 3.13. Let $(\mathcal{S}, \Theta)$ be an MALL proof structure.

Let $x_\&$ be a $\&$ node in $\mathcal{S}$. $x_\&$ is said to be *environment free* if, for all $\lambda \in \Theta$, and for all links $(a, b) \in \lambda$, there exists a path $a \to \cdots \to x_\&$ if and only if there exists a path $b \to \cdots \to x_\&$. If $x_\&$ is not environment free, it is said to be environment linked.

Lemma 3.14. *If $(\mathcal{S}, \Theta)$ is an* MALL *proof net then, for all $\&$ nodes $x_\&$, $x_\&$ is environment free if and only if, for any sequentialization of $(\mathcal{S}, \Theta)$, any $\&$-rule applied on $x_\&$ has an empty environment Γ.*

Proof. Straightforward proof by induction. □

Definition 3.15. Let $(\mathcal{S}, \Theta)$ be an MALL proof structure.

Let $I_{\mathcal{S}\lfloor\Lambda}$ be $G_{\mathcal{S}\lfloor\Lambda}$ extended with all admissible jump edges for all $\{\lambda_1, \lambda_2\} \subseteq \Lambda$ and where $C(I_{\mathcal{S}\lfloor\Lambda})$ contains the premise – and jump – edges incident to all $⅋$ nodes and environment-linked $\&$ nodes of $\mathcal{S} \lfloor \Lambda$, and the jump edges only incident to all environment-free $\&$ nodes of $\mathcal{S} \lfloor \Lambda$.

Lemma 3.16. *If* $(\mathcal{S}, \Theta)$ *is an* MALL *proof net then, for all* $\{\lambda_1, \lambda_2\} \subseteq \Theta$, $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}}$ *is DR-connected.*

Proof. We actually prove the lemma for the graph $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}}$ without jumps. An easy graph-theoretic proof by induction shows that adding the jumps does not DR-disconnect the paired graph.

The proof is by induction on $(\mathcal{S}, \Theta)$, along Definition 1.15. The only critical case is that of a $\&$ rule on $\Gamma, A\&B$, where the $\&$ node $x_\&$ introduced by the rule is environment linked and is toggled by λ_1, λ_2. Assume without loss of generality that $x^l_\& \in \mathcal{S} \downharpoonleft \lambda_1$ and $x^r_\& \in \mathcal{S} \downharpoonleft \lambda_2$.

By Definition 1.15, $\Theta = \Theta_A \uplus \Theta_B$, and $\mathcal{S}$ is $\mathcal{S}_\Gamma \uplus \mathcal{S}_A \uplus \mathcal{S}_B$ (with respective conclusions Γ, A and B) extended with $x_\&$, and $(\mathcal{S}_\Gamma \uplus \mathcal{S}_A, \Theta_A)$, $(\mathcal{S}_\Gamma \uplus \mathcal{S}_B, \Theta_B)$ are both MALL proof nets, and by Lemma 3.14, $\mathcal{S}_\Gamma \neq \emptyset$.

Let $\Lambda_A = \{\lambda \in \Theta_A : \mathcal{S}_\Gamma \downharpoonleft \lambda_1 \cap \mathcal{S}_\Gamma \downharpoonleft \lambda_2 \subseteq \mathcal{S}_\Gamma \downharpoonleft \lambda\}$ and $\Lambda_B = \{\lambda \in \Theta_B : \mathcal{S}_\Gamma \downharpoonleft \lambda_1 \cap \mathcal{S}_\Gamma \downharpoonleft \lambda_2 \subseteq \mathcal{S}_\Gamma \downharpoonleft \lambda\}$. Then, clearly, $\Theta_{\lambda_1,\lambda_2} = \Lambda_A \uplus \Lambda_B$, $\lambda_1 \in \Lambda_A$ and $\lambda_2 \in \Lambda_B$.

Let W^l_1 be the $\&$-resolution of $\mathcal{S}$ defined as follows:

$$\mathcal{S} \downharpoonleft \lambda_1 \cap \mathcal{S} \downharpoonleft \lambda_2 \subseteq W^l_1,$$

$$\forall\& \text{ node } x'_\& \in \mathcal{S}, x'_\& \notin \mathcal{S} \downharpoonleft \lambda_1 \cap \mathcal{S} \downharpoonleft \lambda_2 \Rightarrow {x'}^r_\& \text{ is erased in } W^l_1, \text{ and } x^r_\& \text{ is erased in } W^l_1,$$

and let W^r_1 be defined as follows:

$$\mathcal{S} \downharpoonleft \lambda_1 \cap \mathcal{S} \downharpoonleft \lambda_2 \subseteq W^r_1,$$

$$\forall\& \text{ node } x'_\& \in \mathcal{S}, x'_\& \notin \mathcal{S} \downharpoonleft \lambda_1 \cap \mathcal{S} \downharpoonleft \lambda_2 \Rightarrow {x'}^l_\& \text{ is erased in } W^r_1, \text{ and } x^r_\& \text{ is erased in } W^r_1.$$

Then, by Condition (RES), there exist $\lambda^l_1, \lambda^r_1 \in \Theta$ s.t. $\lambda^l_1 \sqsubseteq W^l_1$ and $\lambda^r_1 \sqsubseteq W^r_1$. Moreover, $\lambda^l_1 \in \Theta_A$, $\lambda^r_1 \in \Theta_A$ and $\mathcal{S} \downharpoonleft \lambda^l_1 \cap \mathcal{S} \downharpoonleft \lambda^r_1 = \mathcal{S} \downharpoonleft \lambda_1 \cap \mathcal{S} \downharpoonleft \lambda_2$. Therefore, $\Lambda_A = \Theta_{\lambda^l_1,\lambda^r_1}$.

Similarly, there exist $\lambda^l_2, \lambda^r_2 \in \Theta$ s.t. $\Lambda_B = \Theta_{\lambda^l_2,\lambda^r_2}$.

By induction hypothesis, $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}} = I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_1,\lambda^r_1}} \cup I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_2,\lambda^r_2}}$, where $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_1,\lambda^r_1}}$ and $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_2,\lambda^r_2}}$ are both DR-connected.

Moreover, by Condition (RES), neither $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_1,\lambda^r_1}}$ nor $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_2,\lambda^r_2}}$ contains a unary couple of edges except for $x_\&$. Therefore, for any switching S of $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}}$, $x^l_\&$ is connected through $S(I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_1,\lambda^r_1}})$ to some vertex $y \in I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_1,\lambda^r_1}} \cap I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_2,\lambda^r_2}} \neq \emptyset$, and back to $x^r_\&$ through $S(I_{\mathcal{S}\downharpoonleft\Theta_{\lambda^l_2,\lambda^r_2}})$. □

Lemma 3.17. *Let* $(\mathcal{S}, \Theta)$ *be an* MALL *proof structure satisfying (RES) and let* $\Lambda \subseteq \Theta$ *with at least two linkings.*

Λ *toggles a* $\&$ *node* $x_\&$ *such that* $x_\&$ *belongs to a switching cycle of* $I_{\mathcal{S}\downharpoonleft\Lambda}$ *if and only if it belongs to a switching cycle of* $H_{\mathcal{S}\downharpoonleft\Lambda}$.

Proof. Condition (RES) implies that no premise edge of any environment-free $\&$ node belongs to any switching cycle of $H_{\mathcal{S}\downharpoonleft\Lambda}$. Therefore, the switching cycles of $H_{\mathcal{S}\downharpoonleft\Lambda}$ are switching cycles of $I_{\mathcal{S}\downharpoonleft\Lambda}$; hence the "if" direction. The "only if" direction proceeds from the fact that the switching cycles of $I_{\mathcal{S}\downharpoonleft\Lambda}$ are switching cycles of $H_{\mathcal{S}\downharpoonleft\Lambda}$. □

Lemmas 3.12 and 3.17 yield the following proposition.

Proposition 3.18. *Let* $(\mathcal{S}, \Theta)$ *be an* MALL *proof structure satisfying (RES).* $(\mathcal{S}, \Theta)$ *satisfies (TOG) iff, for all* $\{\lambda_1, \lambda_2\} \subseteq \Theta$, $\Theta_{\lambda_1,\lambda_2}$ *toggles a* $\&$ *node* $x_\&$ *such that* $x_\&$ *does not belong to any switching cycle of* $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}}$. □

Proposition 3.19. *Let* $(\mathcal{S}, \Theta)$ *be an* MALL *proof structure satisfying (RES) and (MLL). The following algorithm decides whether* $(\mathcal{S}, \Theta)$ *satisfies (TOG) in non-deterministic logarithmic space:*

FOR ALL $\lambda_1, \lambda_2 \in \Theta$
  COMPUTE $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}}$,
  COMPUTE $D(I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}})$ the dependency graph of $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}}$,
  IF $\forall s \in D(I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}})$, $\exists x \in D(I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}})$ such that $\neg$STCONN(s, x)
    THEN REJECT
    ELSE
    LET tog= false
    FOR ALL $\&$ node $x_\&$ in $\mathcal{S}$
      LET $I_{x_\&}$ be $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}}[\forall \mapsto \backslash\!\cdot\,]$ without any premise – or jump – edge to $x_\&$,
      IF no premise-argument or jump-argument of $x_\&$ is connected to $x_\&$ in $I_{x_\&}$
      THEN tog=true
    END FOR ALL
  END IF
  IF tog=false THEN REJECT
END FOR ALL
ACCEPT

Proof. By Proposition 3.19, $(\mathcal{S}, \Theta)$ satisfies (TOG) if and only if, for all $\{\lambda_1, \lambda_2\} \subseteq \Theta$, $\Theta_{\lambda_1,\lambda_2}$ toggles a $\&$ node $x_\&$ such that $x_\&$ does not belong to any switching cycle of $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}}$. By Lemma 3.16, if $(\mathcal{S}, \Theta)$ satisfies (TOG), then $I_{\mathcal{S}\downharpoonleft\Theta_{\lambda_1,\lambda_2}}$ is DR-connected, and, by Corollary 2.5, its dependency graph has a node s from which every node is reachable. Now, if $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$ is DR-connected, a $\&$ node $x_{\&}$ belongs to a switching cycle of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$ if and only if it belongs to a cycle of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}[\forall \mapsto \because]$; therefore the algorithm above decides whether $(\mathcal{S}, \Theta)$ satisfies (TOG).

It is clear that the enumeration of the $\lambda_1, \lambda_2 \in \Theta$ and the computation of $I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}}$ and $D(I_{\mathcal{S}\lfloor\Theta_{\lambda_1,\lambda_2}})$ can be performed in logarithmic space. Since STCONN $\in$ *NL*, the whole algorithm works in *NL*. $\square$

Propositions 2.7, 2.9, 3.10 and 3.19 yield the following result.

Theorem 3.20. *MLL*-CORR*, MELL*-CORR *and MALL*-CORR *are NL-complete under constant-depth reductions.* $\square$

References

[1] A.K. Chandra, L.J. Stockmeyer, U. Vishkin, Constant depth reducibility, SIAM J. Comput. 13 (2) (1984) 423–439.
[2] V. Danos, 1990. Une application de la logique linéaire à l'étude des processus de normalisation (principalement de λ-calcul), Ph.D. Thesis, Université Denis Diderot, Paris 7.
[3] V. Danos, L. Regnier, The structure of multiplicatives, Arch. Math. Logic 28 (3) (1989) 181–203.
[4] L.T. di Falco, The additive multiboxes, Ann. Pure Appl. Logic 120 (1–3) (2003) 65–102.
[5] J.-Y. Girard, Linear logic, Theoret. Comput. Sci. 50 (1) (1987) 1–102.
[6] J.-Y. Girard, Proof-nets: the parallel syntax for proof-theory, in: P. Agliano, A. Ursini (Eds.), Logic and Algebra, vol. 180, Marcel Dekker, New York, 1996.
[7] S. Guerrini, Correctness of multiplicative proof nets is linear, in: Proc. of the Annual Symp. on Logic in Computer Science, LICS'99, IEEE Computer Society Press, 1999, pp. 454–463.
[8] S. Guerrini, A. Masini, Parsing MELL proof nets, Theoret. Comput. Sci. 254 (1–2) (2001) 317–335.
[9] D.j.d. Hughes, R.j. van Glabbeek, Proof nets for unit-free multiplicative-additive linear logic, ACM Trans. Comput. Log. 6 (4) (2005) 784–842.
[10] N. Immerman, Nondeterministic space is closed under complementation, SIAM J. Comput. 17 (5) (1988) 935–938.
[11] B. Jenner, K.-J. Lange, P. McKenzie, Tree isomorphism and some other complete problems for deterministic logspace, DIRO 1059, Univ. de Montréal., 1997.
[12] N.D. Jones, Y.E. Lien, W.T. Laaser, New problems complete for nondeterministic logspace, Math. Syst. Theory 10 (1976) 1–17.
[13] M.I. Kanovich, Horn programming in linear logic is NP-complete, in: Proc. of the Annual Symp. on Logic in Computer Science, LICS'92, IEEE Computer Society Press, 1992, pp. 200–210.
[14] Y. Lafont, From proof-nets to interaction nets, in: J.-Y. Girard, Y. Lafont, L. Regnier (Eds.), in: Advances in Linear Logic, vol. 222, Cambridge University Press, 1995, pp. 225–247.
[15] H.G. Mairson, Normalization bounds for multiplicative linear logic are axiom-sensitive, presentation at GEOCAL'06 Workshop of Implicit Computational Complexity, 2006. Slides available at http://www-lipn.univ-paris13.fr/~baillot/GEOCAL06/SLIDES/Mairson.pdf.
[16] P.-A. Melliès, A topological correctness criterion for non-commutative logic, in: London Mathematical Society Lecture Notes Series, vol. 316, Cambridge University Press, 2004.
[17] A. Murawski, L. Ong, Dominator trees and fast verification of proof nets, in: Proc. of the Annual Symp. on Logic in Computer Science, LICS'00, IEEE Computer Society Press, 2000, pp. 181–191.
[18] A. Murawski, L. Ong, Fast verification of MLL proof nets via IMLL, ACM Trans. Comput. Log. 7 (3) (2006) 473–498.
[19] O. Reingold, Undirected st-connectivity in log-space, in: H.N. Gabow, R. Fagin (Eds.), STOC, ACM, 2005, pp. 376–385.
[20] R. Szelepcsényi, The method of forcing for nondeterministic automata, Bull. EATCS 33 (1987) 96–99.