Global attractor for a modified Swift–Hohenberg equation

Mustafa Polat
Yeditepe University Department of Mathematics, Kayisagi Caddesi, Kayisagi, Istanbul, Turkey

A R T I C L E   I N F O
Article history:
Received 29 January 2008
Received in revised form 12 August 2008
Accepted 10 September 2008

Keywords:
Modified Swift–Hohenberg equation
Global attractor
Absorbing ball
Dissipative systems
Pattern formation systems

A B S T R A C T
In this study, the existence of a global attractor is proven for the modified Swift–Hohenberg equation near the onset of instability with the Diriclet’s boundary conditions. The modified Swift–Hohenberg equation provides a phenomenological model for pattern formation systems.

© 2008 Elsevier Ltd. All rights reserved.


1. Introduction

Swift–Hohenberg type equations arise in the study of convective hydrodynamics [1], plasma confinement in toroidal devices [2], viscous film flow and bifurcating solutions of the Navier–Stokes equations [3]. Recently L.A. Peletier and his coworkers have published several articles on the Swift–Hohenberg equation: for instance the global attractor, the stability of stationary solutions and pattern selections of solutions of this equation have been studied in [4–6].

We consider the following initial-boundary value problem for the modified Swift–Hohenberg equation:

\[ \begin{align*}
    u_t + \Delta^2 u + 2 \Delta u + au + b|\nabla u|^2 + u^3 &= 0, & x \in \Omega, & t \in \mathbb{R}^+ \\
    u(0, x) &= u_0(x), & x \in \Omega, & (1.1) \\
    u &= 0 & \text{and} & \Delta u = 0 & \text{for} & x \in \partial \Omega & (1.2) \\
    u &= 0 & (1.3)
\end{align*} \]

where \( \Omega \) is an open connected bounded domain in \( \mathbb{R}^2 \), \( a \) and \( b \) are arbitrary constants and \( u_0(x) \) is a given function from a suitable phase space. The above problem has been proposed by Arjen Doelman et al. [7] for a pattern formation system with two unbounded spatial directions that is near the onset to instability. Note that, the usual Swift–Hohenberg equation [1] is recovered for \( b = 0 \). The additional term \( b|\nabla u|^2 \), reminiscent of the Kuramoto–Sivashinsky equation (KSE), which arises in the study of various pattern formation phenomena involving some kind of phase turbulence or phase transition [8,9], breaks the symmetry \( u \to -u \). For more references (see [10–12]) and the references therein. The presentation of this paper is as follows: In Section 2, we give the abstract results from the book of Sell–You [10] and also a list of inequalities to be used. In Section 3 the main result of the paper, that is the existence of a global attractor is proved.

2. Preliminaries

The focus of this study is the global stability of the problem (1.1)–(1.3). The study adopts the ideas of Sell–You [10] and Temam [12]. In the sequel we will use the following hypotheses, theorems and inequalities from [10–13].

E-mail address: mpolat@yeditepe.edu.tr.

0898-1221/$ – see front matter © 2008 Elsevier Ltd. All rights reserved.
The Standing Hypothesis A. Let $A$ be a positive, sectorial operator on a Banach space $W$ with associated analytic semigroup $e^{-At}$. Let $V^{2\alpha}$ be the family of interpolation spaces generated by the fractional powers of $A$, where $V^{2\alpha} = \mathcal{D}(A^\alpha)$, for $\alpha \geq 0$. Let $\|A^\alpha u\|_W = \|u\|_{V^{2\alpha}} = \|u\|_{2\alpha}$ denote the norm on $V^{2\alpha}$. (For details see [10, pg 141 and Th. 37.4]).

The Standing Hypothesis B. The operator $A$ is positive, self adjoint, linear with compact resolvent, on a Hilbert space $H$. Consequently $A$ satisfies the Standing Hypothesis A. Moreover, the fractional power spaces $V^{2\alpha}$ defined for all $\alpha \in \mathbb{R}$, and the inner product $\langle u, v \rangle_\alpha := \sum_{i=1}^{\infty} \lambda_i^\alpha u_i v_i$ on $V^{\alpha}$ defines a Hilbert space structure on each $V^{\alpha}$. Also the semigroup $e^{-At}$ is compact, for $t > 0$. (For details see [10, p. 142 and Th. 37.2]).

For the existence we consider the following initial value problem for an abstract nonlinear evolutionary equation of the form

$$\partial_t u + Au = F(u), \quad u \in H, \quad \text{for } u(0) = u_0 \in H \text{ and } t \geq t_0 \geq 0,$$

where $H$ is a Hilbert space. Possible choices for $H$ include $L^2(\Omega) = L^2(\Omega, \mathbb{R})$, $H^1(\Omega)$, $H^1_0(\Omega)$, or $H^2_0(\Omega)$. In this article the norm of these Hilbert spaces will be denoted by $\|\cdot\|_{0,2}$, $\|\cdot\|_{1,2}$ and $\|\cdot\|_{2,2}$ respectively. In addition to this, $L^2(\Omega)$ norms of $u$ will be denoted by $\|\cdot\|_{0,0}$ and the norm of Sobolev spaces $W^{\alpha}_p(\Omega)$ by $\|\cdot\|_{\alpha,p}$. Also $\|u(t)\|_{2,2}$ is equivalent to $\|\Delta u(t)\|_{0,2}$. 

**Definition 2.1** ([10]). Let $I = [t_0, t_0 + \tau)$ be an interval in $\mathbb{R}^+$, where $\tau > 0$. A pair $(u, I)$ is said to be a mild solution of (2.1) in the space $H$ on $I$ if $u(I) \to H$ is a strongly continuous mapping and a solution of the integral equation

$$u(t) = e^{-A(t-t_0)}u_0 + \int_{t_0}^{t} e^{-A(t-s)}F(u(s), s)ds, \quad t \in I.$$  

(2.2)

**Theorem 2.2** ([10]). Let the Standing Hypothesis A be satisfied and assume that $F \in C_{lip}(H^2_0(\Omega), L^2(\Omega))$. Let $K$ be a bounded set in $H^2_0(\Omega)$, and assume that $K$ is an invariant set for the semiflow $V(t)u_0$ given by Eq. (2.1). Then one has $K \subset \mathcal{D}(A) = H^4(\Omega) \cap H^2_0(\Omega)$, and for every $u_0 \in K$, the global mild solution is both a strong and a classical solution of the Eq. (2.1) in $H^2_0(\Omega)$, for all $t \in \mathbb{R}$, with

$$V(.)u_0 \in C(\mathbb{R}, \mathcal{D}(A))$$

and $K$ is a bounded, invariant set in $H^2_0(\Omega)$.

**Definition 2.3.** A semiflow $\sigma$ on a Banach space $W$ is called point dissipative if there is a nonempty bounded set $A$ in $W$ such that $A$ attracts every point in $W$.

**Definition 2.4.** A nonnegative real-valued function $\kappa(B)$ defined for the bounded set $B \subset W$ given by

$$\kappa(B) := \inf \{d : B \text{ has a finite open cover of sets of diameter } < d\}$$

is called the Kuratowski measure of noncompactness.

**Definition 2.5.** A semiflow $\sigma$ on a Banach space $W$ is said to be $\kappa$-contracting if for every bounded set $B \subset W$, one has $\kappa(V(t)B) \to 0$, as $t \to \infty$.

**Definition 2.6.** A semiflow $\sigma$ on $B \subset W$ is said to be ultimately bounded if for every bounded set $B$ in $M$, there is a $\tau = \tau(B) \geq 0$ such that $\gamma^+(V(\tau)B)$ is bounded.

**Theorem 2.7** ([10]). Let $\sigma$ be a $\kappa$-contracting semiflow on a complete metric space $W$. Assume that $\sigma$ is point dissipative and for every compact set $K$ in $W$, there is a $\tau \geq 0$ such that $\gamma^+(V(\tau)K)$ is bounded. Then there is a global attractor $\mathcal{A}$ for $\sigma$, and $\mathcal{A}$ attracts every compact set $K$ in $W$.

Assume in addition that $\sigma$ satisfies one of the following properties:

1. $\sigma$ is compact, or
2. $\sigma$ is ultimately bounded,

then $\mathcal{A}$ attracts every bounded set $B$ in $W$.

**Lemma 2.8** ([10]). Let the Standing Hypothesis A be satisfied and assume that $A$ has compact resolvent. Let $F = F(u)$ satisfy $F \in C_{lip}(H^2_0(\Omega), L^2(\Omega))$ and define $V(t)u_0 = \varphi(u_0, F, t)$. Then for every bounded set $B \subset H^2_0$, there is a time $T = T(B)$, with $0 < T \leq \infty$, and such that for each $t$, with $0 < t < T$, the set $V(t)(B)$ lies in a compact subset of $H^2_0$. 
Gagliardo–Nirenberg Inequality: Let $\Omega$ be an open, bounded domain of Lipschitz class in $\mathbb{R}^n$. Assume $1 \leq p \leq \infty$, $1 \leq q \leq \infty$, $1 \leq r$, $0 < \theta \leq 1$ and let

$$k - \frac{n}{p} \leq \theta \left(m - \frac{n}{q}\right) + (1 - \theta)\frac{n}{r}.$$ 

Then

$$\|u(t)\|_{k,p} \leq C_1(\Omega)\|u(t)\|_{a,r}^{1-\theta}\|u(t)\|_{m,q}^\theta. \quad (2.3)$$

Young Inequality: Let $a, b \in \mathbb{R}^+$, $p \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Then for any $\lambda > 0$ one has

$$ab \leq \frac{\lambda^p a^p}{p} + \frac{b^q}{q\lambda^q}. \quad (2.4)$$

**Theorem 2.9** (Sobolev Imbedding Theorem). Let $1 < p_1 \leq p_2 < \infty$, and $l_1, l_2 \in \mathbb{Z}$. Let $\frac{l_1}{n} - \frac{1}{p_1} \geq \frac{l_2}{n} - \frac{1}{p_2}$. Then

$$W_{p_1}^{l_1} \subset W_{p_2}^{l_2}$$

and if the inequality is strict the embedding is compact; if $p_1 = 2$ then $l_1$ can be taken from $\mathbb{R}$.

**Lemma 2.10** ([12] Uniform Gronwall Lemma). Let $g, h, y$ be three positive locally integrable functions on $(t_0, \infty)$ such that $y'$ is locally integrable on $(t_0, \infty)$ and satisfy

$$y'(t) \leq gy + h \quad \text{for} \quad t \geq t_0 \quad (2.5)$$

with $r, a_1, a_2, a_3$ positive constants. Then $y(t + r) \leq (\frac{a_1}{r} + a_2) \exp(a_1)$ for all $t \geq t_0$.

### 3. Main results

The first step in reducing the problem to the abstract evolution equation is to identify the linear operator $A$. Let $Au = \Delta^2 u$ with the homogeneous boundary conditions. The domain of $A$ is $\mathcal{D}(A) = H^4(\Omega) \cap H_0^2(\Omega)$. It is obvious that the linear operator $A$ is uniformly elliptic, and satisfies the Standing Hypothesis A and the Standing Hypothesis B of [10]. For the nonlinear operator $F(u) = 2\Delta u + au + b|\nabla u|^2 + u^3$ it can easily be verified that $F \in C_{lip}(H_0^2(\Omega), L^2(\Omega))$ and we have

**Theorem 3.1.** For every $u_0 \in H_0^2(\Omega)$, there is a unique, maximally defined, mild solution $V(t)u_0$ in $H_0^2(\Omega)$ of equation $\partial_t u + Au = F(u)$ on the interval $[0, T)$, where $0 < T = T(u_0)$.

The main result of this article is

**Theorem 3.2.** For any $u_0(x) \in H_0^2(\Omega)$, there exists a unique, globally defined, mild solution $V(t)u_0 = \sigma(u_0, t)$ in $H_0^2(\Omega)$ of (1.1)–(1.3), and $V(t)$ is a semigroup on $H_0^2(\Omega)$. Moreover, the semigroup is point dissipative in $H_0^2(\Omega)$ and compact in $H_0^2(\Omega)$ for $t > 0$. Hence the problem (1.1)–(1.3) has a global attractor $\mathcal{A}$ in $H_0^2(\Omega)$.

Proof of this result follows the following two steps

**Step 1:** Dissipation in $L^2(\Omega)$

In this step the dissipation of $u(t)$ in $L^2(\Omega)$ will be proven. To study the dissipation of energy and the global existence of the solution, the energy methods are used as follows: Multiplying the Eq. (1.1) by $u(x, t)$ and integrating over $\Omega$, one gets

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{0,1} + \|\Delta u(t)\|^2_{0,2} + \|u(t)\|^2_{0,4} \leq |a|\|u(t)\|^2_{0,2} + 2\|\nabla u(t)\|^2_{0,2} + |b|\int_\Omega |\nabla u(x, t)|^2 u(x, t) \, dx. \quad (3.1)$$

In order to obtain necessary estimates of solution $u(x, t)$, we use the Gagliardo–Nirenberg inequality with $k = 1, n = 2$, $m = 2, q = 2, r = 3, p = 4, \theta = \frac{1}{10}$ and we get

$$\|\nabla u(t)\|^2_{0,4} = \|u(t)\|^2_{1,4} \leq C_1\|u(t)\|^\frac{1}{2,2} \|u(t)\|^\frac{9}{0,3} \quad (3.2)$$

and

$$\|u(t)\|^2 \|u(t)\|^2_{1,4} \leq C_1 k_2^{14} \|u(t)\|^\frac{14}{2,2} \|u(t)\|^\frac{14}{0,4} \quad (3.3)$$

since $\|u(t)\|_{0,2} \leq k_1 \|u(t)\|_{0,3} \leq k_2 \|u(t)\|_{0,4}$, with $k_1$ and $k_2$ two positive constants. Using Hölder inequality, (3.3) and the Young inequality (2.4) with $p = 10, q = \frac{10}{9}$ we get

$$|b| \int_\Omega |\nabla u(x, t)|^2 |u(x, t)| \, dx \leq |b| \|u(t)\|_{1,4}^2 \|u(t)\|_{0,2}^2 \leq \frac{C_1 k_2^2}{10} \lambda^{\frac{1}{10}} \|u(t)\|_{0,2}^2 + \frac{9|b|C_1}{10 \lambda^{\frac{10}{9}}} \|u(t)\|_{0,4}^\frac{10}{9}. \tag{3.4}$$

Using Gagliardo–Nirenberg and Young inequalities we get

$$2 \|\nabla u(t)\|_{0,2}^2 \leq C_2 \|\Delta u(t)\|_{0,2} \|u(t)\|_{0,2} \leq C_2 \varepsilon_2 \|\Delta u(t)\|_{0,2}^2 + \frac{C_2}{2\varepsilon_2} \|u(t)\|_{0,2}^2. \tag{3.5}$$

Substituting inequalities (3.2)–(3.5) into the inequality (3.1) we get

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{0,2}^2 + \|\Delta u(t)\|_{0,2}^2 + \|u(t)\|_{0,4}^4 \leq \frac{|b|C_0}{10 \lambda^{\frac{1}{10}}} \|u(t)\|_{0,2}^2 + \frac{9|b|C_0}{10 \lambda^{\frac{10}{9}}} \|u(t)\|_{0,4}^\frac{10}{9} + C_2 \varepsilon_2 \|\Delta u(t)\|_{0,2}^2 + \frac{C_2}{2\varepsilon_2} \|u(t)\|_{0,2}^2 + |\lambda| \|u(t)\|_{0,2}^2. \tag{3.6}$$

Now, choose $\varepsilon_2$ and $\lambda$ so that $\frac{|b|C_0}{10 \lambda^{\frac{1}{10}}} + C_2 \varepsilon_2 < 1$ and add $k_1 \|u(t)\|_{0,2}^2$ to both sides of (3.6) and replace $\|u(t)\|_{0,2}^2$ by $\|u(t)\|_{0,4}^2$ in the right hand side of (3.6) we get

$$\frac{d}{dt} \|u(t)\|_{0,2}^2 + k_1 \|u(t)\|_{0,2}^2 \leq -\|u(t)\|_{0,4}^4 + \frac{9|b|C_0}{10 \lambda^{\frac{10}{9}}} \|u(t)\|_{0,4}^\frac{10}{9} + k_2 \|u(t)\|_{0,4}^2. \tag{3.7}$$

From here if $\|u(t)\|_{0,4}$ is large enough then the right hand side of the inequality (3.7) is less than zero and we get

$$\|u(t)\|_{0,2}^2 \leq \|u_0\|_{0,2}^2 e^{-k_1 t} \tag{3.8}$$

and if $\|u(t)\|_{0,4}$ is so small that $-\|u(t)\|_{0,4}^4 + \frac{9|b|C_0}{10 \lambda^{\frac{10}{9}}} \|u(t)\|_{0,4}^\frac{10}{9} + k_2 \|u(t)\|_{0,4}^2 > 0$ then $\|u(t)\|_{0,2}$ is also small since $\|u(t)\|_{0,2} \leq k \|u(t)\|_{0,4}$, thus $u(t)$ has an absorbing set. So the system is dissipative in $L^2(\Omega)$.

Now, we integrate (3.6) with respect to $t$ and using (3.8) we obtain that

$$\int_t^{t+\tau} \|\Delta u(s)\|_{0,2}^2 \, ds \leq rC_2 + \|u(t)\|_{0,2}^2 \leq rC_2 + r^2, \quad \forall r > 0 \tag{3.9}$$

where $r$ is independent of $u_0$ in this sequel which will be used to prove the dissipation of energy in $H_0^2(\Omega)$. In the next section, it is seen that the system is dissipative in $H_0^2(\Omega)$, as well.

Step 2: Dissipation in $H_0^2(\Omega)$

For this, we multiply Eq. (1.1) by $\Delta^2 u(x, t)$ in $L^1(\Omega)$ to get

$$\frac{d}{dt} \|\Delta u(t)\|_{0,2}^2 + \|\Delta^2 u(t)\|_{0,2}^2 + a \|\Delta u(t)\|_{0,2}^2 + (\Delta u(t), \Delta^2 u(t)) + b(\Delta^2 u(t), |\nabla u(t)|^2) + (\Delta u(t), u(t)^3) = 0. \tag{3.10}$$

To obtain an estimate for $u(t)$ in $H_0^2(\Omega)$, the above inequalities and estimates are used as follows:

$$|(\Delta u(t), \Delta^2 u(t)| \leq \|\Delta u(t)\|_{0,2} \|\Delta^2 u(t)\|_{0,2} \leq \frac{\varepsilon_4}{2} \|\Delta u(t)\|_{0,2}^2 + \frac{1}{2\varepsilon_4} \|\Delta^2 u(t)\|_{0,2}^2 \tag{3.11}$$

and using (2.3) and (3.8) one easily gets

$$|\Delta^2 u(t), u^3(t)| \leq C_4 + \varepsilon_5 \|\Delta^2 u(t)\|_{0,2}^2. \tag{3.12}$$

Using Hölder, Gagliardo–Nirenberg and Young’s inequalities

$$\|b| \|\Delta^2 u(t), |\nabla u(t)|^2\| \leq |b| \|\Delta^2 u(t)\|_{0,2} \|\nabla u(t)\|_{0,4}^2 \leq |b| C_3 \|u(t)\|_{0,2} \|\Delta u(t)\|_{0,2} \|\Delta^2 u(t)\|_{0,2} \leq \frac{|b| C_3 \varepsilon_6}{2} \|\Delta^2 u(t)\|_{0,2}^2 + \frac{|b| C_3}{2\varepsilon_6} \|\Delta u(t)\|_{0,2}^2. \tag{3.13}$$
Plugging (3.11)–(3.13) into (3.10) yields
\[
\frac{d}{dt} \|\Delta u(t)\|_{0,2}^2 + \|\Delta^2 u(t)\|_{0,2}^2 + a \|\Delta u(t)\|_{0,2}^2 \\
\leq \frac{\epsilon_4}{2} \|\Delta u(t)\|_{0,2}^2 + \frac{1}{4\epsilon_4} \|\Delta^2 u(t)\|_{0,2}^2 + C_4 + \epsilon_5 \|\Delta^2 u(t)\|_{0,2}^2 \\
+ \frac{|b|C_3\epsilon_6}{2} \|\Delta^2 u(t)\|_{0,2}^2 + \frac{|b|C_3}{2\epsilon_6} \|\Delta u(t)\|_{0,2}^2.
\]
 Choosing \(\epsilon_4, \epsilon_5 \) and \(\epsilon_6\) so that \(\frac{1}{4\epsilon_4} + \epsilon_5 + \frac{|b|C_3\epsilon_6}{2} \leq 1\), one can easily write it as
\[
\frac{d}{dt} \|\Delta u(t)\|_{0,2}^2 \leq C_6 \|\Delta u(t)\|_{0,2}^2 + C_5.
\]
If \(u_0\) is in \(H^2_0(\Omega)\), then the usual Gronwall lemma shows that
\[
\|\Delta u(t)\|_{0,2}^2 \leq \|\Delta u(0)\|_{0,2}^2 \exp(C_6 t) \quad \text{for } t > 0.
\]
A bound valid for all \(t \in \mathbb{R}^+\) is obtained by the application of Lemma 2.10; for an arbitrary fixed \(r > 0\), we find
\[
\|u(t + r)\|_{1,2}^2 \leq \frac{K}{r} \exp(C_6 r), \quad t \geq t_s, \quad K = rC_2 + \rho^2 > 0
\]
provided \(\int_{t}^{t+r} \|\Delta u(s)\|_{0,2}^2 ds \leq K\). Hence (3.17) provides a uniform bound for \(\|\Delta u(t)\|_{0,2}\), \(t > r\), while (3.16) provides a uniform bound for \(\|\Delta u(t)\|_{2,2} = \|u(t)\|_{2,2}\) for \(0 \leq t \leq r\). Thus the existence of an absorbing ball in \(H^2_0(\Omega)\) is proven. This implies that \(V(t) = \text{point dissipative in } H^2_0(\Omega)\). From Lemma 2.8, \(V(t)\) is compact for \(t > 0\). Hence, Theorem 2.7 implies the existence of a global attractor \(\mathcal{A}\) for the problem (1.1)–(1.3) in the space \(H^2_0(\Omega)\). Thus Theorem 3.2 is proven.

Remark. Since the tools we have used work for the periodic boundary values, this result is also valid for the modified Swift–Hohenberg equation with the periodic boundary conditions in the sense [12]. That is, for any \(u_0 \in H^2_{\text{per}}(\Omega)\) there exists a global attractor \(\mathcal{A} \in H^2_{\text{per}}(\Omega)\) under the periodic boundary conditions
\[
\partial^i u(x + L_i e_i, t) = \partial^i u(x, t), \quad x \in \mathbb{R}^2, \quad t > 0, \quad L_i > 0, \quad i = 1, 2, j = 0, 1, 2, 3.
\]

Acknowledgments

The author thanks J. K. Hale and reviewers for their useful comments to improve this article.

References