Structural and Behavioral Equivalences of Tessellation Automata

HISAO YAMADA*

The Moore School of Electrical Engineering,
University of Pennsylvania, Philadelphia, Pennsylvania 19104

AND

SERAFINO AMOROSO

US Army Electronics Command, Fort Monmouth, New Jersey 07703
and
Stevens Institute of Technology, Hoboken, New Jersey 07030

The concepts of structural and behavioral isomorphism on tessellation automata are investigated. Certain equivalence relations preserving one or both forms of isomorphism lead to standardizations of neighborhood structure. The concepts of blocking and the blocked structure play a central role. A weaker form of behavioral isomorphism is also introduced leading to further simplifications of standard neighborhood structure. Finally, a concept of simulation is investigated.

I. INTRODUCTION

In this report we continue our study of the tessellation automaton that was introduced in Yamada–Amoroso (1969) as a generalization of the tessellation structures of Moore (1962). For the convenience of the reader, and since there are some corrections, we now briefly review the formal concepts introduced in Yamada–Amoroso (1969).

The tessellation automaton (TA) is a structure

\[ M = (A, E^d, X, I), \]

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where

1. $A$ is a finite nonempty set called the state alphabet of $M$. $A$ represents the set of states that can be assumed by any machine in the array of machines being modeled.

2. $d$ is a positive integer called the tessellation dimension; and $E^d$, called the tessellation array, is the set of all $d$-tuples of integers. The elements of $E^d$ are used as names for the machines in the array.

3. $X$ is an $n$-tuple of distinct $d$-tuples of integers and is called the neighborhood index for $M$. It is used to define the uniform interconnection pattern among the machines in the array.

Any mapping $c : E^d \to A$ will be called a (array) configuration. $C$ will denote the set of all such mappings. The image of $i \in E^d$ under $c \in C$ is written $c(i)$ and is referred to as the contents of cell $i$ in configuration $c$. If $E^{(d)}$ is the set of all neighborhood indices for $d$-dimensional arrays, then the mapping $N : E^{(d)} \times E^d \to E^{(d)}$ defined as follows: If $X = (\xi_1, \ldots, \xi_n)$ and $i \in E^d$, then $N(X, i) = (i + \xi_1, \ldots, i + \xi_n)$, is used to specify the neighborhood of any cell $i$ relative to some neighborhood index.

Let $N_X : E^d \to (E^d)^n$ be defined by

$$N_X(i) = N(X, i).$$

Each component of $N_X(i)$ is called a neighbor of cell $i$.

Let $c^n : (E^d)^n \to A^n$ be defined by

$$c^n(\rho_1, \ldots, \rho_n) = (c(\rho_1), \ldots, c(\rho_n)),$$

where $c \in C$. The image of $N_X(i)$ under $c^n$ is called the configuration of the neighborhood of cell $i$ in configuration $c$. Finally, let any mapping $\sigma : A^n \to A$ be called a local transformation (for $M$).

A mapping $\tau : C \to C$ defined from a local transformation $\sigma$ as follows will be called a parallel transformation. For any $c \in C$, $c\tau = c'$ (written also $\tau(c) = c'$) if and only if

$$c' : E^d \xrightarrow{N_X} (E^d)^n \xrightarrow{c^n} A^n \xrightarrow{\sigma} A.$$

Alternatively, for any $i \in E^d$

$$c'(i) = \sigma(c^n(N_X(i))).$$

If we call any mapping from $C$ into $C$ a global transformation, then parallel transformations are special global transformations.
We can now complete the definition of the TA $M$ as follows:

4. $T$, called the total input alphabet or total parallel transformation set, is the set of all parallel transformations definable for $M$. $I$, called the input alphabet or transformation set, is an arbitrary nonempty subset of $T$.

Let $A$ be a state alphabet, let $X$ be an $n$ component neighborhood index for $E^d$, let $L = \{ \sigma \mid \sigma : A^n \rightarrow A \}$, and let $T$ be the set of all parallel transformations definable with respect to $A$, $E^d$, and $X$. It will be useful to define the bijection $\delta : L \rightarrow T$ which takes $\sigma$ to the parallel transformation it defines. In particular, we shall often use $\delta^{-1}(I)$ to express the set of local transformations needed to define $I$.

With respect to an arbitrary TAM $M = (A, E_a, X, I)$, cells $i$ and $j$ are said to be immediate neighborhood related if there is a component $\xi_k$ of $X$ such that either $j = i + \xi_k$ or $j = i - \xi_k$. We denote this relation by $R_N$. We say that cells $i$ and $j$ are neighborhood related if either $i = j$ or there is a sequence of cells $k_0, k_1, ..., k_m$ ($m \geq 1$) such that $i = k_0$, $j = k_m$, and $k_q R_N k_{q+1}$ for all $0 \leq q < m$. This latter relation, which is clearly an equivalence relation, is denoted by $R_N^*$. We call the partition $E^d/R_N^* = \{ A_0, A_1, ... \}$ the lamination of the array. If $\#(E^d/R_N^*) > 1$, ($\#$ denotes the cardinality of a set), we say that $M$ is laminated and we call the equivalence classes laminal subarrays. $E^d$ with operator ring $Z$, the set of integers, forms a free module under the operations of componentwise sum of $d$-tuples and multiplication of a $d$-tuple by an integer. $A_0$, the equivalence class in $E^d/R_N^*$ containing $0^d = (0, 0, ..., 0)$, is a free submodule of finite type. (See Section III of Yamada-Amoroso (1969).)

Consider the nonlaminated TA $M_1 = (A_1, E_a, X_1, I_1)$ and $M_2 = (A_2, E_a, X_2, I_2)$, where $\#A_1 = \#A_2$ and where $\#X_1 = \#X_2 = n$. ($\#$ is here extended to apply to lists to indicate length.) A quadruple of mappings $\mu_e = (\mu_a, \mu_e, \mu_x, \mu_x)$ is said to be a structural homomorphism from $M_1$ into $M_2$ if $\mu_a : A_1 \rightarrow A_2$, $\mu_e : E^d \rightarrow E^d$ is injective, $\mu_x : X_1 \rightarrow X_2$ is a bijection, and $X_i$ is the set of components of $X_i$, $i = 1, 2$.

\[ \mu_e : J_1 \rightarrow J_2, \text{ where } J_1 = \delta^{-1}(I_1), \text{ and } J_2 = \delta^{-1}(I_2), \text{ such that for all } k, 1 \leq k \leq d, \text{ and for any } i \in E^d, \]

\[ \mu_e[N(X_1, i)]_k = [N(\mu_e(X_2), \mu_e(i))]_k, \]

where the subscript $k$ denotes the $k$-th component of the $n$-tuple (note $\mu_x(X_1)$ is $\mu_x$ applied to each component of $X_1$); and for any $(a_{i_1}, ..., a_{i_n}) \in A_1^n$, and any $\sigma_j \in J_1$,

\[ \mu_a(\sigma_j(a_{i_1}, ..., a_{i_n})) = (\mu_a(\sigma_j))(\mu_a(a_{i_1}), ..., \mu_a(a_{i_n})). \]
If \( \mu_b \) is a structural homomorphism of \( M_1 \) into \( M_2 \), if each component of \( \mu_\alpha \) is bijective, and if \( \mu^{-1}_b = (\mu^{-1}_\alpha, \mu^{-1}_\epsilon, \mu^{-1}_\epsilon, \mu^{-1}_\epsilon, \mu^{-1}_\epsilon) \) is a structural homomorphism of \( M_2 \) into \( M_1 \), then \( M_1 \) and \( M_2 \) are said to be structurally isomorphic and \( \mu_b \) is then called a structural isomorphism.

Let \( C_1 \) and \( C_2 \) be the sets of (array) configurations for nonlaminated TA \( M_1 = (A_1, E^d, X_1, I_1) \) and \( M_2 = (A_2, E^d, X_2, I_2) \). An ordered pair of mappings

\[
\mu_b = (\mu_\alpha, \mu_\epsilon)
\]

is called a behavioral homomorphism from \( M_1 \) into \( M_2 \) if \( \mu_\alpha : C_1 \rightarrow C_2 \) and \( \mu_\epsilon : I_1 \rightarrow I_2 \) such that for any \( c_1 \in C_1 \) and any \( \tau_1 \in I_1 \),

\[
\mu_\alpha(c_1 \tau_1) = \mu_\alpha(c_1) \mu_\epsilon(\tau_1).
\]

If \( \mu_b \) is a behavioral homomorphism from \( M_1 \) into \( M_2 \), if each component of \( \mu_b \) is bijective, and if \( \mu^{-1}_b = (\mu^{-1}_\alpha, \mu^{-1}_\epsilon) \) is a behavioral homomorphism of \( M_2 \) into \( M_1 \), then \( M_1 \) and \( M_2 \) are said to be behaviorally isomorphic, and \( \mu_b \) is said to be a behavioral isomorphism (from \( M_1 \) onto \( M_2 \)).

In Yamada–Amoroso (1969) it is shown that if \( M_1 \) and \( M_2 \) are structurally isomorphic then they are also behaviorally isomorphic. The result was stated there for TA with arbitrary input alphabets; however, the proof as presented is only correct for total input alphabets. Also, \( \mu_\epsilon \) must be an injection. Finally, considering only nonlaminated TA will save a lot of unnecessary detail later, and in view of Theorem IV.2 of Yamada–Amoroso (1969) the limitation is natural.

II. SOME PRELIMINARY EQUIVALENCES

We begin by considering a number of natural “structural” equivalences over an arbitrary class \([M(A,a,T)]\) of all TA of some fixed dimension \( d \), all having a common state alphabet \( A \), \( \#A > 1 \), and all having total parallel transformation sets.

We might have considered the more natural and larger class resulting from relaxing the requirement for identical state alphabets and assumed only that their cardinalities be identical. However, the class chosen will avoid many unnecessary complications.

Consider the following equivalence relations over \([M(A,a,T)]\):

\[
M_1 R^X M_2 \iff \# X_1 = \# X_2 ,
\]

\[
M_1 R^E M_2 \iff \# (E^d | A_0(X_3)) = \# (E^d | A_0(X_3)) ,
\]

We wish to thank John Whipple of Bell Laboratories for this observation.
and
\[ M_1R_M M_2 \iff A_0(X_1) = A_0(X_2), \]
where in each case, \( X_1 \) and \( X_2 \) are the neighborhood indices for \( M_1 \) and \( M_2 \), and where \( A_0(X) \) denotes the laminal subarray (submodule of \( E^d \)) generated by the components of \( X \).

With \( \mathcal{E}^{(n,d)} \) denoting the set of all \( n \)-component neighborhood indices for \( E^d \), and with \( \lambda_k^{(n)} \), \( 1 \leq k \leq n! \) denoting the distinct permutation operators on the components of any \( X \in \mathcal{E}^{(n,d)} \), we can define another equivalence relation over \([M^{(d,d,T)}] \) as follows: \( M_1R_p M_2 \iff \lambda_k^{(n)} \cdot X_1 = X_2 \) for some \( k \), \( 1 \leq k \leq n! \), where \( X_1, X_2 \) are in \( \mathcal{E}^{(n,d)} \) and are the neighborhood indices for \( M_1 \) and \( M_2 \).

The inclusion (i.e., refinement) relation among the relations introduced so far, and others to be introduced later, are shown in Fig. 4 of Section IX.

**Theorem 1.** For nonlaminated TAM \( M_1 = (A, E^d, X_1, T_1) \) and \( M_2 = (A, E^d, X_2, T_2) \), if \( M_1R \supseteq M_2 \) then \( M_1 \) and \( M_2 \) are structurally, and therefore behaviorally, isomorphic.

\( R_p \) is the only relation introduced so far that implies structural isomorphism.

### III. Equivalence Induced by Coordinate Transformation

Let \( \Theta = \left( \theta_1, \ldots, \theta_d \right) \) be a basis for module \( E^d \). Let the bijection (coordinate transformation) \( \varphi_{\Theta} : E^d \rightarrow E^d \) be defined by \( \varphi_{\Theta}(i) = j \iff j = (j_1, \ldots, j_d) \) where \( i = j_1\theta_1 + \cdots + j_d\theta_d \). We define equivalence relation \( R_{\varphi} \) over \([M^{(d,d,T)}] \) as follows: \( M_1R_{\varphi} M_2 \iff \) there exists a coordinate transformation \( \varphi_{\Theta} \) on \( E^d \) such that \( X_2 = (\varphi_{\Theta}(x_1), \ldots, \varphi_{\Theta}(x_n)) \) where \( X_1 = (x_1, \ldots, x_n) \) and \( X_1, X_2 \) are the neighborhood indices for \( M_1 \) and \( M_2 \).

The following informal remarks should help to motivate this definition. The reader can visualize a two-dimensional array of machines one positioned at each lattice point in the plane. The elements of \( E^2 \) can be considered as naming the machines. The situation where, for any \( i \in E^2 \), \( i \) names the machine situated at lattice point \( i \), can be considered the standard naming of the array. Suppose now that the neighborhood interconnection specified by some \( X = (x_1, \ldots, x_n) \) is now “wired” into the array. Holding the positioning of the machines and their interconnecting wires fixed, if we rename machine \( i \), for each \( i \in E^2 \), by calling it now machine \( \varphi_{\Theta}(i) \), then the
interconnection pattern with respect to the renamed array is now 
\((φ_θ(ξ_1),...,φ_θ(ξ_n))\).

If \(M_1 R \subseteq M_2\), then using the bijection \(φ_θ\) one can easily define a structural isomorphism from \(M_1\) onto \(M_2\). We therefore have

**Theorem 1.** For nonlaminated \(M_1 = (A, E^d, X_1, T_1)\) and \(M_2 = (A, E^d, X_2, T_2)\), if \(M_1 R \subseteq M_2\), then \(M_1\) and \(M_2\) are structurally, and therefore behaviorally, isomorphic.

IV. NEIGHBORHOOD STANDARDIZATIONS PRESERVING STRUCTURAL ISOMORPHISM

Let \(Σ^{(d)}\) denote the set of all neighborhood indices for \(d\)-dimensional arrays, and let \(B(E^d)\) denote the set of all bases of module \(E^d\). For any \(X = (ξ_1, ..., ξ_n)\) in \(Σ^{(d)}\) and any \(Θ = (θ_1, ..., θ_d)\) in \(B(E^d)\), let \(D_k\) be the set of all (distinct) \(k\)-th components of the \(d\)-tuples \(φ_θ(ξ_i), 1 ≤ i ≤ n\). \(φ_θ\) is the (coordinate transformation) mapping defined in Section III. We define \(s_k(Θ, X)\), the \(θ_k\)-spread (or the \(k\)-th spread) of \(φ_θ(X)\) by

\[
s_k(Θ, X) = \max\{\max D_k - \min D_k, \max D_k, -\min D_k\}.
\]

Intuitively, \(s_k(Θ, X)\) is the maximum "separation" between any two cells from among any cell \(i\) and its neighbors, along the \(k\)-th coordinate for "naming" \(φ_θ\). Note that the last two terms in the expression defining \(s_k(Θ, X)\) are for the case where \(i\) is not a neighbor of itself (i.e., \(0^d\) is not a component of \(X\)) and the \(k\)-th components of these cell names are all positive or all negative. The affected cell \(i\) is always included in the spread computations for reasons that should become clear.

For any \(Θ \in B(E^d)\) and any \(X \in Σ^{(d)}\) we define \(S(Θ, X)\) as the \(d\)-tuple \((s_1(Θ, X),...,s_d(Θ, X))\) whose components are the respective \(θ_k\)-spreads of \(φ_θ(X)\). Let \(s_{Θ,X}(1) = \max S(Θ, X)\), i.e., the maximum component value. Let \(s_X(1) = \min\{s_{Θ,X}(1) | Θ \in B(E^d)\}\), i.e., as \(Θ\) is varied over all bases, we want the minimum of the set of all maximum components for some \(Θ\). Let \(Y_X(1) = \{Θ \in B(E^d) | s_{Θ,X}(1) = s_X(1)\}\), i.e., the set of all bases that give this "minimized" maximum component. We shall see later that these sets always contain more than one element. Note also that different bases in \(Y_X(1)\) may have the maximum component value at different component positions.

\(^2\) The reader may wish to skip this section on a first reading.
Choose some $\Theta^{(1)} \in Y_X(1)$ and suppose component $k_1$ of $S(\Theta^{(1)}, X)$ has value $\delta_{\Theta^{(1)}, X}(1)$. Define $\delta_{\Theta^{(1)}, X}(2)$ as $\max_{k \neq k_1} S(\Theta^{(1)}, X)$, i.e., the maximum value of any component of $S(\Theta^{(1)}, X)$ except $k_1$. Let

$$\delta_X(2) = \min\{\delta_{\Theta, X}(2) \mid \Theta \in Y_X(1)\},$$

and let

$$Y_X(2) = \{\Theta \in Y_X(1) \mid \delta_{\Theta, X}(2) = \delta_X(2)\}.$$

In general now, for $1 < w \leq d$, if $\Theta^{(w-1)} \in Y_X(w - 1)$ and if the component positions $k_i, 1 \leq i \leq w - 1$, of $S(\Theta^{(w-1)}, X)$ have the $w - 1$ largest values, then

$$\delta_{\Theta^{(w-1)}, X}(w) = \max_{k \in \{k_i\}} S(\Theta^{(w-1)}, X),$$

i.e., the maximum value of any component not equal to one already minimized. Let $\delta_X(w) = \min\{\delta_{\Theta, X}(w) \mid \Theta \in Y_X(w - 1)\}$. Finally,

$$Y_X(w) = \{\Theta \in Y_X(w - 1) \mid \delta_{\Theta, X}(w) = \delta_X(w)\}.$$

Let $S(\Theta, X)$ be the permutation on the $d$ components of $S(\Theta, X)$ such that they are in decreasing order of magnitude from left to right. If we define a partial ordering on the set of all $d$-tuples of nonnegative integers by $(x_1, \ldots, x_d) \leq (y_1, \ldots, y_d)$ if and only if $x_i \leq y_i, 1 \leq i \leq d$, then we have

**Proposition 1.** For any $X \in \Sigma(d), \Theta \in Y_X(d)$, then there does not exist a $\gamma \in B(E^d)$ such that $S(\gamma, X) \leq S(\Theta, X)$.

Denote by $X_{-0}$ the set of components of $X$ except $0^d$ if $0^d$ is in $X$.

**Proposition 2.** For any $X \in \Sigma(d)$ such that $\#(X_{-0}) \geq 1$, the set of maximum spreads $\delta_{\Theta, X}(1)$ as $\Theta$ varies over $B(E^d)$ is unbounded, i.e., for any $\Theta$ there is a $\gamma$ such that $\delta_{\Theta, X}(1) < \delta_{\gamma, X}(1)$.

Let $[(A, E^d, X, T)]_{R_{\varphi}}$ be an arbitrary equivalence class induced on $[M^{(A, E^d, T)}]$ by $R_{\varphi}$. We define the equivalence relations $R_{X(\delta)}^\gamma, 1 \leq k \leq d$, over $[(A, E^d, X, T)]_{R_{\varphi}}$ by

$$M_1 R_{X(\delta)}^\gamma M_2 \iff (\forall w, 1 \leq w \leq k) \delta_{\Theta^{(w-1)}, X}(w) = \delta_{\Theta^{(w-1)}, X}(w),$$

where $X_1$ and $X_2$ are the neighborhood indices for $M_1$ and $M_2$.

Clearly $R_{X(\delta)} \subseteq R_{X(\delta-1)} \subseteq \cdots \subseteq R_{X(1)}$, and as an immediate corollary of Proposition 2 we have

**Corollary 2.1.** If $\pi$ is a partition induced on $[(A, E^d, X, T)]_{R_{\varphi}}$ by $R_{X(\delta)}^\gamma, 1 \leq k \leq d$, then $\#(\pi) = \kappa_0$. 
PROPOSITION 3. For any $X$ and any $[(A, E^d, X, T)]_{R_{\psi}}$, each equivalence class induced by any $R_{Y_X(k)}$, $1 \leq k \leq d$, is a finite set.

For arbitrary $X \in \mathcal{E}(d)$ and arbitrary $\Theta \in B(E^d)$, let $S(\Theta, X) = (s_1\ldots, s_d)$ and let $S_\lambda(\Theta, X) = (s_{\lambda(1)}\ldots, s_{\lambda(d)})$ be a permutation of the components determined by permutation operator $\lambda$. Then for any $\lambda$ and any $\Theta$, there exists an $\Theta_\lambda \in B(E^d)$ such that $S(\Theta_\lambda, X) = S_\lambda(\Theta, X)$. $\varphi_\lambda$ would clearly permute the coordinate axes in an appropriate way. A fortiori, among the components of $Y_X(d)$, exists a set of bases $\tilde{Y}_X(d)$ such that if $\Theta \in \tilde{Y}_X(d)$, then $S(\Theta, X) = S(\Theta, X)$. The (finitely many) bases in $\tilde{Y}_X(d)$ will be called the min-max standard bases for $X$, and $S(\Theta, X)$, $\Theta \in \tilde{Y}_X(d)$, will be called the min-max standard spread for $X$.

The definition of a standard basis that would minimize, in some meaningful way, the spread of a neighborhood with respect to all its coordinate transformations, could be given in the following alternative way. We could define $\delta_{\Theta,X}(1)$ as the minimum value among the component values in $S(\Theta, X)$, and then by varying $\Theta$ over $B(E^d)$, we could define $H_X(1)$ as the set of all those bases that minimize this (now) minimum component. We can now proceed much as we did above for the $Y_X(k)$, now minimizing the minimum among the components so far not minimized, and obtain the sets of bases $H_X(k)$, $1 \leq k \leq d$.

If we let $\hat{S}(\Theta, X)$ be the permutation on the $d$ components of $S(\Theta, X)$ such that they are increasing in order of magnitude from left to right, then we have

PROPOSITION 4. For any $X \in \mathcal{E}(d)$, if $\Theta \in H_X(d)$, then there does not exist a $\gamma \in B(E^d)$ such that $\hat{S}(\gamma, X) \leq \hat{S}(\Theta, X)$.

Analogously to $\tilde{Y}_X(d)$, we can define $\tilde{H}_X(d)$, which will also be finite. We shall call these the min-min standard bases for $X$, and $\hat{S}(\Theta, X)$, $\Theta \in \tilde{H}_X(d)$, will be called the min-min standard spread for $X$.

PROPOSITION 5. If $X \in \mathcal{E}(d)$ is such that $\#(X) > 1$, then for each $k$, $1 \leq k \leq d$, $H_X(k)$ and $Y_X(k)$ each contain more than one element.

At this time we do not know whether or not a min-min standard basis leads, in fact, to the same neighborhood structure as a min-max standard basis, although we doubt it does. Also, we do not know whether the algorithm to obtain such standard bases is known.
Since no permutation of the components of $X$ seems better than any other, any automaton in a class induced on $[M^{(a,d,T)}]$ by $R_p$ can be chosen as a standard representation.

V. Blockings of the Tessellation Array

Consider an arbitrary submodule $A_0$ of $E^d$, and let $\{A_0, A_1, \ldots\}$ be the partition determined by the quotient module $E^d/A_0$. By a kernel block $K_0$ with respect to $A_0$, we shall mean any subset of $E^d$ that satisfies (a) and (b) below.

(a) \( 0^d \in K_0 \),

(b) For each $A_k \in E^d/A_0$, \( \#(A_k \cap K_0) \geq 1 \) and finite.

Note that kernel blocks are not necessarily finite sets.

For any submodule $A_0$ of $E^d$ and any kernel block $K_0$ with respect to $A_0$, we define a set $B(A_0, K_0)$ of subsets of $E^d$ by

$$ B(A_0, K_0) = \{K_j \mid K_j = K_0 + j \text{ for some } j \in A_0\}, $$

where

$$ K_0 + j = \{i + j \mid i \in K_0\}. $$

We shall refer to $B(A_0, K_0)$ as a (cover) blocking of $E^d$, and the subset elements of a blocking will be called blocks.

**Proposition 1.** $B(A_0, K_0)$ is a partition on $E^d$ if and only if for each $A_k \in E^d/A_0$, \( \#(A_k \cap K_0) = 1 \).

When a kernel block defined from a submodule $A_0$ defines a blocking that is a partition on $E^d$, we shall denote the kernel block by $P_0$ rather than $K_0$. Such a cover blocking $B(A_0, P_0)$ will be called a partition blocking. The following are some easily verified properties of partition blockings.

**Proposition 2.** Let $P_j$ be a block in an arbitrary partition blocking $B(A_0, P_0)$ of $E^d$, then

$$ \#P_0 = \#P_j = \#(E^d/A_0). $$

**Proposition 3.** For $P_j \in B(A_0, P_0)$ and $A_k \in E^d/A_0$,

$$ \#(P_j \cap A_k) = 1. $$
**Proposition 4.** Let $P_j, P_k \in B(\Lambda_0, P_0)$, then there is a unique $i \in \Lambda_0$ such that $P_j = P_k + i$.

It can be shown that for any cover blocking $B(\Lambda_0, K_0)$ of $E^d$, if $\text{rank}(\Lambda_0) = d$, then for any $K_j, K_k \in B(\Lambda_0, K_0)$, $K_j + i = K_k$ implies $i \in \Lambda_0$. The converse is not true however, i.e., $K_j + i = K_k$ implies $i \in \Lambda_0$, does not imply $\text{rank}(\Lambda_0) = d$. A counterexample is given below in Example 4.

From Theorem III.6 of Yamada–Amoroso (1969) and Proposition 2 above, we have

**Proposition 5.** Let $K_0$ be a kernel block defined from some submodule $\Lambda_0$ of $E^d$, then $\text{rank}(\Lambda_0) = d$ if and only if $\# K_0$ is finite.

**Example 1.** Let $\Lambda_0$ be the submodule of $E^2$ generated by $((1, 0), (0, 2))$. Then any of the infinitely many sets

\[
\{(0, 0), (1, 1)\}, \{(0, 0), (2, 1)\}, \{(0, 0), (3, 1)\}, \ldots
\]

is a kernel block yielding a partition blocking.

This example illustrates the following result.

**Proposition 6.** If $\Lambda_0$ is a proper nontrivial submodule of $E^d$, $d > 1$, then there are infinitely many kernel blocks definable from $\Lambda_0$ each yielding a distinct blocking for the fixed $\Lambda_0$.

**Example 2.** Given $K_0 = \{(0, 0), (1, 0), (0, 1), (1, 1)\}$ for $E^2$, there are exactly three distinct partition blockings possible, i.e., three distinct $\Lambda_0^{(i)}$, $i = 1, 2, 3$, such that $K_0$ and $\Lambda_0^{(i)}$ determine a partition blocking. There are exactly five more submodules that together with $K_0$ determine five distinct nonpartition cover blockings. This illustrates

**Proposition 7.** For arbitrary kernel block $K_0$, with respect to $E^d$, if $\# K_0$ is finite, then there are only finitely many submodules $\Lambda_0$ such that $K_0$ and $\Lambda_0$ determine a blocking of $E^d$.

There are cases where, for a finite $P_0$, there is only one $\Lambda_0$ such that $P_0$ and $\Lambda_0$ determine a partition blocking. For example, $P_0 = \{(0, 0), (1, 0), (0, 1)\}$. If $K_0$ (or $P_0$) is infinite, then the number of distinct submodules $\Lambda_0$ such that $\Lambda_0$ and $K_0$ (or $P_0$) determine a blocking (partition blocking) may be finite or infinite.
Example 3. If $K_0 = \{(z, 0) \mid z \in \mathbb{Z}\}$, then any of the infinitely many laminal submodules generated by a single element of the form $(z, 1)$, $z \in \mathbb{Z}$, would determine the same blocking of $E^2$.

Example 4. If $P_0 = \{(z, 0) \mid z \geq 0, z \in \mathbb{Z}\} \cup \{(0, z) \mid z > 0, z \in \mathbb{Z}\}$, then only the $A_0$ generated by $(1, 1)$ would give rise to a blocking of $E^2$ for this $P_0$. $P_0$ can trivially be altered to make this example one for nonpartition blocking.

Any nonempty finite subset of $E_2$ can serve as a kernel block $K_0$ for some cover blocking, but finite subsets of $E^2$ that cannot be kernel blocks for any submodule for a partition blocking exist and are easily constructed. Moreover, even though all blocks of any blocking of $E^2$ must be of the same size and “shape,” and must be periodic in $d$ mutually independent directions, these conditions are not sufficient that such a partition of $E^2$ be a blocking. For example, the partition of the form $\{(m, 4n), (m, 4n + 2)\} \cup \{(m, 4n + 1), (m, 4n + 3)\}$, $m, n \in \mathbb{Z}$, could not be a blocking of $E^2$. $P_0$ would have to be $\{(0, 0), (0, 2)\}$, $\#(E^2/A_0)$ would have to be two, and $(0, 1)$ being in either of these two cosets leads to a contradiction.

Propositions 8, 9, and 10 below will play a role later in Section VIII.

**Proposition 8.** Let $P_0$ be the finitely many cells including $0$ of $E^2$ enclosed by a parallelepiped all of whose edges are parallel to the coordinate axes used for naming the cells. There exists then a laminal submodule $A_0$ that together with $P_0$ determines a partition blocking of $E^2$.

Let $\varphi$ be the coordinate system on $E^2$ determined by basis $\Theta = (\theta_1, \ldots, \theta_4)$, i.e., $\varphi : E^2 \to E^2$ is defined by $\varphi(i) = (z_1, \ldots, z_4)$ if and only if

$$i = z_1\theta_1 + \cdots + z_4\theta_4.$$ 

A collection $B_2$ of subsets of $E^2$ is called a coordinate transformation of a blocking if there exists a blocking $B(A_0, K_0)$ and a coordinate system $\varphi$ such that $B_2 = \{\varphi(K_i) \mid K_i \in B(A_0, K_0)\}$, where $\varphi(K_i) = \{\varphi(i) \mid i \in K_i\}$.

**Proposition 9.** A coordinate transformation of a blocking is a blocking, and if $B_2$ is the blocking determined by $B(A_0, K_0)$ and $\varphi$, then $B_2 = B(\varphi(A_0), \varphi(K_0))$.

On the blocks of a blocking $B(A_0, K_0)$, we can define an operation called block addition as follows: For any blocks $K_u$, $K_v$, and $K_w$, $K_u + K_v = K_w$ if and only if $K_u + i_u = K_v + i_v$, $i_u, i_v \in A_0$, and $K_w = K_0 + i_u + i_v$. 

If $z \in Z$, then by the multiplication of a block $K_i$ by an integer, we mean $zK_i = \{zi \mid i \in K_i\}$, where $zi$ is the usual multiplication of a scalar and a vector.

**Proposition 10.** Under the operations of block addition and multiplication of a block by an integer, $B(A_0, K_0)$ is a free (left) $Z$-module of finite type isomorphic to $A_0$, and also isomorphic to $E^r$, where $r$ is the dimension of $A_0$.

In view of this, we define the dimension of a blocking $B(A_0, K_0)$ to be the dimension of $A_0$.

If $B_1$ and $B_2$ are partition blockings of $E^d$, then $B_1 + B_2$ and $B_1 \cdot B_2$ can be defined as the usual sum and product of partitions, i.e., $B_1 + B_2$ is the partition of $E^d$ such that $i$ and $j$ are in a common block if and only if there is a sequence $i = i_1, i_2, ..., i_k = j$ where $i_p$ is in the same block as $i_{p+1}$ in $B_1$ or $B_2$, $1 \leq p \leq k - 1$; and $B_1 \cdot B_2$ is the partition on $E^d$ such that $i$ and $j$ are in a common block if and only if they are in common blocks in $B_1$ and $B_2$. It is easy to show that neither the sum nor the product of two partition blockings is necessarily a blocking.

The notion of a partition on a set has been generalized to the concept of a set system. The natural sum and product operations for set systems [see, e.g., Hartmanis–Stearns (1966)] again do not necessarily preserve blockings.

It can be shown that for blockings $B_1$ and $B_2$, if $B_1 + B_2$ is a blocking, then its rank is at most the smaller of the ranks of $B_1$ and of $B_2$; and if $B_1 \cdot B_2$ is a blocking, its rank is at least the larger of the ranks of $B_1$ and $B_2$.

VI. The Blocked Union of Blocking

Let $B^{(1)} = B(A^{(1)}_0, K^{(1)}_0)$ be a blocking of rank $r$, let $B^{(2)} = B(A^{(2)}_0, K^{(2)}_0)$ be a blocking of rank $s$, and let $\Theta^{(1)} = (\theta^1, ..., \theta_r)$ be a basis of $A^{(1)}_0$. Let $f_{\Theta^{(1)}} : B^{(1)} \to E^r$ be the mapping defined by $f_{\Theta^{(1)}}(K^{(1)}_j) = (z_1, ..., z_r)$ if and only if $K^{(2)}_j = K^{(1)}_j + j$, $j \in A^{(1)}_0$, and $j = z_1\theta_1 + ... + z_r\theta_r$. It is also natural and convenient to consider $f_{\Theta^{(1)}}$ a mapping on $A^{(1)}_0$ where $f_{\Theta^{(1)}}(j) = (z_1, ..., z_r)$ if $j = z_1\theta_1 + ... + z_r\theta_r$. It will always be clear which domain is being used.

We can now define $K^{(1,2)}_j(\Theta^{(1)})$ as the union of all $K^{(2)}_j \in B^{(1)}$ such that $f_{\Theta^{(1)}}(K^{(2)}_j) \in K^{(2)}_j$, where $K^{(1)}_j \in B^{(1)}, \{K^{(1,2)}_j(\Theta^{(1)}) \mid K^{(2)}_j \in B^{(2)}\}$ will be denoted by $[B^{(1)}, \Theta^{(1)}, B^{(2)}]$. If we define $A^{(1,2)}_0(\Theta^{(1)})$ to be $\{i \in A^{(1)}_0 \mid f_{\Theta^{(1)}}(K^{(2)}_j + i) \in A^{(2)}_0\}$, then the following is easily verified.
LEMMA 1. $A_{0}^{(1,2)}(\Theta^{(1)})$ is a submodule of $E^d$ of rank $s$, where $s$ is the rank of $B^{(2)}$.

Letting $K_{0}^{(1,2)}(\Theta^{(1)})$ denote the union of all $K_{0}^{(1)} \in B^{(1)}$ such that $f_{\Theta^{(1)}}(K_{0}^{(1)}) \in K_{0}^{(2)}$, we then have

PROPOSITION 2. $[B^{(1)}, \Theta^{(1)}, B^{(2)}]$ is a blocking of $E^d$ determined by $A_{0}^{(1,2)}(\Theta^{(1)})$ and $K_{0}^{(1,2)}(\Theta^{(1)})$.

Informally, we might say that Proposition 2 states that “a blocking of a blocking (for any $\Theta^{(1)}$) is a blocking.” More precisely, if $B^{(1)} = B(A_{0}^{(1)}, K_{0}^{(1)})$ is a blocking of $E^d$ of rank $r$, if $B^{(2)} = B(A_{0}^{(2)}, K_{0}^{(2)})$ is a blocking of $E^c$ of rank $s$, and if $\Theta^{(1)}$ is a basis of $A_{0}^{(1)}$, then we call $B(A_{0}^{(1,2)}(\Theta^{(1)}), K_{0}^{(1,2)}(\Theta^{(1)}))$ the blocked union of $B^{(1)}$ with respect to $B^{(2)}$ and $\Theta^{(1)}$. As we mentioned above, we denote this blocked union by $[B^{(1)}, \Theta^{(1)}, B^{(2)}]$.

Let $Q(d, r)$ denote the set of all blockings of rank $r$ for $E^d$, and let $Q = \bigcup_{d=0}^{\infty} \bigcup_{r=0}^{d} Q(d, r)$. Clearly, for any $B^{(1)}, B^{(2)} \in Q$, $[B^{(1)}, \Theta^{(1)}, B^{(2)}]$ is meaningful if and only if $B^{(1)} \in Q(d, r)$, $B^{(2)} \in Q(r, s)$ for some nonnegative integers $d, r, s$ such that $d \geq r \geq s$ and $\Theta^{(1)}$ is a basis for $A_{0}^{(1)}$, the submodule for $B^{(1)}$. In particular, $[B^{(1)}, \Theta^{(1)}, B^{(2)}]$ is meaningful if $B^{(1)}, B^{(2)} \in Q(d, d)$ and $\Theta^{(1)}$ is a basis for $A_{0}^{(1)}$, the submodule for $B^{(1)}$.

For arbitrary blockings $B^{(1)}$ and $B^{(2)}$, even when both $[B^{(1)}, \Theta^{(1)}, B^{(2)}]$ and $[B^{(2)}, \Theta^{(1)}, B^{(1)}]$ are meaningful, they do not necessarily define the same blockings. There are cases where no $\Theta^{(2)}$ exists such that $[B^{(1)}, \Theta^{(1)}, B^{(2)}] = [B^{(2)}, \Theta^{(2)}, B^{(1)}]$.

PROPOSITION 3. For $B^{(1)}, B^{(2)}, B^{(3)} \in Q$, if $[B^{(1)}, \Theta^{(1)}, [B^{(2)}, \Theta^{(2)}, B^{(3)}]]$ and $[B^{(1)}, \Theta^{(1)}, B^{(2)}]$ are meaningful, then

$$[B^{(1)}, \Theta^{(1)}, [B^{(2)}, \Theta^{(2)}, B^{(3)}]] = [[B^{(1)}, \Theta^{(1)}, B^{(2)}], \Theta^{(1,2)}, B^{(3)}],$$

where $\Theta^{(1,2)} = f_{\Theta^{(1)}}(\Theta^{(2)})$, i.e., $\Theta^{(1,2)} = \{i \in A_{0}^{(1)} | f_{\Theta^{(1)}}(B_{0}^{(1)} + i) \in \Theta^{(2)}\}$.

Note that $\Theta^{(1,2)}$ is a basis of a laminal subarray that together with a kernel block determine $[B^{(1)}, \Theta^{(1)}, B^{(2)}]$. This can be established through standard concepts of linear algebra.

PROPOSITION 4. For $B^{(i)} = B(A_{0}^{(i)}, B_{0}^{(i)})$, $i = 1, 2, 3$, in $Q$, and for meaningful $[[B^{(1)}, \Theta^{(1)}, B^{(2)}], \Theta^{(2)}, B^{(3)}]$, if $\Theta = f_{\Theta^{(1)}}(\Theta^{(2)})$, then

$$[[B^{(1)}, \Theta^{(1)}, B^{(2)}], \Theta^{(2)}, B^{(3)}] = [B^{(1)}, \Theta^{(1)}, [B^{(2)}, \Theta, B^{(3)}]].$$
VII. The Blocked Structure and Behaviorally Equivalent TA

Consider a TA $M = ((1, 0), E^2, ((0, 0), (0, 1), (0, -1)), I)$. The neighborhood structure is indicated in Fig. 1(a). If we choose a blocking, e.g., the one shown in Fig. 1(b), then we could consider each block as being a cell capable of assuming four possible states. The blocked array could therefore be considered an array of these cells. To determine its next state, each of these four-state cells would have to have as neighbors the four-state cells indicated in Fig. 1(c). Finally, naming each block with an ordered pair as indicated, e.g., in Fig. 1(d) would yield an array with a neighborhood structure as indicated in Fig. 1(e). It should be intuitively clear from these remarks that a TA $M = (A, E^2, ((0, 0), (0, 1), (1, 0), (1, -1), (0, -1)), I')$ with $\#(A) = 4$ can be defined which is behaviorally equivalent to $M$.

This example should help to motivate the detailed general treatment that we now begin.

![Fig. 1. Blocked structure.](image-url)
Consider an arbitrary TA \( M = (A, E^d, X, I) \) and an arbitrary blocking of rank \( d \). We restrict our attention in what follows to blockings of rank \( d \) since (as we have seen in Section V) this ensures finite blocks; and in anticipation of what is coming, it will ensure the construction of behaviorally isomorphic TA with finite state alphabets.

Continuing, let \( \#(E^d/A_0) = q \), and let \( K_0^{\text{ord}} = (i_{j_1}, \ldots, i_{j_m}) \) be some fixed ordering of the elements of \( K_0 \). Note that \( q = m \) for a partition blocking. For each \( K \in B(A_0, K_0) \), let \( K^{\text{ord}} = (i_{k_1}, \ldots, i_{k_m}) \) where \( i_{k_t} = i_{j_t} + i \), \( 1 \leq t \leq m \), and \( i \in A_0 \). Let \( E^{(b)} = \{ K^{\text{ord}} \mid K \in B(A_0, K_0) \} \). Let \( K(X) = \{ K \in B(A_0, K_0) \mid \text{there is some } i \in K_0 \text{ and some component } \xi \text{ of } X \text{ such that } i + \xi \in K \} \). Let \( K^{\text{ord}}(X) = (K_1, \ldots, K_p) \) be an arbitrary but fixed ordering of \( K(X) \). For each \( K_j \in K(X) \) there is a unique \( \xi_j \in A_0 \) such that \( K_0 + \xi_j = K_j \). Let \( X^{(b)} = (\xi_1, \ldots, \xi_p) \). Note that the components are all distinct and reflect the ordering of \( K^{\text{ord}}(X) \).

Let \( A^{(b)} = A^m \), i.e., the \( m \)-th cross product of \( A \), \( m \) being the number of cells of \( M \) in a block.

Let \( L \) be the set of all local transformations possible for \( M \), i.e., all mappings \( \sigma_j : A^n \rightarrow A \), \( n \) being the number of components of \( X \). Let \( L^{(b)} \) be the set of all mappings \( \sigma_j^{(b)} : (A^m)^p \rightarrow A^m \).

An injection \( \gamma : L \rightarrow L^{(b)} \) is defined as follows. First, let
\[
\beta : \{1, \ldots, m\} \times \{1, \ldots, n\} \rightarrow \{1, \ldots, m\} \times \{1, \ldots, p\}
\]
be the mapping defined by: \( \beta(k, j) = (r, s) \) if and only if for the \( k \)-th cell, \( i \), in \( K_0 \), and the \( j \)-th component, \( \xi \), of \( X \), \( i + \xi \) is the \( s \)-th cell of the \( r \)-th block in \( K^{\text{ord}}(X) \). Using \( \beta \), we now define \( \gamma : \sigma_j = \sigma_j^{(b)} \) if and only if for any \((a_{11}, \ldots, a_{1m}), \ldots, (a_{p1}, \ldots, a_{pm})\) \( (a_{11}, \ldots, a_{1m}), \ldots, (a_{p1}, \ldots, a_{pm}) = (a_1, \ldots, a_m) \) for each \( t \), \( 1 \leq t \leq m \), \( \sigma_j(a_{\beta(t,1)}, \ldots, a_{\beta(t,n)}) = a_t \). The reader should be able to verify that \( \gamma \) is indeed injective. If \( J \) is the subset of \( L \) defining exactly the parallel transformations in \( I \), let \( J^{(b)} \) be the image of \( J \) under \( \gamma \).

The structure \( (A^{(b)}, E^{(b)}, X^{(b)}, J^{(b)}) \) will be called a blocked structure determined by \( M \) and the blocking \( B(A_0, K_0) \). Note that a blocked structure does not fit our definition of a tessellation automaton, but from it we will be able to define a TA behaviorally (but not necessarily structurally) isomorphic to \( M \).

Let \( M^{(b)} = (A^{(b)}, E^{(b)}, X^{(b)}, J^{(b)}) \) be a blocked structure determined by \( M = (A, E^d, X, I) \) and a blocking \( B(A_0, K_0) \). Let \( M_1 = (A_1, E^d, X_1, I_1) \) be a TA defined as follows. \( A_1 \) is any state alphabet such that \( \#A_1 = (\#A)^m \). Let \( \mu_0 : A^{(b)} \rightarrow A_1 \) be any bijection. With \( \Theta = (\theta_1, \ldots, \theta_d) \) as a basis of \( A_0 \), define \( \mu : E^{(b)} \rightarrow E^d \) as follows, \( \mu_0(K_1) = (z_1, \ldots, z_d) \) if \( \xi = (z_1\theta_1 + \cdots + z_d\theta_d) \)
and $K_j = K_0 + \zeta$. With $X$ the set of components of $X$, let $\mu_x : X^{(b)} \to X_1$ be the bijection defined by $\mu_x(\zeta) = \mu_x(K_0 + \zeta)$. With $L_1$ the set of all mappings from $(A_i)^p$ into $A_1$, $p$ being the number of components of $X^{(b)}$, define $\mu_x : J^{(b)} \to L_1$ as follows. For any $\sigma_j^{(b)} \in J^{(b)}$, $\mu_x(\sigma_j^{(b)}) = \sigma_j^{(1)}$ if and only if for any $a^{(b)}$, $a_1^{(b)}$, ..., $a_p^{(b)} \in A^{(b)}$, $\sigma_j^{(b)}(a_1^{(b)}, ..., a_p^{(b)}) = a^{(b)} \Leftrightarrow \sigma_j^{(1)}(\mu_x(a_1^{(b)}), ..., \mu_x(a_p^{(b)})) = \mu_x(a^{(b)})$. Finally, let $J_1$ be the image of $J^{(b)}$ under $\mu_x$, and let $\mu_x : J^{(b)} \to J_1$ be the (bijective) restriction of $\mu_x$. $I_1$ is then the set of all parallel transformations definable from $J_1$.

We call $M_1$ a TA constructible from blocked structure $M^{(b)}$. Note that $M_1$ depends on the choice of $\Theta$.

**Proposition 1.** Let $M^{(b)}$ be a blocked structure determined by TA $M$ and a blocking $B(A_0, K_0)$ of rank $d$. If $M_1$ is a TA constructible from $M^{(b)}$, then $M$ and $M_1$ are behaviorally isomorphic.

**Proof.** With $C$ and $C_1$ the sets of all (array) configurations for $M$ and $M^{(1)}$, and $I$ and $I_1$ their input alphabets, we must show the existence of bijections $\mu_e : C \to C_1$ and $\mu_r : I \to I_1$ such that for any $e \in C$ and any $\tau \in I$,

$$\mu_e(\tau(e)) = \mu_e(\tau)(\mu_r(e)).$$

With $C^{(b)} = \{c^{(b)} \mid c^{(b)} : E^{(b)} \to A^{(b)}\}$, we can define a bijection $\mu' : C \to C^{(b)}$ in the obvious way, i.e., $\mu'(c) = c^{(b)}$ if and only if $c^{(b)}((i_1, ..., i_m)) = (c(i_1), ..., c(i_m))$. A bijection $\mu' : C^{(b)} \to C_1$ exists by associating with each $c^{(b)} \in C^{(b)}$, $c_1 \in C_1$ as follows. Each $i \in E^d$ is the image of a unique $B_i$ under $\mu_e : E^{(b)} \to E^d$. If $c^{(b)}(B_i) = a^{(b)}$, then $c_1(i) = \mu_x(a^{(b)})$. We define $\mu_c : C \to C_1$ as the composition of $\mu'$ and $\mu''$, i.e.,

$$\mu_c : C \xrightarrow{\mu'} C^{(b)} \xrightarrow{\mu''} C_1.$$

Let $\rho : J \to J_1$ be defined by

$$\rho : J \xrightarrow{\nu} J^{(b)} \xrightarrow{\mu_x} J_1.$$

$\mu_r : I \to I_1$ is defined from $\rho$ in the obvious way, and is clearly a bijection.

The rest of the proof, though tedious, is straightforward. ::

If $M_1$ is constructible from $M$ by blocking, then the number of components in the respective neighborhoods need not be the same (see the example at the beginning of this section). This leads to

**Proposition 2.** There exist TA $M$ and $M_1$ such that $M$ and $M_1$ are behaviorally but not structurally isomorphic.
VIII. Neighborhood Standardizations by Blocking

In this section we show by means of blocking that any TA is behaviorally isomorphic to a TA with a neighborhood index of any of a number of standard forms.

By a \((d\text{-dimensional})\) partial Moore neighborhood index we shall mean an index \(X = (\xi_1, \xi_2, \ldots, \xi_n)\) where each component is an element of the set \(\{(z_1, \ldots, z_d) \mid z_j \in \{-1, 0, 1\}, 1 \leq j \leq d\}\), and \(\xi_i \neq \xi_k\) if \(i \neq k\).

A partial Moore neighborhood (for a \(d\)-dimensional TA) with (all) \(3^d\) components will be called a Moore neighborhood index. This is the generalization to \(d\)-dimensions of what was used in Moore (1962) for the two-dimensional case.

Using Propositions V.8 and VII.1, the following can be established.

**Proposition 1.** For any TA \(M_1\), there exists a behaviorally isomorphic TA \(M_2\) with a partial Moore neighborhood index, and \(M_2\) is constructible from \(M_1\) through a blocked structure arising from a partition blocking.

Trivially, the word *partial* can be dropped from the above statement. One could add the required components and let all parallel transformations be independent of these added components. This concept of "dummy neighbors" was discussed in Section III of Yamada–Amoroso (1969).

If \(M_1\) with partial Moore neighborhood index \(X_1\) arises by a partition blocking from \(M_2\) with neighborhood index \(X_2\), then if the structure of \(X_2\) is sufficiently complex, the structure of \(X_1\) can be much simpler. It should be noted that there are cases however where \(X_1\) turns out more complex than \(X_2\), even if \(X_1\) is reduced (i.e., even if all dummy neighbors are removed). An example is shown below in Fig. 2.

![Diagram](image)

(a) \(E^2\) AND \(X\)

(b) \(E^{(b)}\) AND \(X^{(b)}\)

Fig. 2. An increase in the number of neighbors by blocking.
An appropriate coordinate transformation before blocking can lead, in some cases, to a simpler neighborhood index. An example of this is shown below in Fig. 3.

![Diagram](image)

**FIG. 3.** The effect of an appropriate coordinate transformation before blocking.

We now proceed to show that appropriate use of coordinate transformation and a partition blocking can always lead to a behaviorally isomorphic TA with a proper partial Moore neighborhood. First some preliminary concepts. Let $X = (\xi_1, ..., \xi_n)$ be an arbitrary neighborhood index in $d$ dimensions. $X$ will be called an *equisignum* neighborhood index if each component $\xi_j = (z_1, ..., z_a)$ is such that all nonzero $z_i$, $1 \leq i \leq d$, are of the same sign.

**Lemma 2.** For any TA $M_1$, there exists a behaviorally isomorphic TA $M_2$ with an equisignum neighborhood index.

**Proof.** Let $M_1$ and $M_2$ have neighborhood indices $X_1$ and $X_2$, respectively.

If $d = 1$ any neighborhood index is equisignum; hence we can assume $d \geq 2$. From Proposition 1 and the transitivity of a behavioral isomorphism we can assume further that $X_1$ is a partial Moore neighborhood index. Let $X_2 = \varphi_{\Theta}(X_1)$, i.e., the neighborhood index that arises from a coordinate transformation where $\Theta = (\theta_1, ..., \theta_d)$ is the basis defined by

$$
\theta_1 = (1, 0, 0, 0, ..., 0, 0),
$$
$$
\theta_2 = (-1, 1, 0, 0, ..., 0, 0),
$$
$$
\theta_3 = (-1, -1, 1, 0, ..., 0, 0),
$$
$$
... 
$$
$$
\theta_d = (-1, -1, -1, -1, ..., -1, 1).
$$
Suppose \( \varphi(i) = j \), where \( i = (a_1, \ldots, a_d) \) and \( j = (b_1, \ldots, b_d) \), then

\[
(a_1, \ldots, a_d) = b_1 \theta_1 + b_2 \theta_2 + \cdots + b_d \theta_d.
\]

This implies that

\[
\begin{align*}
b_d &= a_d, \\
b_{d-1} &= a_{d-1} + a_d, \\
b_{d-2} &= a_{d-2} + a_{d-1} + 2a_d, \\
b_{d-3} &= a_{d-3} + a_{d-2} + 2a_{d-1} + 4a_d, \\
&\quad \vdots \\
b_2 &= a_2 + a_3 + 2a_4 + \cdots + 2^{d-3}a_d, \\
b_1 &= a_1 + a_2 + 2a_3 + 4a_4 + \cdots + 2^{d-2}a_d.
\end{align*}
\]

Clearly each \( b_i, 1 \leq i \leq d \), is either 0 or has the same sign as \( a_d \). The proof is completed by Proposition 9 of Section V.

Lemma 2 can be strengthened such that \( M_1 \mathcal{R} M_2 \).

By a \((d\text{-dimensional})\) partial equisignum standard neighborhood index we mean an equisignum neighborhood index which is also a partial Moore neighborhood index. By a \((d\text{-dimensional})\) equisignum standard neighborhood index we mean a "largest" partial equisignum standard neighborhood index, i.e., one with \((2^{d+1} - 1)\) components. Clearly the equisignum standard is a partial Moore but not a (total) Moore neighborhood index.

Using exactly the same construction process required to establish Proposition 1, we can also establish:

**Lemma 3.** For any TA \( M_1 \) with an equisignum neighborhood index, there exists a behaviorally isomorphic TA \( M_2 \) with a partial equisignum standard neighborhood index, and \( M_2 \) is constructible from \( M_1 \) from a blocked structure arising from a partition blocking.

The above results lead easily to the following.

**Theorem 4.** For any TA \( M_1 \), there exists a behaviorally isomorphic TA \( M_2 \) with a partial equisignum standard neighborhood index, and \( M_2 \) is constructible from \( M_1 \) by some coordinate transformation and some blocked structure arising from a partition blocking.

We conjecture that Theorem 4 is as much as can be said concerning the reduction of neighborhood structure preserving behavioral isomorphism.
relative to the \textit{partition} blocking concept. This conjecture is a special case of the following more general open problem.

Let $R^n$ be an $n$-dimensional Euclidean space and let $G$ be a group with $n$ generators operating on $R^n$, i.e., for all $g \in G$ and all $x \in R^n$, $gx \in R^n$ and the mapping that takes $x$ to $gx$ is a homeomorphism from $R^n$ to $R^n$. Further, $(g_1g_2)x = g_1(g_2(x))$ and $ex = x$, for all $x \in R^n$, $e$ being the identity for $G$. Let $F \subset R^n$ be a fundamental region of $G$ over $R^n$, i.e., $GF = R^n$ and for all $g \neq e$, $gF \cap F = \emptyset$. (Note the relation between fundamental region and kernel block.) Clearly, $B = \{gF \mid g \in G\}$ is a partition of $R^n$ and the classes of $B$ form a group under the operation defined by $b_1b_2 = g_1g_2F$ if $b_1 = g_1F$ and $b_2 = g_2F$. Assume $F$ consists of $m$ connected regions, $1 \leq m$. Define a neighborhood relation $\rho$ on $B$ by: $b_1 \rho b_2$ if and only if there is an $x_1 \in b_1$ such that any $\epsilon$-neighborhood $N_{\epsilon}$ of $x_1$ (in the sense of analysis) contains an element $x_2 \in b_2$, or there exists an $x_2 \in b_2$ such that any $\epsilon$-neighborhood of $x_2$ contains an element $x_1 \in b_1$. That is, $b_1 \rho b_2 \iff (\exists x_1 \in b_1)(\exists x_2 \in b_2)(x_2 \in N_{\epsilon}(x_1)$ or $x_1 \in N_{\epsilon}(x_2))$. Then $\rho$ is reflexive and symmetric. Let $\nu(b_i) = \{b_j \in B \mid b_i \rho b_j\}$. It can be shown that $\#\nu(b_i)$ is constant for all $b_i \in B$, and we denote this constant by $\#(G,F)$. The problem can now be stated.

Among all possible groups $G_i^{(n)}$ operating on $R^n$, and among all fundamental regions $F_i^{(n)}$ associated with $G_i^{(n)}$, for each $n$, find a pair $(G_p^{(n)}, F_p^{(n)})$ such that $\#(G_p^{(n)}, F_p^{(n)})$ is minimum.

An obvious generalization is to remove the restriction to a Euclidean space. However, it may then no longer be relevant to the notion of tessellation automata as we defined it.

Nonpartition cover blockings can lead to the following further reduction in neighborhood structure.

By a \textit{(d-dimensional) von Neumann} neighborhood index we mean a $(2d+1)$-tuple $X = (\xi_1, \ldots, \xi_{2d+1})$ where one component is $0^d$ and the rest are each of the form $(0, \ldots, 0, z, 0, \ldots, 0)$ where $z \in \{1, -1\}$, i.e., each has exactly one nonzero component, that being either 1 or $-1$. $X$ is called a \textit{partial} von Neumann neighborhood index if all its components are components of some von Neumann neighborhood index. This is a generalization to $d$ dimensions of the neighborhood used in von Neumann (1966).

**Theorem 5.** \textit{For any TA $M_1$, there exists a behaviorally isomorphic TA $M_2$ with a (partial) von Neumann neighborhood index, and $M_2$ is constructible from a blocked structure (arising from a cover blocking).}

The proof of this result can easily be translated into our framework from the proof given, e.g., in Cole (1966).

We conjecture that Theorem 5 is as much as can be said concerning the reduction of neighborhood structure preserving behavioral isomorphism.

IX. EQUIVALENCE INDUCED BY NEIGHBORHOOD SHIFTS

Let $X_1$ and $X_2$ be neighborhood indices of $d$-dimensional TA. We say that $X_2$ arises from $X_1$ by shifting the affected cell by $\rho$, where $\rho \in \mathbb{Z}^d$, if $X_2 = (\xi_1 - \rho, \ldots, \xi_n - \rho)$, where $X_1 = (\xi_1, \ldots, \xi_n)$. We express this by writing $X_2 = X_1 - \rho$.

A $\rho$-shift of a configuration can be defined as follows. Let $c_j$ and $c_k$ be arbitrary mappings from $E^d$ into $A$ and $\rho \in \mathbb{Z}^d$, then $c_k$ is a $\rho$-shift of $c_j$ if and only if for every $i \in E^d$,

$$c_k(i) = c_j(i - \rho).$$

PROPOSITION 1. If $M_1 = (A, E^d, X_1, I_1)$ and $M_2 = (A, E^d, X_2, I_2)$ where $X_2 = X_1 - \rho$ for some $\rho \in \mathbb{Z}^d$, and if $\tau_1 \in T_1$ and $\tau_2 \in T_2$ are both defined from the same local transformation, then for any $c : E^d \rightarrow A$, $c\tau_2$ is a $\rho$-shift of $c\tau_1$.

This proposition states that configurations are merely shifted and not "distorted" when transformed by corresponding transformations in TA whose neighborhoods are related by shifting of the affected cell. The laminations for two such TA, however, can be quite different. For example, it can be shown that for a two-dimensional TA with a neighborhood index $X = ((0,0), (2,0), (0,-6), (2,-6))$, by shifting the affected cell, i.e., by considering all indices $X - \rho$, $\rho \in \mathbb{Z}^2$, exactly eight different laminal sub-modules $A_0(X - \rho)$ can be generated. Working out the details of this example can illustrate a number of facts about this shifting phenomenon. For example, even if shifting the affected cell changes the lamination number $n_1 = \#(E^d/A_0(X))$ to $n_2 = \#(E^d/A_0(X - \rho))$, neither of $n_1$ or $n_2$ may divide the other, and even if $n_1$ divides $n_2$ it may not be the case that one of $A_0(X)$ or $A_0(X - \rho)$ contains the other. Also, not every divisor of $\#(E^d/A_0(X))$ is represented among the numbers $\#(E^d/A_0(X - \rho))$ as $\rho$ is varied. Finally, it is not always possible to obtain a nonlaminal TA from a laminated one by shifting the affected cell (even though the rank of $A_0$ equals the rank of $E^d$).

Some further properties of this neighborhood shifting are listed below.
PROPOSITION 2. If $0^d$ is a component of $X_1$ and $X_2 = X_1 - \rho$, then
$$E^d/A_0(X_1) \subseteq E^d/A_0(X_2).$$

PROPOSITION 3. If $X_2 = X_1 - \rho$, then $\rho \in A_0(X_1)$ if and only if $A_0(X_2) \subseteq A_0(X_1)$.

COROLLARY 3.1. If $X_2 = X_1 - \rho$, $0^d$ is not a component of $X_1$ and is a component of $X_2$, then $A_0(X_1) \subseteq A_0(X_2)$.

COROLLARY 3.2. If $X_2 = X_1 - \rho$ and $0^d$ is a component of $X_1$ and $X_2$, then $A_0(X_1) = A_0(X_2)$.

The converse of Corollary 3.2 is not true. For example, with $X_1 = ((1, 0), (0, 1))$ and $X_2 = ((-1, 0), (0, -1))$, $A_0(X_1) = A_0(X_2)$.

If $X_2 = X_1 - \rho$ and $0^d$ is not a component of $X_2$, then there are cases where $A_0(X_1) \supset A_0(X_2)$, $A_0(X_1) \subset A_0(X_2)$, $A_0(X_1) = A_0(X_2)$, or where neither is contained in the other. ($\supset$ signifies proper containment.)

PROPOSITION 4. If $X_2 = X_1 - \rho$ for some $\rho \in A_0(X_1)$, $0^d$ is a component of $X_1$, and $0^d$ is not a component of $X_2$, then
$$A_0(X_1) = A_0(X_2).$$

Clearly, from Proposition 1, if one were only interested in “patterns” of array configurations, one would equate two TA that differ only by a shifting of the affected cell. In line with this, we define the following equivalence relation over $[M(A,a,T)]$.

$$M_1 R_s M_2 \iff \text{for some } \rho \in \mathbb{Z}^d, X_2 = X_1 - \rho \text{ where } X_1, X_2 \text{ are the neighborhood indices for } M_1, M_2.$$

THEOREM 5. For arbitrary TA $M_1, M_2 \in [M(A,a,T)]$, $M_1 R_s M_2$ does not imply that $M_1$ and $M_2$ are behaviorally isomorphic; hence, not structurally isomorphic as well.

A proof of this can easily be formulated from the following remark. A “nontrivial” configuration $c$ which is a fixed point for a transformation $\tau$ would not be a fixed point on a different TA related by $R_s$ for the corresponding transformation.

By the sum of two relations $R_1$ and $R_2$ we mean $x(R_1 + R_2)y$ if and only if there is a sequence $x = x_1, x_2, ..., x_n = y$ such that either $x_iR_1x_{i+1}$ or $x_iR_2x_{i+1}$ for all $i$, $1 \leq i \leq n - 1$. 
The inclusion lattice for all the relations introduced and certain meaningful sums and products is shown in Fig. 4, where $R_U$ is the universal relation over $[M^{(a,d,T)}]$. $R_i$ is below $R_j$ in the diagram if $R_i$ is a proper refinement of $R_j$. Further, if two relations are not on a common descending path, then neither relation is included in the other.

![Diagram of the inclusion lattice of equivalence relations over $[M^{(a,d,T)}]$.]

A standard representation of an equivalence class of $R_s$ over $[M^{(a,d,T)}]$ could reasonably be defined as a TA such that if its neighborhood index components are considered as lattice points in $d$-space, their center of gravity would be closest to the origin. It is easily seen that this standard representation would often not be uniquely determined.

**X. Weak Behavioral Homomorphism and Further Neighborhood Reduction**

Consider a TA $M = (A, E^d, X, I)$ with a designated $a_0 \in A$ that we shall refer to as the quiescent symbol. Let $c_0, c_1, \ldots$ be a sequence of configurations of $M$ such that for $k = 0, 1, 2, \ldots$, $c_{k+1} = c_k \tau_{j_k}$, $\tau_{j_k} \in I$, and $c_\ell(i) \neq a_0$ for only finitely many $i \in E^d$. Suppose finally that for each $c_k$ in the sequence,
the “patterns” of nonquiescent symbols are clustered near cell 0^d and that in each case they are bounded within some fixed area.

If a hardware version of \( M \) were actually constructed, the tessellation array would be approximated by a finite array of cells. For this situation, \( R_s \) would clearly not be a natural equivalence since the patterns of nonquiescent symbols might drift right off the finite array if the neighborhood were changed by a \( p \)-shift.

At other times, e.g., in the setting of Yamada–Amoroso (1970), \( R_s \) is an extremely natural relation.

In line with this, we now introduce a weaker form of a behavioral isomorphism. As we shall see below, this form of isomorphism can be made to relate TA insensitive to “pattern shifts.”

Let \( C, C' \) and \( I, I' \) be the complete sets of array configurations and the parallel transformation sets for two \( d \)-dimensional TA \( M \) and \( M' \). Let \( R \) and \( R' \) be two equivalence relations defined on the respective sets \( C \) and \( C' \) such that for any \([c_1] \in C/R\) and any \( \tau \in I \), \([c_1] \tau \subseteq [c_2] \) for some \([c_2] \in C/R\), and for any \([c_1'] \in C'/R'\) and any \( \tau' \in I' \), \([c_1'] \tau' \subseteq [c_2'] \) for some \([c_2'] \in C'/R'\).

An ordered triple of mappings \( \mu_{wb} = (\mu_c, \mu_\tau, \mu_E) \) is said to be a weak behavioral homomorphism from TA \( M \) to TA \( M' \) (both \( d \)-dimensional), denoted by

\[
M \sim_{\mu_{wb}} M'
\]

if and only if

\[
\begin{align*}
\mu_c : C & \rightarrow C', \\
\mu_\tau : I & \rightarrow I', \\
\mu_E : C/R & \rightarrow C'/R',
\end{align*}
\]

such that for any \( c_1 \in C \) and any \( \tau_1 \in I \),

\[
\mu_E([c_1 \tau_1]) = [\mu_c(c_1) \mu_\tau(\tau_1)].
\]

If \( \mu_c, \mu_\tau, \) and \( \mu_E \) are bijective, and if \( \mu_{wb}' = (\mu_c^{-1}, \mu_\tau^{-1}, \mu_E^{-1}) \) is a weak behavioral homomorphism from \( M' \) to \( M \), then \( M \) and \( M' \) are said to be weakly behaviorally isomorphic, or \( \mu_{wb} \) is said to be a weak behavioral isomorphism from \( M \) to \( M' \).

**Proposition 1.** A behavioral homomorphism is a weak-behavioral homomorphism.

The converse is not true.
PROPOSITION 2. If TA $M$ and $M'$ are shift equivalent, then they are weakly behaviorally isomorphic.

A neighborhood index $X = (\xi_1, ..., \xi_n)$ is said to be positive if for each component $\xi_i$, each of its components is nonnegative, i.e., if $\xi_i = (z_{1i}, ..., z_{di})$, then $z_{ji} \geq 0$, $1 \leq j \leq d$.

COROLLARY 2.1. For any TA $M = (A, E^d, X, I)$, there exists a TA $M' = (A, E^d, X', I')$ such that $M$ is weakly behaviorally isomorphic to $M'$, $X'$ is positive, and $MR_s M'$.

By a (d-dimensional) partial positive standard neighborhood index, we mean a positive neighborhood index which is also a partial Moore neighborhood index. By a (d-dimensional) positive standard neighborhood index we mean a "largest" partial positive index, i.e., one with $2^d$ components.

PROPOSITION 3. For an arbitrary TA $M = (A, E^d, X, I)$, there exists a TA $M' = (A', E^d, X', I')$ such that $M$ and $M'$ are weakly behaviorally isomorphic, $X'$ is a partial positive standard neighborhood index, and $M'$ arises from $M$ by a shift of the affected cell and a partition blocking.

By a (d-dimensional) standard neighborhood index we mean a (d + 1)-tuple that is some permutation of $(0^d, (1, 0, ..., 0), (0, 1, 0, ..., 0), ..., (0, ..., 0, 1))$. Note that a standard index is a positive index that is also a partial von Neumann index.

THEOREM 4. For any TA $M = (A, E^d, X, I)$ there exists a TA $M' = (A', E^d, X', I')$ such that $M$ and $M'$ are weakly behaviorally isomorphic, $X'$ is a standard neighborhood index, and $M'$ arises from $M$ by a shift of the affected cell and a cover blocking.

Consider again the class of TA with a quiescent symbol $a_0$ and let $\overline{C}$ be the complete set of finitely supported array configurations, i.e., $\overline{C} = \{c \in C \mid c(i) \neq a_0 \text{ for only finitely many } i \in E^d\}$. The reader should be able to verify the following, through the use of the density of patterns to be represented in a given finite area [see Moore (1962)].

THEOREM 5. With respect to $\overline{C}$ only, the standard neighborhood index is the simplest index to which any neighborhood index can be reduced preserving weak behavioral isomorphism between the respective TA.

The case for general $C$ will be treated in the sequel. All neighborhood reductions discussed so far are summarized in Fig. 5.
XI. Decomposition and Simulation

In this section we deal with a construction process that is a reversal of the process of inducing a behaviorally isomorphic TA by blocking.

Let $C_1$ and $C_2$ be the complete array configuration sets for TA $M_1 = (A_1, E^a, X_1, I_1)$ and $M_2 = (A_2, E^b, X_2, I_2)$, respectively.

We define a behavioral isomorphism from $M_1$ into $M_2$, denoted by $M_1 \simeq_{b} M_2$, as follows: $M_1 \simeq_{b} M_2$ if and only if there exist injections $\mu_c : C_1 \rightarrow C_2$ and $\mu_I : I_1 \rightarrow I_2$ such that for any $c_1 \in C_1$ and $\tau_1 \in I_1$,

$$\mu_c(c_1\tau_1) = \mu_c(c_1) \cdot \mu_I(\tau_1).$$

We will also denote this by saying that $M_2$ simulates $M_1$ (in real time).
**Theorem 1.** For any TA $M_1$ and for any state alphabet $A_2$ such that $\#A_2 > 1$, there exists a TA $M_2$ with state alphabet $A_2$ such that $M_1 \subset M_2$.

**Proof.** If $\#A_2 > \#A_1$, it is a trivial matter to construct an $M_2$ satisfying the theorem. Assume, therefore, that $\#A_2 < \#A_1$.

Choose the smallest integer $n_0$ such that

$$(\#A_2)^{\left(n_0 - 1\right)} \geq \#A_1,$$

where $d$ is the tessellation dimension of $M_1$ (and $M_2$). Let $P_0 \subset E^d$ be defined by

$$P_0 = \{(i_1, i_2, \ldots, i_d) \mid 0 \leq i_k < n_0\},$$

and let submodule $A_0 \subset E^d$ be defined by the basis $\Theta = \{n_0 e_1, \ldots, n_0 e_d\}$, where $\{e_1, \ldots, e_d\}$ is the standard basis of $E^d$. Then $B(A_0, P_0)$ is a partition blocking of $E^d$. Define the bijection $f_0 : E^d \to B(A_0, P_0)$ by $f_0(k) = P_k \mapsto P_k = P_0 + n_0 k$. Let 0, 1 $\in A_2$ be two distinguished symbols of $A_2$. For the kernel block $P_0$ of $B(A_0, P_0)$, the set $W_0(P_0)$ of 0-wall cells of $P_0$ is defined by

$$W_0(P_0) = \{(0, i_2, i_3, \ldots, i_d) \mid 0 \leq i_r < n_0, 2 \leq r \leq d\},$$

and the set $W_1(P_0)$ of 1-wall cells of $P_0$ is defined by

$$W_1(P_0) = \{(1, i_2, i_3, \ldots, i_d) \mid 1 \leq i_r < n_0, 2 \leq r \leq d\},$$

and the set $W_0(P_0)$ of 0-wall cells of $P_0$ is defined by

$$W_0(P_k) = \{(i_1, i_2, \ldots, i_d) \mid 1 \leq i_r < n_0, 2 \leq r \leq d\},$$

and the set $W_1(P_k)$ of 1-wall cells of $P_k$ is defined by

$$W_1(P_k) = \{(i_1, i_2, \ldots, i_d) \mid 1 \leq i_r < n_0, 2 \leq r \leq d\},$$

For any $P_0 \in B(A_0, P_0)$, the set $W_0(P_0)$ of 0-wall cells of $P_0$ is defined by

$$W_0(P_k) = \{i \in E^d \mid (\exists j \in W_0(P_0))(i = j + k)\}.$$
and the set $W_1(P_k)$ of 1-wall cells of $P_k$ is defined by

$$W_1(P_k) = \{ i \in \mathbb{E}^d \mid (\exists j \in W_1(P_0))(i = j + k) \}.$$ 

Intuitively, $W_0 = \bigcup_{P_a \in \mathbb{R}(\mathbb{A}, \mathbb{p})} W_0(P_k)$ makes up walls of cells which are on the boundaries of blocks, and $W_1 = \bigcup_{P_a \in \mathbb{R}(\mathbb{A}, \mathbb{p})} W_1(P_k)$ makes up the inner lining cells just inside of $W_0$ for each block. Clearly, $W_0(P_k) \subset P_k$, $W_1(P_k) \subset P_k$, and $W_0(P_k) \cap W_1(P_k) = \emptyset$.

The set $D(P_k) = P_k - W_0(P_k) - W_1(P_k)$ for each $k$ will be referred to as the data cells of $P_k$. Clearly, each $D(P_k)$ is an array of $(n_0 - 3)^d$ cells “inside” of $W_1(P_k)$. By a code we shall mean a mapping $\gamma : D(P_0) \rightarrow A_a$, i.e., a code is a configuration of the cells in $D(P_0)$. The set $\Gamma$ of all codes, the dictionary, clearly has cardinality $(#A_a)^{(n_0-3)^d}$. By an encoding we shall mean an injection $\gamma : A_1 \rightarrow \Gamma$. Note that our initial choice of $n_0$ guarantees the existence of such a $\gamma$.

We now define an injection $\mu_\gamma : C_1 \rightarrow C_a$ ($C_i$ being the complete set of array configurations for $M_i$, $i = 1, 2$) as follows: For any $c_1 \in C_1$, $\mu_\gamma(c_1) = c_2$ implies, for any $k$, $c_2(i) = 0$ if $i \in W_0(P_k)$, $c_2(i) = 1$ if $i \in W_1(P_k)$, and if for any $k$, $c_2(k) = a$, then for any $i \in D(P_0) + n_0k$, $c_2(i) = (\gamma(a))(i - n_0k)$. In other words, the information that $c_1(i) = a$ is held in $D(P_0) + n_0k$ being in a configuration mirroring the encoding of $a$.

Our remaining tasks are to define $X_2$ and $T_2$ for $M_2$, and injection $\mu_\gamma : I_1 \rightarrow T_2$. Intuitively, each cell of $M_2$ must have enough neighbors to determine its position in a $D(P_0)$, and enough to be able to take part in the configuration change of $D(P_k)$ to mirror the corresponding change in cell $k$ of $M$, where $D(P_k) = D(P_0) + n_0k$. It is sufficient therefore that $X_2$ contains all cells in $\{(z_1, \ldots, z_d) \mid |z_i| \leq n_0, 1 \leq i \leq d\}$ (this can determine its position in a $P_k$) and all data cells in each $P_j$ such that $P_j = P_0 + n_0(r + k)$ where $r$ is a neighbor of $k$ in $M_1$.

With $T_2$ the total set of parallel transformations for $X_2$ and $A_2$, we define $\mu_\gamma : I_1 \rightarrow T_2$ such that for any $\tau_i \in I$, and for any $c_1 \in C_1$, $\mu_\gamma(c_1\tau_i) = \mu_a(c_1)\mu_\gamma(\tau_i)$. The construction of $\mu_\gamma$ is tedious but clearly possible.

Had the $#A_2$ been greater than two, we could have used a designated symbol $a_m \in A_2$ to mark the cells in $\{0^d + n_0k \mid k \in \mathbb{Z}^d\}$ and then used $A_2 - \{a_m\}$ to encode the cell states of $M_1$. The “blocks” $P_k$ would be located from the $a_m$ symbols eliminating the need for the $W_0(P_k)$ and $W_1(P_k)$ cells. However, when $#A_2 = 2$, we no longer have the luxury of the dis-

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4 (Added in proof): The configurations in $C_2$ involving 0-walls and 1-walls do not have finite support. It is clearly possible to avoid such configurations by the choice of more complex transformations in $T_2$ which would build 0-walls and 1-walls when they are needed, and eliminate then when they are no longer needed.
tesellation automata

29

tinguished \( a_m \), and the boundaries of \( P_k \) must be established by a uniquely identifiable patterns in coding. The pattern of \( W_0(P_h) \) and \( W_1(P_h) \) walls employed in the above proof is sufficient for the purpose, but it does not necessarily lead to an \( M_2 \) with a minimal size.

This fact leads to the general problem of designing locally uniquely decodable \( d \)-dimensional minimal codes. In other words, given two alphabets \( A_1 \) and \( A_2 \) such that \( \#A_1 > \#A_2 \), and dimension \( d \), find a partition blocking \( B(A_0, P_0) \) with the minimal \( \#P_0 \) such that (a) the dictionary \( \Gamma \) of codes \( \eta : P_0 \rightarrow A_2 \) has cardinality \( \#\Gamma \geq \#A_1 \) (note the walls of 0-cells and 1-cells are now eliminated), and (b) no matter how codes are assigned over the \( P_k \), the boundaries of \( P_k \) should be uniquely determined from the values of cells within a fixed finite neighboring region of \( P_k \).

Suppose now, for \( \#A_2 = n \), we construct a set \( A \) of length \( t \) words in \( A_2 \) such that for all pairs of words \( a_1a_2 \ldots a_t \) and \( b_1b_2 \ldots b_t \) in \( A \), none of the overlaps \( a_2a_3 \ldots a_t b_1, a_2a_3 \ldots a_t b_2, \ldots, a_2a_3 \ldots a_t b_{t-1} \) are in \( A \). Such a \( A \) is called a comma-free dictionary of length \( k \) in \( A_2 \), because any string of words \( \omega_1 \ldots \omega_r \in A^* \) can be locally uniquely deconcatenated. Let \( W(n, t) \) denote the greatest number of words that such a dictionary can contain. For various pairs of \( (n, t) \) such a number is known (see, e.g., Jiggs, 1963).

Coming back to our problem, it can be restated in the following form: Given the size \( t \) of the kernel block \( P_0 \) and \( A_2 \), which shape of \( P_0 \) permits the largest dictionary \( A \) of locally uniquely decodable codes? We conjecture that, for every dimension \( d \), kernel block size \( t \), and size \( n \) of the alphabet \( A_2 \), if we take \( P_0 \) to be length one along all coordinate axes, except a coordinate axis that has length \( t \), and use comma-free dictionary \( A \) of length \( t \) words in \( A_2 \), then it will be at least as efficient as any other coding scheme. We also conjecture that this comma-free dictionary for "linear" block \( P_0 \) is by far the largest among all possible shapes of \( P_0 \). We do not have a proof for this, but the reasoning behind this conjecture is that this choice of \( P_0 \) makes it necessary to establish a boundary only in one direction, while any other choice of \( P_0 \) will make it necessary to establish boundaries in other directions as well, which would most likely impose additional constraints on \( A \).

A systematic construction of comma-free dictionaries for various partition blockings in higher dimensions appears to be mathematically challenging.

Given a blocking \( B(A_0, P_0) \) and a compatible dictionary \( A \), whether or not \( A \) is comma-free is always decidable. However, for large \( P_0 \) and \( A_2 \), exhaustive checking becomes tedious, and the search for a simpler test appears to be warranted.

Another question concerned with TA decomposition is whether or not
an arbitrary partition blocking $B(A_0, P_0)$ is equal to the blocked union of some $B(A_0^{(1)}, P_0^{(1)})$ with respect to some $B(A_0^{(2)}, P_0^{(2)})$ and $\Theta^{(1)}$, where $\#P_0^{(1)}$ and $\#P_0^{(2)}$ are both less than $\#P_0$. If not, we say $B(A_0, P_0)$ is a prime blocking. The reader should be able to prove the following:

**Proposition 2.** For an arbitrary $d$-dimensional TA, $d > 1$, there are infinitely many prime partition blocking of $E^d$.

**XII. Concluding Remarks**

The main purpose of this report was to present some results obtained in our attempt to make precise the intuitive concept of two tessellation automata being "essentially" the same. Clearly, the concept of behavioral isomorphism is essential for our purposes.

On the other hand, it seems likely that alternative definitions for the concept of "structural" isomorphism are possible.

We have seen that TA related by $R_1 + R_2$ are structurally isomorphic. We would be interested in knowing whether or not a structural isomorphism between $M_1$ and $M_2$ implies $M_1$ can be changed to $M_2$ by a neighborhood permutation and a coordinate transformation.

The concepts of structural and behavioral homomorphisms, which for us were just preliminary to the corresponding isomorphisms, are clearly worth studying further.

We believe that the blocking concept is "exactly the right concept" for the trade-off of cell states and neighborhood structure preserving behavioral isomorphism. We feel that a careful clarification of this statement would warrant the further research implied.

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**References**


