JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 177, 386–411 (1993)

Multiplicativity Factors for Orlicz Space Function Norms

RICHARD ARENS

Department of Mathematics, University of California, Los Angeles, California 90024

MOSHE GOLDBERG

Department of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

AND

W. A. J. LUXEMBURG

Department of Mathematics, California Institute of Technology, Pasadena, California 91125

Submitted by Dorothy Maharam Stone

Received January 20, 1992

Let ρ_φ be a function norm defined by a Young function φ with respect to a measure space $(\mathbf{T}, \Omega, m)$, and let L^φ be the Orlicz space determined by ρ_φ. If L^φ is an algebra, then a constant $\mu > 0$ is called a multiplicativity factor for ρ_φ if $\rho_\varphi(fg) \leqslant \mu \rho_\varphi(f)\, \rho_\varphi(g)$ for all $f, g \in L^\varphi$. The main objective of this paper is to give conditions under which L^φ is indeed an algebra, and to obtain in this case the best (least) multiplicativity factor for ρ_φ. The first of our principal results is that L^φ is an algebra if and only if

$$m_{\inf} \equiv \inf\{m(\mathbf{A}) > 0 : \mathbf{A} \in \Omega\} > 0$$

or

$$x_\infty(\varphi) \equiv \sup\{x \geqslant 0 : \varphi(x) < \infty\} < \infty.$$

Our second main result states that if L^φ is an algebra and $(\mathbf{T}, \Omega, m)$ is free of infinite atoms, then the best multiplicativity factor for ρ_φ is $\varphi^{-1}(1/m_{\inf})$ if $m_{\inf} > 0$, and $x_\infty(\varphi)$ if $m_{\inf} = 0$. © 1993 Academic Press, Inc.

0022-247X/93 $5.00

1. Introduction

Throughout this paper let $(\mathbf{T}, \Omega, m)$ be a measure space, where $\mathbf{T}$ is a nonempty set, Ω a σ-algebra of subsets of $\mathbf{T}$, and m a nontrivial, countably additive, nonnegative measure. Let $\mathcal{M} = \mathcal{M}(\mathbf{T}, \Omega, m)$ denote the class of all $\mathbb{F}$-valued, Ω-measurable functions on $\mathbf{T}$. Then (e.g., [Z1]) $\mathcal{M}$ is a function algebra with respect to the usual pointwise operations.

Following [LZ1] we call a mapping

$$\rho\colon \mathcal{M} \to [0, \infty]$$

a *function norm* if for $f, g \in \mathcal{M}$ and $\alpha \in \mathbb{F}$,

$$\rho(f) = 0 \qquad \text{if and only if} \quad f = 0 \text{ a.e.}, \tag{1.1a}$$

$$\rho(\alpha f) = |\alpha|\, \rho(f), \tag{1.1b}$$

$$\rho(f + g) \leqslant \rho(f) + \rho(g), \tag{1.1c}$$

$$\rho \text{ is } \textit{monotonic}, \text{ i.e., } |f| \leqslant |g| \text{ a.e. implies } \rho(f) \leqslant \rho(g). \tag{1.1d}$$

We readily see that ρ is *absolute*, that is

$$\rho(|f|) = \rho(f), \qquad f \in \mathcal{M}.$$

Indeed, for $f \in \mathcal{M}$ set $g = |f|$. Then $|f| = |g|$; hence $\rho(f) \leqslant \rho(g) \leqslant \rho(f)$, and the assertion follows.

As $\rho(f)$ is not necessarily finite for every $f \in \mathcal{M}$, we consider the class

$$\mathcal{M}_\rho = \mathcal{M}_\rho(\mathbf{T}, \Omega, m) \equiv \{f \in \mathcal{M} : \rho(f) < \infty\}$$

of functions on which ρ is finite.

As is customary, we identify in $\mathcal{M}$ the equivalence classes of functions equal a.e. on $\mathbf{T}$. If $[f]$ is the equivalence class to which f belongs, then obviously

$$g \in [f] \text{ implies } \rho(g) = \rho(f),$$

so we define

$$\rho([f]) = \rho(f), \qquad f \in \mathcal{M}.$$

The partitioning of $\mathcal{M}$ into classes partitions $\mathcal{M}_\rho$ as well, so we set

$$L_\rho = L_\rho(\mathbf{T}, \Omega, m) \equiv \{[f] : f \in \mathcal{M}_\rho\} = \{[f] : f \in \mathcal{M}, \rho(f) < \infty\}$$

and readily obtain:

THEOREM 1.1 [LZ1, Theorem 3.5]. *L_ρ is a linear space over $\mathbb{F}$, closed under absolute values; and ρ is an absolute, monotonic norm on L_ρ.*

We recall that an $\mathbb{F}$-valued function f is *m-essentially bounded* if for some constant $\gamma > 0$,

$$m\{t \in \mathbf{T} : |f(t)| > \gamma\} = 0.$$

As usual, we denote by $L^\infty = L^\infty(\mathbf{T}, \Omega, m)$ the algebra of equivalence classes of all $\mathbb{F}$-valued, m-essentially bounded functions on $\mathbf{T}$ and let

$$\|f\|_\infty \equiv \inf\{\gamma > 0 : m\{t \in \mathbf{T} : |f(t)| > \gamma\} = 0\}$$

be the norm on L^∞.

With this common definition of L^∞ we can easily show that L_ρ is often large enough to contain L^∞:

THEOREM 1.2. [AGL, Theorem 2.2(a)]. *Let ρ be a function norm. Then the following are equivalent:*

(i) $L^\infty \subseteq L_\rho$.

(ii) *L_ρ contains the constant functions.*

(iii) *L_ρ contains e, the function of constant value 1.*

We call our function norm ρ *σ-subadditive* if

$$\{f_n\}_1^\infty \subset \mathcal{M},\ f_n \geqslant 0, \text{ implies } \rho\left(\sum_1^\infty f_n\right) \leqslant \sum_1^\infty \rho(f_n).$$

Combining Theorems 4.2 and 4.8 of [LZ1] we get:

THEOREM 1.3 [LZ1]. *L_ρ is complete (with respect to ρ) if and only if ρ is σ-subadditive.*

Suppose now that L_ρ is an algebra. Then as usual, we call the function norm ρ *submultiplicative* if

$$\rho(fg) \leqslant \rho(f)\,\rho(g) \qquad \forall f, g \in L_\rho.$$

Similarly, if L_ρ is closed under squaring (i.e., $f \in L_\rho$ implies $f^2 \in L_\rho$), then we say that ρ is *subquadrative* if

$$\rho(f^2) \leqslant \rho(f)^2 \qquad \forall f \in L_\rho.$$

Given a function norm ρ and a constant $\mu > 0$ then evidently, $\rho_{(\mu)} \equiv \mu\rho$ is a function norm too and

$$L_\rho = L_{\rho_{(\mu)}}.$$

If L_ρ is an algebra then $\rho_{(\mu)}$ may or may not be submultiplicative on L_ρ. If it is, we call μ a *multiplicativity factor* or simply an M-factor for ρ.

Analogously, if L_ρ is closed under squaring and $\rho_{(\lambda)} \equiv \lambda\rho$ is subquadrative on L_ρ for $\lambda > 0$, we call λ a *quadrativity factor* or a *Q-factor* for ρ.

We see at once that *if L_ρ is an algebra, then $\mu > 0$ is an M-factor for ρ if and only if*

$$\rho(fg) \leqslant \mu\rho(f)\,\rho(g) \qquad \forall f, g \in L_\rho.$$

Likewise, *if L_ρ is closed under squaring, then $\lambda > 0$ is a Q-factor for ρ if and only if*

$$\rho(f^2) \leqslant \lambda\rho(f)^2 \qquad \forall f \in L_\rho.$$

Evidently, if α_0 is an M- or a Q-factor for ρ, then so is every $\alpha \geqslant \alpha_0$. So with the above definitions, we entertained in [AGL] the following two questions: Under what conditions is L_ρ an algebra, and in this case what are the best (least) M- and Q-factors for ρ?

In answering these questions we proved:

THEOREM 1.4 [AGL, Theorem 2.5]. *Let ρ be a σ-subadditive function norm. Then the following are equivalent*:

(a) *L_ρ is closed under squaring.*

(b) *L_ρ is closed under multiplication, hence an algebra.*

(c) $L_\rho \subseteq L^\infty$.

THEOREM 1.5 [AGL, Theorem 2.6]. *Let ρ be a σ-subadditive function norm, and let L_ρ satisfy the equivalent conditions* (a)–(c) *in Theorem* 1.4. *Then*

(i) *L_ρ is a subalgebra of L^∞.*

(ii) *ρ has M- hence Q-factors on L_ρ.*

(iii) *The sets of M- and Q-factors for ρ coincide.*

(iv) *If $L_\rho \neq \{0\}$ then,*

$$\mu_\rho \equiv \sup\{\|f\|_\infty : f \in L_\rho, \rho(f) \leqslant 1\} \tag{1.2}$$

is an M-factor for ρ.

(v) *If $L_\rho = \{0\}$, then every $\mu > 0$ is an M-factor for ρ.*

Note that if $L_\rho = \{0\}$, then μ_ρ in (1.2) vanishes, hence it is not an M-factor by definition.

We call a set $\mathbf{A} \in \Omega$ an *infinite atom* if $m(\mathbf{A}) = \infty$, and for every $\mathbf{B} \in \Omega$ with $\mathbf{B} \subseteq \mathbf{A}$, either $m(\mathbf{B}) = \infty$ or $m(\mathbf{B}) = 0$.

Further, following [LZ4] we call a function norm ρ *saturated* if for every set $\mathbf{A} \in \Omega$ of finite positive measure, there exists a measurable subset $\mathbf{B} \subseteq \mathbf{A}$ of positive measure, such that the characteristic function $\chi_{\mathbf{B}}$ is in L_ρ, i.e., $\rho(\chi_{\mathbf{B}}) < \infty$.

With these definitions we provided conditions under which μ_ρ in (1.2) is the best M-factor for ρ:

THEOREM 1.6. [AGL, Theorem 2.9]. *Let* $(\mathbf{T}, \Omega, m)$ *be free of infinite atoms, and let* ρ *be a* σ*-subadditive, saturated function norm. If* L_ρ *satisfies the equivalent conditions* (a)–(c) *in Theorem* 1.4, *then*:

(i) $L_\rho \neq \{0\}$.

(ii) L_ρ *is a subalgebra of* L^∞.

(iii) ρ *has M- hence Q-factors on* L_ρ.

(iv) *The sets of M- and Q-factors for* ρ *coincide.*

(v) *The best* (*least*) *M- and Q-factors for* ρ *on* L_ρ *are both given by* μ_ρ *in* (1.2).

2. Orlicz Space Function Norms

We recall that a mapping

$$\varphi\colon [0, \infty) \to [0, \infty]$$

is a *Young function* if

$$\varphi(0) = 0 \text{ and } \varphi \text{ does not vanish identically,} \tag{2.1a}$$

$$\varphi \text{ is not identically } \infty \text{ for } x > 0, \tag{2.1b}$$

$$\varphi \text{ is monotone increasing,} \tag{2.1c}$$

$$\varphi \text{ is convex where it is finite,} \tag{2.1d}$$

$$\varphi \text{ is left-continuous.} \tag{2.1e}$$

The convexity of φ implies that

$$\varphi \text{ is continuous on each interval } [0, a] \text{ whereon it is finite.} \tag{2.2}$$

Further, it is not hard to see that

$$\varphi(x) \xrightarrow[x \to \infty]{} \infty. \tag{2.3}$$

Let $(\mathbf{T}, \Omega, m)$ and $\mathcal{M} = \mathcal{M}(\mathbf{T}, \Omega, m)$ be as in Section 1. Given a Young function φ we define a mapping

$$\rho_\varphi : \mathcal{M} \to [0, \infty]$$

by

$$\rho_\varphi(f) = \inf\left\{\tau > 0 : \int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \leqslant 1\right\}, \qquad f \in \mathcal{M}, \tag{2.4}$$

and prove:

LEMMA 2.1. *Let φ be a Young function and let $f \in \mathcal{M}$. Then*

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \leqslant 1 \qquad \forall \tau > 0 \text{ with } \tau \geqslant \rho_\varphi(f).$$

Proof. If $\rho_\varphi(f) = \infty$, there is nothing to prove, so suppose $\rho_\varphi(f) < \infty$. Then by (2.4) and (2.1c),

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \leqslant 1 \qquad \forall \tau > \rho_\varphi(f). \tag{2.5}$$

Finally, let $\rho_\varphi(f) > 0$ and take $\tau \downarrow \rho_\varphi(f)$. Then by (2.1c) and (2.1e),

$$\varphi\left(\frac{|f(t)|}{\tau}\right) \uparrow \varphi\left(\frac{|f(t)|}{\rho_\varphi(f)}\right) \qquad \text{on } \mathbf{T}.$$

So by Levi's Theorem[1] and (2.5),

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\rho_\varphi(f)}\right) dm = \lim_{\tau \downarrow \rho_\varphi(t)} \int_{\mathbf{T}} \rho\left(\frac{|f(t)|}{\tau}\right) dm \leqslant 1. \quad \blacksquare$$

We can now prove:

THEOREM 2.1 (compare [Z2, Theorem 131.5]). *ρ_φ in (2.4) is a function norm.*

Proof. If $f = 0$ a.e. then clearly $\rho_\varphi(f) = 0$. Conversely, if $0 \neq f \in \mathcal{M}$ then we can select a sufficiently small $\varepsilon > 0$ and a set $\mathbf{A} \in \Omega$, such that $m(\mathbf{A}) > 0$ and

$$|f(t)| \geqslant \varepsilon \qquad \text{for} \quad t \in \mathbf{A}.$$

[1] Also known as the Monotone Convergence Theorem.

Hence for $\tau > 0$, (2.1c) implies

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \geqslant \int_{\mathbf{A}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \geqslant \varphi\left(\frac{\varepsilon}{\tau}\right) m(\mathbf{A}). \tag{2.6}$$

Now if $\rho_\varphi(f) = 0$ then by Lemma 2.1,

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \leqslant 1 \qquad \forall \tau > 0. \tag{2.7}$$

So (2.6) and (2.7) imply

$$\varphi\left(\frac{\varepsilon}{\tau}\right) m(\mathbf{A}) \leqslant 1 \qquad \forall \tau > 0. \tag{2.8}$$

On the other hand, by (2.3),

$$\varphi\left(\frac{\varepsilon}{\tau}\right) m(\mathbf{A}) \xrightarrow[\tau \to 0]{} \infty,$$

a contradiction to (2.8), so (1.1a) holds.

As ρ_φ obviously satisfies (1.1b), we next prove (1.1c). If either $f = 0$ or $g = 0$ a.e. then there is nothing to prove, so assume $0 \neq f,\ g \in \mathcal{M}$. Then $\rho_\varphi(f) > 0$ and $\rho_\varphi(g) > 0$; so by Lemma 2.1 again,

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\rho_\varphi(f)}\right) dm \leqslant 1, \qquad \int_{\mathbf{T}} \varphi\left(\frac{|g(t)|}{\rho_\varphi(g)}\right) dm \leqslant 1. \tag{2.9}$$

Setting $\rho_\varphi(f) + \rho_\varphi(g) \equiv \gamma > 0$, we write

$$\rho_\varphi(f) = \alpha\gamma, \qquad \rho_\varphi(g) = \beta\gamma, \qquad \alpha + \beta = 1. \tag{2.10}$$

Thus, by (2.1c), (2.1d), (2.10), and (2.9),

$$\begin{aligned}
&\int_{\mathbf{T}} \varphi\left(\frac{|f(t) + g(t)|}{\gamma}\right) dm \\
&\quad \leqslant \int_{\mathbf{T}} \varphi\left(\frac{|f(t)| + |g(t)|}{\gamma}\right) dm \\
&\quad = \int_{\mathbf{T}} \varphi\left(\alpha \frac{|f(t)|}{\alpha\gamma} + \beta \frac{|g(t)|}{\beta\gamma}\right) dm \\
&\quad \leqslant \alpha \int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\alpha\gamma}\right) dm + \beta \int_{\mathbf{T}} \varphi\left(\frac{|g(t)|}{\beta\gamma}\right) dm \\
&\quad = \alpha \int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\rho_\varphi(f)}\right) dm + \beta \int_{\mathbf{T}} \varphi\left(\frac{|g(t)|}{\rho_\varphi(g)}\right) dm \leqslant \alpha + \beta = 1;
\end{aligned}$$

hence

$$\rho_\varphi(f+g) \leqslant \gamma = \rho_\varphi(f) + \rho_\varphi(g).$$

Finally, let $f, g \in \mathcal{M}$ satisfy $|f| \leqslant |g|$ a.e. Again, if $g = 0$ a.e., there is nothing to prove; so let $g \neq 0$. Then $\rho_\varphi(g) > 0$; hence

$$\frac{|f(t)|}{\rho_\varphi(g)} \leqslant \frac{|g(t)|}{\rho_\varphi(g)} \qquad \text{a.e.} \tag{2.11}$$

By (2.11), (2.1c), and Lemma 2.1,

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\rho_\varphi(g)}\right) dm \leqslant \int_{\mathbf{T}} \varphi\left(\frac{|g(t)|}{\rho_\varphi(g)}\right) dm \leqslant 1,$$

so by (2.4), $\rho_\varphi(g) \geqslant \rho_\varphi(f)$. ∎

Theorems 2.1 and 1.1 imply now that

$$L_{\rho_\varphi} = \{[f] : f \in \mathcal{M}, \rho_\varphi(f) < \infty\}$$

is a linear space, closed under absolute values; and ρ_φ is an absolute, monotonic norm on L_{ρ_φ}.

The norm ρ_φ in (2.4) is usually called the *Luxemburg norm* on L_{ρ_φ} (e.g., [KR, p. 78; Z2, p. 582]). From the historical point of view it is interesting to observe that ρ_φ is the *Minkowski gauge function* determined by the convex set

$$\left\{f \in \mathcal{M} : \int_{\mathbf{T}} \varphi(|f(t)|)\, dm \leqslant 1\right\}.$$

PROPOSITION 2.1. *Let φ be a Young function, and let $f \in \mathcal{M}$. Then $f \in L_{\rho_\varphi}$ if and only if*

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm < \infty \qquad \textit{for some} \quad \tau > 0. \tag{2.12}$$

Proof. If $f \in L_{\rho_\varphi}$ then $\rho_\varphi(f) < \infty$; so (2.4) implies (2.12). Conversely, suppose (2.12) holds. By (2.1a)–(2.1c), and (2.2), as $\tau \uparrow \infty$ we have

$$\varphi\left(\frac{f(t)}{\tau}\right) \downarrow 0.$$

So by Levi's Theorem,

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \xrightarrow[\tau \uparrow \infty]{} 0;$$

hence the integral is bounded by 1 for some $\tau > 0$, and by (2.4), $\rho_\varphi(f) < \infty$. ▌

As in the statement and proof of Proposition 2.1, we often use f for $[f]$. This is commonly done in measure theoretic contexts.

The class of functions $f \in \mathcal{M}$ that satisfy (2.12) is known (e.g., [O]) as the *Orlicz space* determined by φ and is traditionally denoted by L^φ. By Proposition 2.1 we have

$$L_{\rho_\varphi} = L^\varphi,$$

and we henceforth prefer L^φ as the name of this space.

Note that if p, $1 \leqslant p < \infty$, is fixed then

$$\varphi(x) = x^p, \qquad x \geqslant 0,$$

is a Young function and

$$\begin{aligned}\rho_\varphi(f) &= \inf\left\{\tau > 0 : \int_{\mathbf{T}} \left|\frac{f(t)}{\tau}\right|^p dm \leqslant 1\right\} \\ &= \inf\left\{\tau > 0 : \left(\int_{\mathbf{T}} |f(t)|^p \, dm\right)^{1/p} \leqslant \tau\right\} \\ &= \left(\int_{\mathbf{T}} |f(t)|^p \, dm\right)^{1/p} \equiv \|f\|_p, \qquad f \in \mathcal{M}.\end{aligned}$$

So $L^p(\mathbf{T}, \Omega, m)$, the classical L^p space, is an Orlicz space.

If our Young function is given by

$$\varphi(x) = \begin{cases} 0, & 0 \leqslant x \leqslant 1 \\ \infty, & x > 1, \end{cases}$$

we get

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm = \begin{cases} 0, & \tau \geqslant \|f\|_\infty, \\ \infty, & \tau < \|f\|_\infty. \end{cases}$$

Hence

$$\rho_\varphi(f) = \|f\|_\infty, \qquad f \in \mathcal{M};$$

and so L^∞ is also an Orlicz space.

We prove next:

THEOREM 2.2. *Let φ be a Young function. Then ρ_φ is σ-subadditive and L^φ is complete.*

Proof. Take $\{f_j\}_1^\infty \subset \mathcal{M}$, $f_j \geqslant 0$. If $\sum_1^\infty \rho_\varphi(f_j) = \infty$ then there is nothing to prove. So suppose

$$\sum_{j=1}^{\infty} \rho_\varphi(f_j) = \alpha < \infty, \tag{2.13}$$

where without loss of generality we may exclude $\alpha = 0$. Set

$$g = \sum_{j=1}^{\infty} f_j,$$

$$g_n = \sum_{j=1}^{n} f_j, \qquad n = 1, 2, 3, \ldots.$$

Clearly

$$\rho_\varphi(g_n) = \rho_\varphi\left(\sum_{j=1}^{n} f_j\right) \leqslant \sum_{j=1}^{n} \rho_\varphi(f_j) \leqslant \sum_{j=1}^{\infty} \rho_\varphi(f_j) = \alpha. \tag{2.14}$$

As $f_j \geqslant 0$, we have $g_n \uparrow g$, and since φ is monotone increasing and left continuous,

$$\varphi\left(\frac{g_n(t)}{\alpha}\right) \uparrow \varphi\left(\frac{g(t)}{\alpha}\right) \qquad \text{on } \mathbf{T}.$$

By Levi's Theorem, (2.14), and Lemma 2.1, therefore,

$$\int_{\mathbf{T}} \varphi\left(\frac{g(t)}{\alpha}\right) dm = \lim_{n \to \infty} \int_{\mathbf{T}} \varphi\left(\frac{g_n(t)}{\alpha}\right) dm \leqslant 1.$$

Hence (2.4) and (2.13) imply

$$\rho_\varphi(g) \leqslant \alpha = \sum_1^\infty \rho_\varphi(f_j);$$

so ρ_φ is σ-subadditive and Theorem 1.3 completes the proof. ∎

We remark that the σ-subadditivity of ρ_φ can also be deduced by combining [Z2, Theorem 131.6; LZ2, Theorem 5.3; LZ1, Theorem 4.2].

Given a Young function φ we set now

$$x_0(\varphi) = \sup\{x \geqslant 0 : \varphi(x) = 0\}, \tag{2.15a}$$

$$x_\infty(\varphi) = \sup\{x \geqslant 0 : \varphi(x) < \infty\}. \tag{2.15b}$$

By (2.1a)–(2.1c), we immediately obtain

$$0 \leqslant x_0(\varphi) \leqslant x_\infty(\varphi) \leqslant \infty; \tag{2.16a}$$

$$x_0(\varphi) < \infty, \qquad x_\infty(\varphi) > 0; \tag{2.16b}$$

$$\varphi(x) > 0 \qquad \forall x > x_0(\varphi). \tag{2.16c}$$

Next, we define the *left inverse* of a Young function φ by

$$\varphi^{-1}(x) = \sup\{y \geqslant 0 : \varphi(y) \leqslant x\}, \qquad x \geqslant 0.$$

Clearly, φ^{-1}, like φ, is monotone increasing and left-continuous. Further,

$$0 < \varphi^{-1}(x) < \infty \qquad \forall x > 0; \tag{2.17a}$$

$$x_0(\varphi) \leqslant \varphi^{-1}(x) \leqslant x_\infty(\varphi) \qquad \forall x \geqslant 0; \tag{2.17b}$$

$$x \leqslant \varphi^{-1}(\varphi(x)) \qquad \forall 0 \leqslant x < x_\infty(\varphi); \tag{2.17c}$$

and

$$\varphi(\varphi^{-1}(x)) \leqslant x \qquad \forall x \geqslant 0 \tag{2.17d}$$

with equality if and only if φ is continuous at x.

Note that if

$$\varphi(x) \xrightarrow[x \uparrow x_\infty(\varphi)]{} \infty,$$

then φ^{-1} is the usual inverse of φ when φ is restricted to $[x_0(\varphi), x_\infty(\varphi)]$. If, however,

$$x_\infty(\varphi) < \infty \qquad \text{and} \qquad \varphi(x) \xrightarrow[x \uparrow x_\infty(\varphi)]{} \gamma < \infty,$$

then

$$\varphi^{-1}(x) = x_\infty(\varphi) \qquad \forall x \geqslant \gamma.$$

With this we can prove now an Orlicz space extension of Theorem 1.2:

THEOREM 2.3. *Let φ be a Young function. Then*

(a) *The following are equivalent*:

(i) $L^\infty \subseteq L^\varphi$.

(ii) *L^φ contains the constant functions.*

(iii) *L^φ contains e, the function of constant value* 1.

(iv) $x_0(\varphi) > 0$ *or*[2] $m(\mathbf{T}) < \infty$.

[2] The conditions $x_0(\varphi) > 0$ and $m(\mathbf{T}) < \infty$ are obviously independent, just as φ and m are.

(b) *If* $x_0(\varphi)>0$ *then*

$$x_0(\varphi)\,\rho_\varphi(f)\leqslant\|f\|_\infty \qquad \forall f\in L^\infty; \tag{2.18a}$$

and if $m(\mathbf{T})<\infty$ *then* $\varphi^{-1}(1/m(\mathbf{T}))>0$ *and*

$$\varphi^{-1}\left(\frac{1}{m(\mathbf{T})}\right)\rho_\varphi(f)\leqslant\|f\|_\infty \qquad \forall f\in L^\infty. \tag{2.18b}$$

Proof. (a) As (i)–(iii) are equivalent by Theorem 1.2, we prove the equivalence of (iii) and (iv).

Indeed, let (iii) hold. Then by Lemma 2.1,

$$\varphi\left(\frac{1}{\rho_\varphi(e)}\right)m(\mathbf{T})=\int_{\mathbf{T}}\varphi\left(\frac{e(t)}{\rho_\varphi(e)}\right)dm\leqslant 1. \tag{2.19}$$

Now, if $x_0(\varphi)=0$, then $\varphi(1/\rho_\varphi(e))>0$; hence $m(\mathbf{T})<\infty$ by (2.19), so (iii) implies (iv).

Conversely, suppose (iv) holds. Since $\varphi(x_0(\varphi))=0$ and $x_0(\varphi)<\infty$, $x_0(\varphi)>0$ implies

$$\int_{\mathbf{T}}\varphi\left(\frac{e(t)}{1/x_0(\varphi)}\right)dm=\int_{\mathbf{T}}\varphi(x_0(\varphi))\,dm=0<\infty;$$

so $e\in L^\varphi$ by Proposition 2.1. If on the other hand $m(\mathbf{T})<\infty$, then by (2.1b) we can choose $\tau>0$ such that $\varphi(1/\tau)<\infty$; so

$$\int_{\mathbf{T}}\varphi\left(\frac{e(t)}{\tau}\right)dm=\varphi\left(\frac{1}{\tau}\right)m(\mathbf{T})<\infty,$$

and again $e\in L^\varphi$.

(b) Let $x_0(\varphi)>0$ and take $0\neq f\in L^\infty$. Then

$$\frac{|f(t)|}{\|f\|_\infty}x_0(\varphi)\leqslant x_0(\varphi) \qquad \text{a.e.};$$

hence

$$\varphi\left(\frac{|f(t)|}{\|f\|_\infty}x_0(\varphi)\right)=0 \qquad \text{a.e.},$$

and so

$$\int_{\mathbf{T}}\varphi\left(\frac{|f(t)|}{\|f\|_\infty}x_0(\varphi)\right)dm=0\leqslant 1.$$

By (2.4) therefore,

$$\rho_\varphi(f) \leqslant \frac{\|f\|_\infty}{x_0(\varphi)},$$

and (2.18a) follows.

Finally, suppose $m(\mathbf{T}) < \infty$ and let $0 \neq f \in L^\infty$. Since m is a nontrivial measure, $m(\mathbf{T}) > 0$, so we have $\varphi^{-1}(1/m(\mathbf{T})) > 0$ by (2.17a). As

$$\frac{|f(t)|}{\|f\|_\infty} \leqslant 1 \qquad \text{a.e.,}$$

we use the monotonicity of φ and (2.17d) to obtain

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\|f\|_\infty} \varphi^{-1}\left(\frac{1}{m(\mathbf{T})}\right)\right) dm$$
$$\leqslant \int_{\mathbf{T}} \varphi\left(\varphi^{-1}\left(\frac{1}{m(\mathbf{T})}\right)\right) dm \leqslant \int_{\mathbf{T}} \frac{1}{m(\mathbf{T})}\, dm = 1.$$

So by (2.4) again,

$$\rho_\varphi(f) \leqslant \frac{\|f\|_\infty}{\varphi^{-1}\left(\dfrac{1}{m(\mathbf{T})}\right)}$$

and we are done. ∎

We recall that a set $\mathbf{A} \in \Omega$ is a *finite atom* if $0 < m(\mathbf{A}) < \infty$, and for every $\mathbf{B} \in \Omega$ satisfying $\mathbf{B} \subseteq \mathbf{A}$, either $m(\mathbf{B}) = 0$ or $m(\mathbf{B}) = m(\mathbf{A})$.

Further, $(\mathbf{T}, \Omega, m)$ is *completely atomic* if for every $\mathbf{B} \in \Omega$ with $m(\mathbf{B}) > 0$ there exists a finite atom $\mathbf{A} \in \Omega$ such that $\mathbf{A} \subseteq \mathbf{B}$.

For the sake of completeness we now state and prove the following facts:

LEMMA 2.2. (a) *If $\mathbf{B} \in \Omega$ is a set of positive measure containing no atoms, then there exists a sequence $\{\mathbf{B}_n\}_1^\infty$ of mutually disjoint sets of positive measure such that*

$$m(\mathbf{B}_n) \xrightarrow[n \to \infty]{} 0.$$

(b) *If*

$$m_{\inf} \equiv \inf\{m(\mathbf{B}) : \mathbf{B} \in \Omega, m(\mathbf{B}) > 0\} = 0, \tag{2.20}$$

then the conclusion of part (a) *holds.*

(c) *If* $g \in L^1(\mathbf{T}, \Omega, m)$, *then its support*

$$\mathbf{G} = \{t \in \mathbf{T} : g(t) \neq 0\}$$

is of σ-finite measure; i.e., there exist a sequence of at most countably many disjoint sets $\{\mathbf{G}_n\}$ *of finite measure such that* $\mathbf{G} = \bigcup \mathbf{G}_n$.

(d) *If* $m_{\text{inf}} > 0$, *then* $(\mathbf{T}, \Omega, m)$ *is completely atomic.*

(e) *If* $(\mathbf{T}, \Omega, m)$ *is completely atomic and* $\mathbf{G}$ *is a set of σ-finite measure, then* $\mathbf{G}$ *is a union of at most countably many atoms.*

(f) *If* $h \in \mathscr{M}(\mathbf{T}, \Omega, m)$ *and* $\mathbf{A} \in \Omega$ *is an atom, then* h *takes a constant value a.e. on* $\mathbf{A}$.

Proof. (a) Set $\mathbf{C}_1 = \mathbf{B}$. As $\mathbf{C}_1$ is of positive measure containing no atoms, we can find a measurable set $\mathbf{E} \subset \mathbf{C}_1$ with $0 < m(\mathbf{E}) < m(\mathbf{C}_1)$ where either $m(\mathbf{E}) < \frac{1}{2}m(\mathbf{C}_1)$ or $m(\mathbf{C}_1 \backslash \mathbf{E}) < \frac{1}{2}m(\mathbf{C}_1)$. Let $\mathbf{C}_2$ be that part of $\mathbf{C}_1$ with $m(\mathbf{C}_2) < \frac{1}{2}m(\mathbf{C}_1)$. Since $\mathbf{C}_2$ is free of atoms and $m(\mathbf{C}_2) > 0$, we can repeat the process, obtaining a nested sequence $\{\mathbf{C}_n\}$ satisfying $0 < m(\mathbf{C}_n) \downarrow 0$. Hence the required sequence is

$$\mathbf{B}_n = \mathbf{C}_n \backslash \mathbf{C}_{n-1}, \qquad n = 1, 2, 3, \ldots.$$

(b) If $m_{\text{inf}} = 0$ then there exists a sequence $\{\mathbf{B}\}_1^\infty$ of sets of positive measure such that

$$m(\mathbf{B}_n) \xrightarrow[n \to \infty]{} 0.$$

If every $\mathbf{B}_n$ is an atom, then since distinct atoms are disjoint, we are done. If not, then Ω contains a set $\mathbf{B}$ free of atoms with $m(\mathbf{B}) > 0$, and (b) follows from (a).

(c) Let g be an integrable function, and put

$$\mathbf{E}_n = \left\{t \in \mathbf{T} : \frac{1}{n} \leqslant |g(t)| \leqslant n\right\}, \qquad n = 1, 2, 3, \ldots.$$

Then

$$\frac{1}{n} m(\mathbf{E}_n) \leqslant \int_{\mathbf{T}} |g(t)| \, dm < \infty;$$

so $m(\mathbf{E}_n) < \infty$. Since $\mathbf{E}_n \uparrow \mathbf{G}$, we take

$$\mathbf{G}_1 = \mathbf{E}_1; \qquad \mathbf{G}_n = \mathbf{E}_n \backslash \mathbf{E}_{n-1}, \qquad n = 2, 3, 4, \ldots$$

and (c) follows.

(d) If $(\mathbf{T}, \Omega, m)$ is not completely atomic, then there exists a set $\mathbf{A} \in \Omega$ free of atoms with $m(\mathbf{A}) > 0$. Hence $m_{\text{inf}} = 0$ by part (a), and (d) follows.

(e) For each $n = 1, 2, 3, \ldots$, set

$$\Gamma_n = \left\{ \mathbf{A} \in \Omega \colon \mathbf{A} \text{ an atom in } \mathbf{G}; m(\mathbf{A}) \geqslant \frac{1}{n} \right\}.$$

Since $\mathbf{G}$ is of σ-finite measure, then $\mathbf{G}$ is a union of at most countably many disjoint sets $\{\mathbf{G}_j\}$ of finite measure. As $m(\mathbf{G}_j) < \infty$ and since distinct atoms are disjoint, it follows that Γ_n contains at most countably many atoms. Hence $\mathbf{H} \equiv \bigcup \Gamma_n$ is a union of at most countably many atoms, consisting of all the atoms in $\mathbf{G}$. If $\mathbf{H} \subsetneq \mathbf{G}$, then there exists a set $\mathbf{E} \subset \mathbf{G}$ with $m(\mathbf{E}) > 0$ and $\mathbf{E} \cap \mathbf{H} = \emptyset$. As $(\mathbf{T}, \Omega, m)$ is completely atomic, we can find an atom $\mathbf{A} \subset \mathbf{E}$ hence $\mathbf{A} \subsetneq \mathbf{H}$, a contradiction; so $\mathbf{G} = \mathbf{H}$ and (e) holds.

(f) Assume $\mathbf{A} \in \Omega$ is an atom and h, a measurable function, is not a constant a.e. on $\mathbf{A}$. Then either h_1, the real part of h, or h_2, the imaginary part of h, is not constant a.e. on $\mathbf{A}$. Suppose h_1 is not constant a.e. Then there is a real γ such that the set

$$\mathbf{E} \equiv \{ t \in \mathbf{A} : h_1(t) < \gamma \}$$

satisfies

$$m(\mathbf{E}) > 0, \qquad m(\mathbf{A} \backslash \mathbf{E}) > 0.$$

Since $\mathbf{A}$ is an atom we obtain

$$m(\mathbf{E}) = m(\mathbf{A} \backslash \mathbf{E}) = m(\mathbf{A});$$

a contradiction, so (f) follows. ∎

We are now ready to prove an Orlicz space version of Theorem 1.4:

THEOREM 2.4 (First Main Theorem). *Let φ be a Young function. Then the following are equivalent*:

(a) *L^φ is closed under squaring.*

(b) *L^φ is closed under multiplication, hence an algebra.*

(c) *$L^\varphi \subseteq L^\infty$.*

(d) *$m_{\text{inf}} > 0$ or[3] $x_\infty(\varphi) < \infty$.*

Proof. By Theorem 2.2, ρ_φ is σ-subadditive. So in view of Theorem 1.4, it suffices to show the equivalence of (c) and (d).

Assume first that (d) does not hold; that is, $m_{\text{inf}} = 0$ and $x_\infty(\varphi) = \infty$. Since $m_{\text{inf}} = 0$, it follows by Lemma 2.2(b) that there exists a sequence $\{\mathbf{B}_n\}_1^\infty$ of mutually disjoint sets with

$$0 < m(\mathbf{B}_n) \leqslant 2^{-n}, \qquad n = 1, 2, 3, \ldots. \tag{2.21a}$$

[3] The conditions $m_{\text{inf}} > 0$ and $x_\infty(\varphi) < \infty$ are independent, as m and φ are.

As $x_\infty(\varphi) = \infty$, there exists an increasing sequence $\{\gamma_n\}_1^\infty$ of positive real numbers satisfying

$$\varphi(\gamma_n) = n, \qquad n = 1, 2, 3, \dots. \tag{2.21b}$$

Let χ_n be the characteristic function of $\mathbf{B}_n$ and put

$$f = \sum_1^\infty \gamma_n \chi_n. \tag{2.21c}$$

Then $f \notin L^\infty$, since obviously $\gamma_n \to \infty$ and $m(\mathbf{B}_n) > 0$ for all n. On the other hand, being a countable sum of measurable functions, f is measurable. By (2.21) therefore,

$$\int_{\mathbf{T}} \varphi(f(t))\, dm = \sum_1^\infty \varphi(\gamma_n)\, m(\mathbf{B}_n) \leqslant \sum_1^n \frac{n}{2^n} < \infty;$$

thus $f \in L^\varphi$ by Proposition 2.1. Since $f \notin L^\infty$, (c) does not hold; so (c) implies (d).

We next prove that (d) implies (c). First, suppose $m_{\inf} > 0$ and let $0 \neq f \in L^\varphi$. Then by Lemma 2.1,

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\rho_\varphi(f)}\right) dm \leqslant 1; \tag{2.22}$$

so putting

$$h(t) \equiv \frac{|f(t)|}{\rho_\varphi(f)}, \tag{2.23}$$

we obtain

$$g \equiv \varphi(h) \in L^1(\mathbf{T}, \Omega, m).$$

Now set

$$\mathbf{G} = \{t \in \mathbf{T} : h(t) > x_0(\varphi)\}.$$

Then $\mathbf{G}$ is the support of g; so $\mathbf{G}$ is of σ-finite measure by Lemma 2.2(c). As $m_{\inf} > 0$, $(\mathbf{T}, \Omega, m)$ is completely atomic by Lemma 2.2(d); so Lemma 2.2(e) implies that $\mathbf{G} = \bigcup \mathbf{A}_n$ is a union of at most countably many (disjoint) atoms. Since f is measurable, so is h; hence by Lemma 2.2(f), h takes constant values on atoms. We thus have

$$h(t) = \delta_n \qquad \text{a.e. on } \mathbf{A}_n,\ n = 1, 2, 3, ..., \tag{2.24}$$

where the δ_n are (nonnegative) constants. Since the atoms $\mathbf{A}_n$ are disjoint, (2.22)–(2.24) and the monotonicity of φ imply

$$\sum_n \varphi(\delta_n)\, m(\mathbf{A}_n) = \int_{\mathbf{G}} \varphi(h(t))\, dm \leqslant \int_{\mathbf{T}} \varphi(h(t))\, dm \leqslant 1.$$

It follows that

$$\varphi(\delta_n) \leqslant \frac{1}{m(\mathbf{A}_n)} \leqslant \frac{1}{m_{\text{inf}}}, \qquad n = 1, 2, 3, ...$$

Hence by the monotonicity of φ^{-1} and (2.17c),

$$\delta_n \leqslant \varphi^{-1}(\varphi(\delta_n)) \leqslant \varphi^{-1}\left(\frac{1}{m_{\text{inf}}}\right), \qquad n = 1, 2, 3, ...; \tag{2.25}$$

so by (2.24) and (2.25),

$$h(t) \leqslant \varphi^{-1}\left(\frac{1}{m_{\text{inf}}}\right) \qquad \text{a.e. on } \mathbf{G}. \tag{2.26a}$$

By the definition of $\mathbf{G}$ and (2.17b), however,

$$h(t) \leqslant x_0(\varphi) \leqslant \varphi^{-1}\left(\frac{1}{m_{\text{inf}}}\right) \qquad \text{on } \mathbf{T} \setminus \mathbf{G}. \tag{2.26b}$$

Thus (2.26) and (2.17a) yield

$$h(t) \leqslant \varphi^{-1}\left(\frac{1}{m_{\text{inf}}}\right) < \infty \qquad \text{a.e. on } \mathbf{T}; \tag{2.27}$$

so $f \in L^\infty$, and (c) holds when $m_{\text{inf}} > 0$ in (d).

Finally, suppose $x_\infty(\varphi) < \infty$, and take $0 \neq f \notin L^\infty$. For $\tau > 0$ we write

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm = \int_{\mathbf{A}_\tau} \varphi\left(\frac{|f(t)|}{\tau}\right) dm + \int_{\mathbf{T} \setminus \mathbf{A}_\tau} \varphi\left(\frac{|f(t)|}{\tau}\right) dm,$$

where

$$\mathbf{A}_\tau = \left\{ t \in \mathbf{T} : \frac{|f(t)|}{\tau} > x_\infty(\varphi) \right\}.$$

As $f \notin L^\infty$, it follows that $m(\mathbf{A}_\tau) > 0$ for all $\tau > 0$; hence

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \geqslant \int_{\mathbf{A}_\tau} \varphi\left(\frac{|f(t)|}{\tau}\right) dm = \infty \cdot m(\mathbf{A}_\tau) = \infty.$$

Consequently $f \notin L^\varphi$; so again (c) holds. ∎

We proceed to prove:

LEMMA 2.3. *Let φ be a Young function.*

(a) *If $m_{\inf} > 0$ then $L^\varphi \subseteq L^\infty$, and*

$$\|f\|_\infty \leqslant \varphi^{-1}\left(\frac{1}{m_{\inf}}\right) \rho_\varphi(f) \qquad \forall f \in L^\varphi. \tag{2.28a}$$

(b) *If $x_\infty(\varphi) < \infty$ then again $L^\varphi \subseteq L^\infty$, and*

$$\|f\|_\infty \leqslant x_\infty(\varphi)\, \rho_\varphi(f) \qquad \forall f \in L^\varphi. \tag{2.28b}$$

Proof. Suppose $m_{\inf} > 0$. Take $0 \neq f \in L^\varphi$ and let h be the function in (2.23). Then by (2.27),

$$|f(t)| \leqslant \varphi^{-1}\left(\frac{1}{m_{\inf}}\right) \rho_\varphi(f) \qquad \text{a.e.}$$

and (a) follows.

Next, let $x_\infty(\varphi) < \infty$. Choose $0 \neq f \in L^\varphi$ and put

$$\mathbf{A} = \{t \in \mathbf{T} : |f(t)| > x_\infty(\varphi)\, \rho_\varphi(f)\}.$$

Since

$$\varphi\left(\frac{|f(t)|}{\rho_\varphi(f)}\right) = \infty \qquad \forall t \in \mathbf{A},$$

Lemma 2.1 yields

$$\infty \cdot m(\mathbf{A}) = \int_{\mathbf{A}} \varphi\left(\frac{|f(t)|}{\rho_\varphi(f)}\right) dm \leqslant \int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\rho_\varphi(f)}\right) dm \leqslant 1.$$

Consequently $m(\mathbf{A}) = 0$; so by the definition of $\mathbf{A}$,

$$|f(t)| \leqslant x_\infty(\varphi)\, \rho_\varphi(f) \qquad \text{a.e. on } \mathbf{T},$$

and (b) follows. ∎

The conditions $m_{\inf} > 0$ and $x_\infty(\varphi) < \infty$ in Theorem 2.4(d) together with this lemma make possible an Orlicz space analogue of Theorem 1.5:

THEOREM 2.5. *Let φ be a Young function, and let L^φ satisfy the equivalent conditions* (a)–(d) *in Theorem* 2.4. *Then*

(i) *L^φ is a subalgebra of L^∞.*

(ii) *ρ_φ has M- hence Q-factors on L^φ.*

(iii) *The sets of M- and Q-factors for ρ_φ coincide.*

(iv) *If $m_{\text{inf}} > 0$, then $\varphi^{-1}(1/m_{\text{inf}})$ is an M-factor for ρ_φ.*

(v) *If $x_\infty(\varphi) < \infty$, then $x_\infty(\varphi)$ is an M-factor for ρ_φ.*

Proof. Part (i) is just (b) and (c) of Theorem 2.4. By Theorem 2.2, ρ_φ is σ-subadditive so (ii) and (iii) follow from Theorem 1.5. Finally, if $L^\varphi = \{0\}$ then (iv) and (v) hold in a trivial manner since every positive number is now an M-factor. If, however, $L^\varphi \neq \{0\}$, Theorem 1.5(iv) implies that

$$\mu_\varphi \equiv \sup\{\|f\|_\infty : f \in L^\varphi, \rho_\varphi(f) \leqslant 1\} \tag{2.29}$$

is an M-factor for ρ_φ. Hence, if $m_{\text{inf}} > 0$, then (2.29) and (2.28a) imply (iv); and if $x_\infty(\varphi) < \infty$, then (2.29) and (2.28b) imply (v). ∎

The following observation is perhaps of independent interest:

PROPOSITION 2.2. *Let φ be a Young function and let $\mathbf{B} \in \Omega$ satisfy $0 < m(\mathbf{B}) < \infty$. Then $\chi_\mathbf{B} \in L^\varphi$ and*

$$\rho_\varphi(\chi_\mathbf{B}) = \left[\varphi^{-1}\left(\frac{1}{m(\mathbf{B})}\right)\right]^{-1}.$$

Proof. By the definitions of ρ_φ and φ^{-1},

$$\begin{aligned}
\rho_\varphi(\chi_\mathbf{B}) &= \inf\left\{\tau > 0 : \int_\mathbf{T} \varphi\left(\frac{\chi_\mathbf{B}}{\tau}\right) dm \leqslant 1\right\} \\
&= \inf\left\{\tau > 0 : \varphi\left(\frac{1}{\tau}\right) m(\mathbf{B}) \leqslant 1\right\} \\
&= \inf\left\{\frac{1}{x} > 0 : \varphi(x) \leqslant \frac{1}{m(\mathbf{B})}\right\} \\
&= \left[\sup\left\{y > 0 : \varphi(y) \leqslant \frac{1}{m(\mathbf{B})}\right\}\right]^{-1} = \left[\varphi^{-1}\left(\frac{1}{m(\mathbf{B})}\right)\right]^{-1}. \quad ∎
\end{aligned}$$

With this proposition we get:

COROLLARY 2.1. *Let φ be a Young function. If $(\mathbf{T}, \Omega, m)$ is free of infinite atoms, then ρ_φ is saturated and $L^\varphi \neq \{0\}$.*

Proof. If $(\mathbf{T}, \Omega, m)$ is free of infinite atoms, then for every $\mathbf{A} \in \Omega$ with $m(\mathbf{A}) > 0$ there exists a subset $\mathbf{B} \subseteq \mathbf{A}$, $\mathbf{B} \in \Omega$, with $0 < m(\mathbf{B}) < \infty$. Hence by Proposition 2.2, $\chi_{\mathbf{B}} \in L^{\varphi}$ and the corollary follows. ∎

We are now in the position to determine the best M- and Q-factors for ρ_{φ} in case L^{φ} is an algebra and $(\mathbf{T}, \Omega, m)$ has no infinite atoms. This provides the following Orlicz space version of Theorem 1.6:

THEOREM 2.6 (Second Main Theorem). *Let φ be a Young function, and let $(\mathbf{T}, \Omega, m)$ be free of infinite atoms. If L^{φ} satisfies the equivalent conditions* (a)–(d) *in Theorem* 2.4, *then*:

(i) $L^{\varphi} \neq \{0\}$.

(ii) *L^{φ} is a subalgebra of L^{∞}.*

(iii) *ρ_{φ} has M- hence Q-factors on L^{φ}.*

(iv) *The sets of M- and Q-factors for ρ_{φ} coincide.*

(v) *If $m_{\text{inf}} > 0$, then the best (least) M- and Q-factors are both given by $\varphi^{-1}(1/m_{\text{inf}})$.*

(vi) *If $m_{\text{inf}} = 0$, then the best M- and Q-factors are given by $x_{\infty}(\varphi)$.*

Proof. The heart of the theorem is (v) and (vi), since (i) follows from Corollary 2.1, (ii) holds by (b)–(c) in Theorem 2.4, and (iii) and (iv) are contained in Theorem 2.5.

To prove (v) suppose $m_{\text{inf}} > 0$. By Theorem 2.2 and Corollary 2.1, ρ_{φ} is σ-subadditive and saturated. Thus, Theorem 1.6 implies that the best M- and Q-factors for ρ_{φ} are both given by μ_{φ} is (2.29). So by Theorem 2.5(iv),

$$\mu_{\varphi} \leqslant \varphi^{-1}\left(\frac{1}{m_{\text{inf}}}\right).$$

To prove the converse inequality, observe that for every $\varepsilon > 0$ there exists a set $\mathbf{A}_{\varepsilon} \in \Omega$ satisfying

$$0 < m(\mathbf{A}_{\varepsilon}) \leqslant m_{\text{inf}} + \varepsilon. \tag{2.30}$$

Now take

$$\tau_{\varepsilon} = \left[\varphi^{-1}\left(\frac{1}{m_{\text{inf}} + \varepsilon}\right)\right]^{-1} > 0$$

and set

$$f_{\varepsilon} = \frac{\chi_{\varepsilon}}{\tau_{\varepsilon}},$$

where χ_ε is the characteristic function of $\mathbf{A}_\varepsilon$. Then by Proposition 2.2, (2.30), and the monotonicity of φ^{-1},

$$\rho_\varphi(f_\varepsilon) = \frac{1}{\tau_\varepsilon}\rho_\varphi(\chi_\varepsilon) = \frac{1}{\tau_\varepsilon}\left[\varphi^{-1}\left(\frac{1}{m(\mathbf{A}_\varepsilon)}\right)\right]^{-1}$$

$$= \frac{\varphi^{-1}(1/(m_{\inf}+\varepsilon))}{\varphi^{-1}(1/m(\mathbf{A}_\varepsilon))} \leqslant 1. \tag{2.31}$$

Further,

$$\|f_\varepsilon\|_\infty = \frac{\|\chi_\varepsilon\|_\infty}{\tau_\varepsilon} = \varphi^{-1}\left(\frac{1}{m_{\inf}+\varepsilon}\right). \tag{2.32}$$

Hence by (2.29), (2.31), and (2.32)

$$\mu_\varphi \geqslant \varphi^{-1}\left(\frac{1}{m_{\inf}+\varepsilon}\right) \qquad \forall \varepsilon > 0.$$

Letting ε tend to zero and using the left-continuity of φ^{-1} we get

$$\mu_\varphi \geqslant \varphi^{-1}\left(\frac{1}{m_{\inf}}\right)$$

and (v) follows.

In order to prove (vi) suppose $m_{\inf} = 0$. Then by hypothesis, $x_\infty(\varphi) < \infty$. As in the previous part of the proof, the best M- and Q-factors for ρ_φ are given by μ_φ in (2.29). So by Theorem 2.5(v),

$$\mu_\varphi \leqslant x_\infty(\varphi).$$

Since $m_{\inf} = 0$, Ω contains sets of arbitrarily small positive measure. Hence for $0 < \varepsilon \leqslant x_\infty(\varphi)$ there exists a set $\mathbf{B}_\varepsilon$ with

$$m(\mathbf{B}_\varepsilon) > 0, \qquad \varphi(x_\infty(\varphi) - \varepsilon)\, m(\mathbf{B}_\varepsilon) \leqslant 1.$$

Let χ_ε be the characteristic function of $\mathbf{B}_\varepsilon$ and set

$$f_\varepsilon = (x_\infty(\varphi) - \varepsilon)\, \chi_\varepsilon.$$

Then

$$\int_{\mathbf{T}} \varphi(f_\varepsilon(t))\, dm = \varphi(x_\infty(\varphi) - \varepsilon)\, m(\mathbf{B}_\varepsilon) \leqslant 1;$$

so

$$\rho_\varphi(f_\varepsilon) \leqslant 1. \tag{2.33}$$

Moreover,

$$\|f_\varepsilon\|_\infty = x_\infty(\varphi) - \varepsilon. \tag{2.34}$$

So by (2.29), (2.33), and (2.34),

$$\mu_\varphi \geqslant x_\infty(\varphi) - \varepsilon;$$

hence

$$\mu_\varphi \geqslant x_\infty(\varphi)$$

and (vi) follows. ▮

For example, fix p, $1 \leqslant p < \infty$, and consider the Young function

$$\varphi(x) = x^p, \qquad x \geqslant 0,$$

and its corresponding Orlicz space $L^\varphi = L^p(\mathrm{T}, \Omega, m)$. Since $x_\infty(\varphi) = \infty$, Theorem 2.4 implies that L^p is an algebra (contained in L^∞) if and only if $m_{\inf} > 0$; and in this case, by Theorem 2.6, the best M- and Q-factors for $\|\cdot\|_p$ are both given by

$$\varphi^{-1}\left(\frac{1}{m_{\inf}}\right) = \left(\frac{1}{m_{\inf}}\right)^{1/p}.$$

Now if m is the Lebesgue or any other nonatomic measure, then $m_{\inf} = 0$; so L^p is not an algebra and the question of M-factors is irrelevant.

On the other hand, if $\mathrm{T} = \mathbb{Z}^+ = \{1, 2, 3, ...\}$, and m is the counting measure assigning to each subset of $\mathbb{Z}^+$ its cardinality, then $\mathscr{M}$ is the algebra of all $\mathbb{F}$-valued sequences $a = \{\alpha_j\}_1^\infty$, L^∞ is the algebra of all bounded sequences, and L^p is the space of all sequences satisfying

$$\|a\|_p = \left(\sum_1^n |\alpha_j|^p\right)^{1/p} < \infty.$$

So $m_{\inf} = 1$, L^p is a subalgebra of L^∞, and $\|\cdot\|_p$ is in fact submultiplicative.

Another application is accomplished by collecting our results in Theorem 2.3 and Lemma 2.3, and obtaining the following characterization of the Young functions φ for which $L^\varphi = L^\infty$:

409/177/2-7

THEOREM 2.7. *Let φ be a Young function. Then $L^{\varphi} = L^{\infty}$ if and only if any of the following four cases holds*:

(a) $0 < x_0(\varphi) \leqslant x_{\infty}(\varphi) < \infty$, *in which case*

$$x_0(\varphi)\,\rho_{\varphi}(f) \leqslant \|f\|_{\infty} \leqslant x_{\infty}(\varphi)\,\rho_{\varphi}(f) \qquad \forall f \in L^{\varphi}. \tag{2.35a}$$

(b) $m(\mathbf{T}) < \infty$ *and* $x_{\infty}(\varphi) < \infty$, *in which case* $\varphi^{-1}(1/m(\mathbf{T})) > 0$ *and*

$$\varphi^{-1}\left(\frac{1}{m(\mathbf{T})}\right) \rho_{\varphi}(f) \leqslant \|f\|_{\infty} \leqslant x_{\infty}(\varphi)\,\rho_{\varphi}(f) \qquad \forall f \in L^{\varphi}. \tag{2.35b}$$

(c) $x_0(\varphi) > 0$ *and* $m_{\text{inf}} > 0$, *in which case*

$$x_0(\varphi)\,\rho_{\varphi}(f) \leqslant \|f\|_{\infty} \leqslant \varphi^{-1}\left(\frac{1}{m_{\text{inf}}}\right) \rho_{\varphi}(f) \qquad \forall f \in L^{\varphi}. \tag{2.35c}$$

(d) $m(\mathbf{T}) < \infty$ *and* $m_{\text{inf}} > 0$, *in which case* $\varphi^{-1}(1/m(\mathbf{T})) > 0$ *and*

$$\varphi^{-1}\left(\frac{1}{m(\mathbf{T})}\right) \rho_{\varphi}(f) \leqslant \|f\|_{\infty} \leqslant \varphi^{-1}\left(\frac{1}{m_{\text{inf}}}\right) \rho_{\varphi}(f) \qquad \forall f \in L^{\varphi}. \tag{2.35d}$$

We point out that cases (a)–(d) above overlap since they all contain the special case where $0 < x_0(\varphi) \leqslant x_{\infty}(\varphi) < \infty$, $m(\mathbf{T}) < \infty$, and $m_{\text{inf}} > 0$.

We also note that by (2.17b), all the inequalities in (2.35) are strengthenings of the inequality

$$x_0(\varphi)\,\rho_{\varphi}(f) \leqslant \|f\|_{\infty} \leqslant x_{\infty}(\varphi)\,\rho_{\varphi}(f).$$

Further, by part (a) of the theorem, if

$$x_0(\varphi) = x_{\infty}(\varphi) = \alpha \qquad (0 < \alpha < \infty \text{ by } (2.16\text{b}))$$

then

$$\rho_{\varphi}(f) = \alpha^{-1}\,\|f\|_{\infty} \qquad \forall f \in L^{\varphi}.$$

Finally, we mention that in case (d) of the theorem we may use parts (d) and (e) of Lemma 2.2 to conclude that $\mathbf{T}$ is a union of at most countably many atoms. Since in this case $m_{\text{inf}} > 0$ and $m(\mathbf{T}) < \infty$, it follows that in fact $\mathbf{T}$ is a union of finitely many (mutually disjoint and finite) atoms, say $\mathbf{A}_1, ..., \mathbf{A}_q$. Hence if $f \in \mathscr{M} = \mathscr{M}(\mathbf{T}, \Omega, m)$, then by Lemma 2.2(f),

$$f = \sum_{j=1}^{q} \gamma_i \chi_j \qquad \text{a.e.}$$

where the γ_j are constants and the χ_j are the characteristic functions of $\mathbf{A}_j$. This shows that $\mathcal{M}$ is q-dimensional; thus $\mathcal{M}$ is isomorphic to $\mathbb{F}^q$ and all norms on $\mathcal{M}$ are equivalent.

We would like to show now that the class of Luxemburg norms ρ_φ is properly contained in the class of function norms. To this end we quote:

THEOREM 2.8 [Z2, Theorem 131.6]. *Let φ be a Young function. Then ρ_φ has the Fatou property, i.e., whenever $\{f_n\}_1^\infty \subset L_\rho$, $f_n \geq 0$, is a monotone increasing sequence such that*

$$\sup_n \rho(f_n) < \infty,$$

then

$$f \equiv \lim_{n \to \infty} f_n \in L_\rho \qquad \text{and} \qquad \rho(f) = \lim_{n \to \infty} \rho(f_n).$$

Now take $\mathbf{T} = \mathbb{Z}^+ = \{1, 2, 3, ...\}$ and let m be the counting measure. Then, as earlier, $\mathcal{M} = \mathcal{M}(\mathbb{Z}^+, \Omega, m)$ is the algebra of sequences over $\mathbb{F}$. Define

$$\rho(a) = \sup_j |\alpha_j| + \limsup_{j \to \infty} |\alpha_j|, \qquad a = \{\alpha_j\}_1^\infty \in \mathcal{M}.$$

Then surely ρ is a function norm on $\mathcal{M}$. On the other hand for $k = 1, 2, 3, ...$, consider the sequences

$$a_k = \{\alpha_{k_j}\}_{j=1}^\infty = \begin{cases} 1, & j \leq k \\ 0, & \text{otherwise.} \end{cases}$$

Then $a_k \uparrow e = (1, 1, 1, ...)$, and

$$\rho(a_k) = 1, \quad k = 1, 2, 3, ...; \qquad \rho(e) = 2.$$

Hence ρ does not have the Fatou property; so by Theorem 2.8, ρ is not a Luxemburg norm.

We conclude the paper by discussing the following remote possibilities:

THEOREM 2.9. *Let φ be a Young function. Then*

(a) *$L^\varphi = \{0\}$ if and only if $\mathbf{T}$ is an infinite atom and $x_0(\varphi) = 0$.*

(b) *If $\mathbf{T}$ is an infinite atom then $L^\varphi = L^\infty$ if and only if $x_0(\varphi) > 0$.*

Proof. (a) If $\mathbf{T}$ is not an infinite atom, then there exists a set $\mathbf{B} \in \Omega$ with $0 < m(\mathbf{B}) < \infty$; and so by Proposition 2.2, $\chi_\mathbf{B} \in L^\varphi$, hence $L^\varphi \neq \{0\}$.

Conversely, let $\mathbf{T}$ be an infinite atom and suppose $x_0(\varphi)=0$. If $0 \neq f \in \mathscr{M}$, then for sufficiently small $\varepsilon > 0$ there exists $\mathbf{B} \in \Omega$ with $m(\mathbf{B}) > 0$ such that

$$|f(t)| > \varepsilon \qquad \text{on } \mathbf{B}.$$

By hypothesis, $m(\mathbf{B}) = \infty$ and

$$\varphi(x) > 0 \qquad \forall x > 0.$$

Hence for all $\tau > 0$,

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \geqslant \int_{\mathbf{B}} \varphi\left(\frac{\varepsilon}{\tau}\right) dm = \varphi\left(\frac{\varepsilon}{\tau}\right) \cdot \infty = \infty.$$

Consequently, $\rho_\varphi(f) = \infty$; thus $f \notin L^\varphi$ and so $L^\varphi = \{0\}$.

(b) Let $\mathbf{T}$ be an infinite atom. If $L^\varphi = L^\infty$ then $x_0(\varphi) > 0$ by (a) since $L^\infty \neq \{0\}$. Conversely, assume $x_0(\varphi) > 0$. Then $L^\infty \subseteq L^\varphi$ by Theorem 2.3. To show that $L^\varphi \subseteq L^\infty$, choose $0 \neq f \notin L^\infty$ and write

$$\int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm = \int_{\mathbf{A}_\tau} \varphi\left(\frac{|f(t)|}{\tau}\right) dm + \int_{\mathbf{T} \setminus \mathbf{A}_\tau} \varphi\left(\frac{|f(t)|}{\tau}\right) dm,$$

where

$$\mathbf{A}_\tau = \left\{ t \in \mathbf{T} : \frac{|f(t)|}{\tau} > x_0(\varphi) + 1 \right\}, \qquad \tau > 0.$$

Since $f \notin L^\infty$, then $m(\mathbf{A}_\tau) > 0$; and as $\mathbf{T}$ is an infinite atom we actually have $m(\mathbf{A}_\tau) = \infty$. Hence,

$$\begin{aligned} \int_{\mathbf{T}} \varphi\left(\frac{|f(t)|}{\tau}\right) dm &\geqslant \int_{\mathbf{A}_\tau} \varphi\left(\frac{|f(t)|}{\tau}\right) dm \\ &\geqslant \varphi(x_0(\varphi) + 1)\, m(\mathbf{A}_\tau) = \infty, \qquad \forall \tau > 0; \end{aligned}$$

so $f \notin L^\varphi$, and the theorem follows. ∎

REFERENCES

[AGL] R. ARENS, M. GOLDBERG, AND W. A. J. LUXEMBURG, Multiplicativity factors for function norms, *J. Math. Anal. Appl.* **177** (1993), 368–385.

[KR] M. A. KRASNOSEL'SKII AND YA. B. RUTICKII, "Convex Functions and Orlicz Spaces," Noordhoff, Groningen, 1961.

[LZ1] W. A. J. LUXEMBURG AND A. C. ZAANEN, Notes on Banach functions spaces, I, *Indag. Math.* **25** (1963), 135–147.

[LZ2] W. A. J. LUXEMBURG AND A. C. ZAANEN, Notes on Banach functions spaces, II, *Indag. Math.* **25** (1963), 148–153.

[LZ3] W. A. J. LUXEMBURG AND A. C. ZAANEN, Notes on Banach functions spaces, III, *Indag. Math.* **25** (1963), 239–250.

[LZ4] W. A. J. LUXEMBURG AND A. C. ZAANEN, Notes on Banach functions spaces, IV, *Indag. Math.* **25** (1963), 251–263.

[O] W. ORLICZ, Über eine gewisse Klasse von Räumen von Typus B, *Bull. Inter. Acad. Polon. Sci. Lett. Classe Math. Naturelles Sér. A* (1932), 207–220.

[Z1] A. C. ZAANEN, "Integration," 2nd ed., North-Holland, Amsterdam, 1967.

[Z2] A. C. ZAANEN, "Riesz Spaces, II," North-Holland, Amsterdam, 1983.