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Some Observations on the Asymptotic Behaviour of the Solutions of the Equation $\dot{x} = A(t)x(\lambda t) + B(t)x(t)$ $\lambda > 0$*

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1. INTRODUCTION

The equation

$$\dot{x} = A(t)\,x(\lambda t) + B(t)\,x(t) \tag{1}$$

has been studied, under special hypothesis, in many papers, for theoretical reasons as well as for some applications. For references on this see [1]. The paper [1] is devoted to relating the asymptotic properties of solutions of Eq. (1) to the behaviour of solutions of the equation

$$\dot{x} = B(t)\,x(t) \tag{2}$$

when A, B are $n \times n$ constant matrices and B is diagonalizable. In this paper we are going to generalize the results obtained in [1] to the case that $A(t)$, $B(t)$ are $n \times n$ bounded and measurable matrices for $t \in [t_0, +\infty)$, $t_0 \geq 0$. In particular we do not make any assumption on the structure of the matrix $B(t)$. The technique that we shall use is, with minor modifications, the same used in [1].

This paper is organized in this way:

The remaining part of this section is devoted to recall some properties of Eq. (1) and the definition and some properties of the characteristic exponents of Eq. (2). Section (2) contains the main results of this paper, which generalize analogous results proved in [1] for the case $0 < \lambda < 1$. In Section 3 the case that $\lambda > 1$ is considered. The results of this section are given without proofs.

Now we consider Eq. (1) with $\lambda \in (0, 1)$. If $\varphi(t)$ is a continuous function on $[\lambda t_0, t_0]$, Eq. (1) has a unique solution $x(t; \varphi)$ which is continuous on $[\lambda t_0, +\infty)$, absolutely continuous on $[t_0, +\infty)$ and such that $x(t; \varphi) = \varphi(t)$ for $t \in [\lambda t_0, t_0]$. When $t_0 = 0$, we assume that $[\lambda t_0, t_0] = \{0\}$ and that $\varphi(t)$ is a vector $\varphi(0)$. The existence and uniqueness result just asserted is proved, for example, in

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[2, Theorems 2.5, 2.6]. Let $f(t)$ be a continuous n-vector valued function on $[t_0, +\infty)$. The characteristic exponent of f is

$$\chi(f) = \limsup_{t\to+\infty}\{\log |f(t)|\}/t$$

([3, 4]. $|\cdot|$ is the norm in R^n).

The characteristic exponents of solutions of Eq. (2) are called the characteristic exponents of Eq. (2). Their number is at most n, so that we can denote them as $\chi_1, \dots, \chi_r$, $r \leqslant n$, and we intend that $\chi_i < \chi_{i+1}$. We shall consider also the adjoint equation of Eq. (2)

$$\dot{y} = -B^*(t)\, y(t) \tag{3}$$

the characteristic exponents of Eq. (3) will be denoted $\chi_1', \dots, \chi_r'$, ordered so that $\chi_i' > \chi_{i+1}'$. Observe that they are exactly r, as those of (2). It may be useful to recall that $\chi_i + \chi_i' \geqslant 0$ for any i. $\chi_i + \chi_i' = 0$ for any i means that Eq. (2), (3) are regular, and this happens in particular if B is a constant matrix (and in this case the χ_i's are the distinct real parts of eigenvalues of B).

2. The case $0 < \lambda < 1$

In this section we generalize Theorems 2, 3 in [1] to the case that $A(t)$ and $B(t)$ are bounded measurable functions.

THEOREM 1. *Assume that* $0 < \chi_1 < \cdots < \chi_r$, *and that* $\lambda\chi_r + \chi_1' < 0$. *Let* $G(t, s)$ *be the evolution matrix of Eq.* (2). *Then for every solution* $x(t)$ *of Eq.* (1) *there exists* $v_x \in R^n$ *such that*

$$\lim_{t\to+\infty} G(t_0, t)\, x(t) = v_x .$$

Proof. It will be useful to consider the following situation: Assume that it is possible to find a decomposition of R^n, $R^n = \bigoplus_{1i}^{k} \mathscr{R}_i$ such that $B(t) = \mathscr{B}_1(t) + \cdots + \mathscr{B}_k(t)$, $\mathscr{B}_i(t)\, x = 0$ if $x \in \mathscr{R}_j$, $i \neq j$, $\mathscr{B}_i(t)\, x \in \mathscr{R}_i$ for any t if $x \in \mathscr{R}_i$ (i.e. the matrix $B(t)$ may be diagonalized. Of course, in general, it will be $k = 1$).

If x is an n-vector, x_i denotes the projection of x on the space $\mathscr{R}_i$, so that we can consider the k equations

$$\dot{x}_i(t) = \mathscr{B}_i x_i(t). \tag{4}$$

Of course, if χ is a characteristic exponent of one of the Equations (4), χ is also a characteristic exponent of Eq. (2). Write $A(t) = (A_{ij}(t))$ in the obvious way. If $x(t)$ is a solution of Eq. (1) its i-th component $x_i(t)$ is a solution of the equation

$$\dot{x}_i(t) = \sum_{1j}^{k} A_{ij}(t)\, x_j(\lambda t) + \mathscr{B}_i(t)\, x_i(t).$$

If $G_i(t, s)$ is the evolution matrix of the i-th of the Eq. (4)

$$G_i(t_0, t)\, x_i(t) = G_i(t_0, \tau)\, x_i(\tau) + \int_\tau^t G_i(t_0, s)\, A(s)\, G_i(\lambda s, t_0)\, G_i(t_0, \lambda s)\, x(\lambda s)\, ds. \tag{5}$$

Let $I_k = [t_0 + 1/\lambda^k, t_0 + 1/\lambda^{k+1}]$ and $M_k{}^i = \sup |G_i(t_0, t)\, x_i(t)|$ on I_k. If $\tau = t_0 + 1/\lambda^k$, $t \in I_k$

$$|G_i(t_0, t)\, x_i(t)| \leqslant M_k{}^i \left[1 + \| A \| \int_{t_0+1/\lambda^k}^t |G_i(t_0, s)|\, |G_i(\lambda s, t_0)|\, ds\right]$$

$$\| A \| = \sup |A(t)| \qquad t \geqslant t_0 .$$

Now consider that $\chi(G_i(t_0, t)) \leqslant \chi_1'$ so that

$$|G_i(t_0, t)| \leqslant \alpha_\epsilon \exp[(\chi_1' + \epsilon)\, t] \qquad \text{for any } \epsilon > 0 \quad \text{and some} \quad \alpha_\epsilon .$$

Analogously,

$$|G_i(\lambda t, t_0)| \leqslant \alpha_\epsilon' \exp[(\lambda \chi_n + \lambda\epsilon)\, t]$$

so that

$$|G_i(t_0, t)\, x_i(t)| \leqslant M_k{}^i \left[1 + \| A \|\, \alpha_\epsilon \alpha_\epsilon' \int_{t_0+1/\lambda^k}^t \exp\{(\chi_1' + \lambda\chi_n)\, s + \epsilon(1 + \lambda)\, s\}\, ds\right]$$

$$\leqslant M_k{}^i(1 + H \exp(\eta/\lambda^k))$$

with $H = -\{\| A \|\, \alpha_\epsilon \alpha_\epsilon' \exp[(\chi_1' + \lambda\chi_n + \epsilon(1 + \lambda))\, t_0]\}/[\chi_1' + \lambda\chi_n + \epsilon(1 + \lambda)]$, at least if $\eta = \chi_1' + \lambda\chi_n + \epsilon(1 + \lambda) < 0$, (i.e. for small ϵ). From this we have

$$M_{k+1}^i \leqslant M_0 \prod_{1r}^{\infty} [1 + H \exp(\eta/\lambda^r)] = M.$$

Now we look again at (5). We have that

$$|G_i(t_0, t)\, x_i(t) - G_i(t_0, \tau)\, x_i(\tau)| \leqslant \| A \|\, MH \int_\tau^t \exp(\eta s)\, ds$$

so that $\lim_{t\to+\infty} G_i(t_0, t)\, x_i(t)$ exists and is a $v_x{}^i \in \mathscr{R}_i$ because the integral on the left is convergent. The theorem is proved with $v_x = \sum_{1i}^k v_x{}^i$.

COROLLARY 1. *Under the hypothesis of Theorem* 1, *if there is*

$$\chi = \lim_{t\to+\infty}(\log \mid G(t, t_0)\mid)/t \tag{6}$$

then $\chi(x) \leqslant \chi$ *for any solution* $x(t)$ *of Eq.* (1).

Proof. We have already seen in the proof of Theorem 1 that for any solution x of (1), $\mid G(t_0, t)\, x(t)\mid$ is bounded. $G(t_0, t)$ is an invertible matrix, $G^{-1}(t_0, t) = G(t, t_0)$ so that

$$\mid G(t, t_0)\mid^{-1} \mid x(t)\mid \leqslant \mid G(t_0, t)\, x(t)\mid \leqslant M.$$

Condition (6) implies ([4, page 457]) $\chi(\mid G(t, t_0)\mid^{-1}) = -\chi$ and that $\chi(\mid G(t, t_0)\mid^{-1} \mid x(t)\mid) = -\chi + \chi(\mid x(t)\mid)$. This last number is not positive, because $\mid G(t, t_0)\mid^{-1} \mid x(t)\mid$ is bounded. This proves the corollary.

In particular, if B is a constant matrix, we can assume that it is written in Jordan form. Then condition (6) holds not only for $G(t, t_0)$ but also for every $G_i(t, t_0)$ and χ is in this case the real part of the eigenvalue of the i-th block, say $\chi_{(i)}$, so that $\chi(\mid x_i(t)\mid) \leqslant \chi_{(i)}$. Furthermore we have

COROLLARY 2. *If B is a constant matrix which satisfies the hypothesis of Theorem* 1, *every solution of Eq.* (1) *satisfies*

$$\chi(x) \leqslant \chi_r$$

and for every χ_i *there is a vector* α_i *such that*

$$\lim_{t\to+\infty} \exp(-\chi_i t)\, \alpha_i^* x(t)$$

exists for at least one solution of Eq. (1).

Proof. The first assertion is obvious because

$$\chi(x) \leqslant \max \chi_i = \chi_r$$

from the observation at the end of Corollary 1.

If B is in Jordan form, then $G_i(t_0, t) = P_i(t) \exp(-\lambda(t - t_0))$, where λ is an eigenvalue of B, say of real part χ, and $P_i(t)$ is a triangular matrix which is a polynomial in t, with diagonal entries equal to 1. Let $x_i(t) = (x_i^1(t),..., x_i^s(t))^*$ be a solution of (1). The last component of $G_i(t_0, t)\, x_i(t)$ is $x_i^s(t) \exp(-\lambda(t - t_0))$ and this function has a limit for $t \to +\infty$ (from Theorem 1). So the corollary is proved if B is in Jordan form and, after a change of coordinates, in the general case.

Observe that if $r = n$ (i.e. B is diagonalizable) this corollary is Theorem 2 in [1]. Now we look for a generalization of Theorem 3 in [1] to the Equation (1).

First of all observe that if $x(t)$ is a solution of (2), then $y(t) = x(e^t)\exp(-\alpha t)$ is a solution of

$$\dot{y} = (B(e^t)\, e^t - \alpha I)\, y(t) \tag{7}$$

(I is the $n \times n$ identity matrix). Then

$$|\, y(t)| \leqslant K \exp\{-\alpha t + \beta e^t\} \qquad \text{if} \qquad |\, x(t)| < K \cdot \exp\{\beta t\}.$$

Now we can prove the following theorem

THEOREM 2. *Assume that Eq.* (2) *is exponentially stable, i.e.*

$$|\, G(t, s)| \leqslant K \exp(\beta(t - s)) \qquad \beta < 0, \quad t_0 \leqslant s \leqslant t.$$

If $\sigma = -\log \lambda > 0$ *and if* α *satisfies*

$$\sigma\alpha > [\log \| A \| - \log(-\beta)]$$

there exists a positive number L *such that*

$$|\, x(t)| \leqslant L \cdot t^{[\alpha + (\log k)/\sigma]}$$

for every solution $x(t)$ *of* (1).

Proof. If $x(t)$ is a solution of (1) then $w(t) = x(e^t)\exp(-\alpha t)$ is a solution of the equation

$$\dot{w}(t) = (B(e^t)\, e^t - \alpha I)\, w(t) + e^t e^{-\alpha\sigma} A(e^t)\, w(t - \sigma)$$

so that

$$w(t) = E(t, \tau)\, w(\tau) + \int_\tau^t E(t, s)\, e^s e^{-\alpha\sigma} A(e^s)\, w(s - \sigma)\, ds.$$

Here $E(t, \tau)$ is the evolution matrix of Eq. (7) so that

$$|\, E(t, \tau)| = |\, \exp(-\alpha(t - \tau))\, G(e^t, e^\tau)| \leqslant K \exp\{\beta(e^t - e^\tau) - \alpha(t - \tau)\}$$

Let now $M_k = \sup |\, w(t)|$ for $t \in I_k = [t_0 + k\sigma, t_0 + (k + 1)\,\sigma]$. If $\tau = \tau_k = t_0 + k\sigma$ and $t \in I_k$

$$\begin{aligned} |\, w(t)| \leqslant M_{k-1} K \Big[& \exp\{\beta(e^t - e^{\tau_k}) - \alpha(t - \tau_k)\} \\ & + e^{-\sigma\alpha} \| A \| \int_{\tau_k}^t e^s \exp\{\beta(e^t - e^s) - \alpha(t - s)\}\, ds \Big] \end{aligned} \tag{10}$$

so that from (8) and (10)

$$\begin{aligned} |\, w(t)|/K \leqslant M_{k-1} \Big\{ & \exp\Big[\beta(e^t - e^{\tau_k}) - \alpha(t - \tau_k) \\ & + \int_{\tau_k}^t (-\beta)\, e^s \exp[\beta(e^t - e^s) - \alpha(t - s)]\, ds\Big]\Big\} \qquad t \in I_k\,. \end{aligned}$$

This is inequality 4.9 in [1] and so, as in [1]

$$M_k/K \leqslant M_{k-1}(1 + O(e^{-\tau_k})).$$

From this we have

$$M_k \leqslant K^k M_0 \prod_{0r}^{k} (1 + O(e^{-\tau_r})).$$

That is, $| w(t)| \leqslant NK^k$ holds for $t \in I_k$ because $\prod_{0r}^{\infty} (1 + O(e^{-\tau_r}))$ is convergent. For $t \in I_k$, $k < (t - t_0)/\sigma$, so that

$$| w(t)| < L[K^{1/\sigma}]^t \qquad \text{for } t \in [t_0, +\infty), \quad L = N[K^{-t_0/\sigma}].$$

From the definition of $w(t)$

$$| x(t)| \leqslant L[K^{1/\sigma}]^{\log t}\, t^{\alpha} = L \cdot t^{[\alpha+(\log k)/\sigma]}.$$

Observe that in [1] $G(t, s) = \mathrm{diag}(\exp(\beta_i(t - s))$, so that $K = 1$.

3. The case $\lambda > 1$

Now we consider the case $\lambda > 1$. If $\lambda > 1$ theorems of existence and unicity of solutions do not hold any more, and only for special initial conditions $\varphi(t)$ will Eq. (1) have at least one solution $x(t; \varphi)$. Every solution $x(t)$ of (1) satisfies

$$\dot{x}(1/t) = -(A(1/t)/t^2)\, x(1/\mu t) - (B(1/t)/t^2)\, x(1/t)$$

$\mu = 1/\lambda < 1$, at least if $t_0 > 0$.

However the asymptotic properties of $x(t)$ cannot be obtained from this equation with the help of Theorems 1, 2, even if these theorems hold. The following theorems generalize Theorems 5, 6 in [1].

Theorem 3. *Assume that Eq.* (3) *is exponentially stable and* K, β *are chosen so that* $| G(s, t)| \leqslant K \exp(-\beta(t - s))$ $t \geqslant s \geqslant t_0$, $\beta > 0$. *If* $x(t)$ *is a solution of Eq.* (1) *such that* $\lim_{t\to+\infty} t^{-\alpha}x(t) = 0$

$$\alpha \leqslant (\log \beta - \log \| A \| - \log K)/\log \lambda$$

then $x(t) = 0$ *for every* t.

Theorem 4. *If* $\chi_1 < \chi_2 < \cdots < \chi_r < 0$, $\lambda\chi_r + \chi_1' < 0$, *and if* $x(t)$ *is a solution of Eq.* (1) *such that* $\lim_{t\to+\infty} G(t_0, t)\, x(t) = 0$, *then* $x(t) = 0$ *for every* $t \in [t_0, +\infty)$.

We omit the proofs which are obtained, with few changes, from the proofs of the analogous results in [1].

We observe only that from Theorem 3 we have the following corollary:

COROLLARY 3. *If B is a constant matrix and if $G(t, s)$ satisfies*

$$|G(t,s)| \leqslant H \exp(-\beta(t-s)) \qquad t \geqslant s \geqslant t_0, \quad \beta > 0 \tag{11}$$

$$|G(t,s)| \leqslant H \exp(\gamma(s-t)) \qquad s \geqslant t \geqslant t_0 \tag{12}$$

and if $-\lambda\beta + \gamma < 0$, then only the null solution of Eq. (1) *decays faster then* $\exp(-\gamma t)$.

Proof. B is a constant matrix, so that from (11), (12) we have $\chi_r \leqslant -\beta$, $\chi_1' = -\chi_1 \leqslant \gamma$. Then $\lambda\chi_r + \chi_1' \leqslant -\lambda\beta + \gamma < 0$. If $\lim_{t\to+\infty}(e^{\gamma t}x(t)) = 0$ then $\lim_{t\to+\infty}(G(t_0, t)\, x(t)) = 0$ and $x(t)$ must be zero from Theorem 3.

REFERENCES

1. E.-B. LIM, Asymptotic behavior of solutions of the functional differential equation $x'(t) = Ax(\lambda t) + Bx(t)$, $\lambda > 0$, *J. Math. Anal. Appl.* **55** (1976), 794–806.
2. M. N. OGUZTÖRELI, "Time Lag Control Systems," Academic Press, New York, 1966.
3. B. E. BILOV, *et al.*, "Theory of Liapunov Exponents," [Russian] Izdatel'stvo "Nauka," Moscow, USSR, 1966.
4. G. SANSONE AND R. CONTI, "Non-Linear Differential Equations," Pergamon Press, Oxford, 1964.