Partitions of graphs into one or two independent sets and cliques

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Abstract

It is shown in this note that it can be recognized in polynomial time whether the vertex set of a finite undirected graph can be partitioned into one or two independent sets and one or two cliques. Such graphs generalize bipartite and split graphs and the result also shows that it can be recognized in polynomial time whether a graph can be partitioned into two split graphs.

A time bound $O(n^3)$ is given for the recognition of graphs which can be partitioned into two independent sets and one clique (one independent set and two cliques, resp.), and a time bound $O(n^4)$ is given for the recognition of graphs which can be partitioned into two independent sets and two cliques.

Keywords: Partitions of graphs into independent sets and cliques; Generalization of bipartite and split graphs; Polynomial time recognition

1. Introduction

This paper investigates the recognition complexity of a common generalization of bipartite and split graphs, namely graphs which can be partitioned into one or two independent vertex sets and one or two cliques. It is shown that those graphs can be recognized in polynomial time. This contrasts to the well-known NP-completeness of the 3-colorability problem for graphs (cf. [4, 3]).

Throughout this note all graphs are finite, simple (i.e. without self-loops and multiple edges) and undirected. Let $G = (V, E)$ be a graph and $n = |V|$, $m = |E|$, $V' \subseteq V$ is *independent* iff for all $u, v \in V'$ $uv \notin E$. $V' \subseteq V$ is a *clique* iff for all $u, v \in V'$ with $u \neq v$ $uv \in E$. $V_1, \dots, V_k$ is a *partition of* V iff for all $i, j \in \{1, \dots, k\}$ with $i \neq j$ $V_i \cap V_j = \emptyset$ and $\bigcup_{i=1}^{k} V_i = V$. A partition $I_1, \dots, I_k, C_1, \dots, C_l$ of V with independent sets $I_j, j \in \{1, \dots, k\}$, and cliques $C_i, i \in \{1, \dots, l\}$, is a (k, l)-*partition* of G.

For a graph $G = (V, E)$ let $\bar{G} = (V, \bar{E})$ with $\bar{E} = \{uv: u, v \in V \text{ and } u \neq v \text{ and } uv \notin E\}$ denote the *complement graph* of G.

As usual let K_i denote an induced clique of size i, let C_k denote an induced cycle with k vertices and let $2K_2$ denote the complement of C_4: $2K_2 = \bar{C}_4$ (a graph with 4 vertices and two disjoint edges).

Let (k, l) denote the set of all graphs $G = (V, E)$ for which there is a (k, l)-partition of the vertex set V. For $V' \subseteq V$ let $G(V')$ denote the subgraph of G induced by V'. For short we sometimes write $V' \in (k, l)$ meaning $G(V') \in (k, l)$. Thus $(2, 0)$ is the class of *bipartite graphs* and $(1, 1)$ is the class of *split graphs*. It is well known that bipartiteness of a graph can be recognized in linear time. For the notion of split graphs, cf. [2, 5] where a linear time recognition algorithm for split graphs is given which uses their degree sequence characterization.

By the following standard construction it can be easily seen that the recognition problem for (k, l) is NP-complete if $k \geqslant 3$ or $l \geqslant 3$: A graph G is in $(k, 0)$ iff G is k-colorable, and so the problem is NP-complete for $k \geqslant 3$. Further, $G \in (k, l)$ iff $G \cup K_{k+1} \in (k, l+1)$ where K_{k+1} is a clique of size $k + 1$. Finally by using that $G \in (k, l) \Leftrightarrow \bar{G} \in (l, k)$ the assertion follows.

For $v \in V$ let $N(v) = \{u: u \in V \text{ and } vu \in E\}$ (the *neighborhood* of v) and $\bar{N}(v) = \{u: u \in V \text{ and } u \neq v \text{ and } uv \notin E\}$ (the *non-neighborhood* of v).

It is well known [2] that G is a split graph iff G contains no induced C_4, C_5, and $2K_2$ — a property which is used in the proof of Lemma 1.

Let $\omega(G) = \max\{|V'|: V' \text{ a clique of } G\}$ denote the *maximum clique size* of G. Let $\mathcal{N}$ denote the set of positive integers.

2. An $O(n^3)$ time bounded recognition for $(2, 1)$ and $(1, 2)$

The basic principle of our approach for $(2, 1)$ and $(1, 2)$ is a vertex classification according to the following two neighborhood conditions:

(N1) $N(v) \in (1, 1)$,

(N2) $\bar{N}(v) \in (2, 0)$.

Evidently, if a graph $G = (V, E)$ has a $(2, 1)$-partition I_1, I_2, C then for all vertices $v \in C$ condition (N2) is fulfilled and for all vertices $v \in I_1 \cup I_2$ condition (N1) is fulfilled. An immediate consequence is that if G has a vertex $v \in V$ with $N(v) \notin (1, 1)$ and $\bar{N}(x) \notin (2, 0)$ then $G \notin (2, 1)$ holds.

Assume now that G has no such vertex, i.e. every vertex $v \in V$ fulfills at least one of the conditions (N1), (N2). Assume furthermore that G has a $(2, 1)$-partition I_1, I_2, C. If $N(v) \notin (1, 1)$ then necessarily v belongs to the clique C. If $\bar{N}(v) \notin (2, 0)$ then necessarily v belongs to the bipartite part $I_1 \cup I_2$ of that $(2, 1)$-partition of G.

Thus for vertices which fulfill exactly one of the conditions (N1), (N2) it is determined to which part of the $(2, 1)$-partition they belong (if there is such a partition). This means that the decision of '$G \in (2, 1)$?' can be started by first assigning:

(A1) $C^S := \{v: N(v) \notin (1, 1) \text{ and } \bar{N}(v) \in (2, 0)\}$ and checking whether C^S is a clique.

(A2) $I_1^S \cup I_2^S := \{v: N(v) \in (1, 1) \text{ and } \bar{N}(v) \notin (2, 0)\}$ and checking whether $I_1^S \cup I_2^S$ is bipartite.

If C^S is no clique or $I_1^S \cup I_2^S$ is not bipartite then evidently the correct answer is '$G \notin (\mathbf{2}, \mathbf{1})$'. Otherwise continue by inserting the vertices for which both conditions (N1), (N2) are fulfilled. Let

$$R := \{v \colon N(v) \in (\mathbf{1}, \mathbf{1}) \text{ and } \bar{N}(v) \in (\mathbf{2}, \mathbf{0})\}.$$

If $R = \emptyset$ then I_1^S, I_2^S, C^S is a $(2, 1)$-partition of G. Otherwise we try an extension of the already existing bipartite $I_1^S \cup I_2^S$ and clique C^S to a $(2, 1)$-partition by inserting R and checking whether the bipartite part remains bipartite and the clique remains a clique, respectively. Hereby the following observation is useful:

Proposition 1. *There is an extension of the independent sets I_1^S, I_2^S and the clique C^S to a $(2, 1)$-partition I_1, I_2, C of G iff*

$$R \subseteq \bigcap_{v \in C^S} N(v) \cup \bigcap_{v \in I_1^S} \bar{N}(v) \cup \bigcap_{v \in I_2^S} \bar{N}(v) \qquad (*)$$

holds.

Proof. '$\Rightarrow$': If I_1, I_2, C is a $(2, 1)$-partition of G with $C^S \subseteq C$ and $I_j^S \subseteq I_j$, $j \in \{1, 2\}$, then also $R \subseteq I_1 \cup I_2 \cup C$. Let $R = R_C \cup R_{I_1} \cup R_{I_2}$ with $R_C = R \cap C$, $R_{I_j} = R \cap I_j, j \in \{1, 2\}$. Then $C = C^S \cup R_C$ is a clique and thus $R_C \subseteq \bigcap_{v \in C^S} N(v)$. Also $I_j = I_j^S \cup R_{I_j}, j \in \{1, 2\}$, are independent sets and thus $R_{I_j} \subseteq \bigcap_{v \in I_j^S} \bar{N}(v)$ holds. Thus $(*)$ is fulfilled.

'$\Leftarrow$': Let $(*)$ be fulfilled and let $R_C = R \cap \bigcap_{v \in C^S} N(v)$, $R_{I_1} = (R \setminus R_C) \cap \bigcap_{v \in I_1^S} \bar{N}(v)$, $R_{I_2} = (R \setminus (R_C \cup R_{I_1})) \cap \bigcap_{v \in I_2^S} \bar{N}(v)$. Then $R = R_C \cup R_{I_1} \cup R_{I_2}$ is a partition of R with the property: $I_j^S \cup R_{I_j}, j \in \{1, 2\}$, are independent sets and $C^S \cup R_C$ is a clique. Thus $I_j^S \cup R_{I_j}, j \in \{1, 2\}$, and $C^S \cup R_C$ form an extension of I_1^S, I_2^S, C^S, and this extension is a $(2, 1)$-partition of G. $\qquad \square$

This is the basis for the following simple recognition algorithm for $(\mathbf{2}, \mathbf{1})$:

Algorithm 1

Input: *A graph* $G = (V, E)$
Output: *A decision of* '$G \in (\mathbf{2}, 1)$?' *and a* $(2, 1)$-*partition* I_1, I_2, C *if* $G \in (\mathbf{2}, \mathbf{1})$.

(0)　$I_1 := \emptyset; I_2 := \emptyset; C := \emptyset;$
(1)　**for all** $v \in V$ *check whether* $N(v) \in (\mathbf{1}, \mathbf{1})$, $\bar{N}(v) \in (\mathbf{2}, \mathbf{0})$;
(2)　**if** *there is a vertex* $v \in V$ *with* $N(v) \notin (\mathbf{1}, \mathbf{1})$ *and* $\bar{N}(v) \notin (\mathbf{2}, \mathbf{0})$ **then**
(3)　　$G \notin (\mathbf{2}, \mathbf{1}).$ *STOP*
　　else
　　　begin
(4)　　　$C^S := \{v \colon N(v) \notin (\mathbf{1}, \mathbf{1}) \text{ and } \bar{N}(v) \in (\mathbf{2}, \mathbf{0})\};$
(5)　　　**if** C^S *is no clique* **then** $G \notin (\mathbf{2}, \mathbf{1}).$ *STOP*
(6)　　　$I_1^S \cup I_2^S := \{v \colon N(v) \in (\mathbf{1}, \mathbf{1}) \text{ and } \bar{N}(v) \notin (\mathbf{2}, \mathbf{0})\};$

(7)   **if** $I_1^S \cup I_2^S$ *is not bipartite* **then** $G \notin (\mathbf{2}, \mathbf{1})$. *STOP*

(8) $R := \{v: N(v) \in (\mathbf{1}, \mathbf{1}) \text{ and } \bar{N}(v) \in (\mathbf{2}, \mathbf{0})\}$;

(9)   **if** R *is empty* **then** *STOP with* $G \in (\mathbf{2}, \mathbf{1})$ *and the* $(\mathbf{2}, \mathbf{1})$-*partition*

(10) $I_1 := I_1^S,\ I_2 := I_2^S,\ C := C^S$

        else

(11)   **if** $R \subseteq \bigcap_{v \in C^S} N(v) \cup \bigcap_{v \in I_1^S} \bar{N}(v) \cup \bigcap_{v \in I_2^S} \bar{N}(v)$ **then** *extend* I_1^S, I_2^S, C^S
          to a $(\mathbf{2}, \mathbf{1})$-*partition (as described in the proof of Proposition 1)*

(12)   **else** $G \notin (\mathbf{2}, \mathbf{1})$. *STOP*
      end;

Theorem 1. *Algorithm* 1 *checks in* $O(n^3)$ *steps whether a graph* G *is in* $(\mathbf{2}, \mathbf{1})$.

Proof. *Correctness*: Up to step (10) the correctness of the algorithm immediately follows from the preceding arguments. The correctness of the rest follows from Proposition 1.

Time bound: We do not discuss the trivial steps. (1) can be carried out in $O(n^3)$ steps since for each vertex $v \in V$ it can be tested in $O(n^2)$ steps whether v fulfills the conditions (N1), (N2). (5) and (7) can be checked easily in time $O(n^2)$. (11) can be done in $O(n^2)$ steps testing for each $u \in R$ whether it belongs to one of the sets $\bigcap_{v \in C^S} N(v)$, $\bigcap_{v \in I_1^S} \bar{N}(v)$, $\bigcap_{v \in I_2^S} \bar{N}(v)$. $\square$

Since $G \in (\mathbf{1}, \mathbf{2})$ iff $\bar{G} \in (\mathbf{2}, \mathbf{1})$ we have the following

Corollary 1. *It can be recognized in* $O(n^3)$ *steps whether a graph* G *is in* $(\mathbf{1}, \mathbf{2})$.

3. An $O(n^4)$ time bounded recognition for $(\mathbf{2}, \mathbf{2})$

Now we show that an approach very similar to the $(\mathbf{2}, \mathbf{1})$ recognition works also for $(\mathbf{2}, \mathbf{2})$ recognition. The basic principle for $(\mathbf{2}, \mathbf{2})$ is a vertex classification according to the following two neighborhood conditions:

(N3) $N(v) \in (\mathbf{1}, \mathbf{2})$,

(N4) $\bar{N}(v) \in (\mathbf{2}, \mathbf{1})$.

Evidently, if a graph $G = (V, E)$ has a $(\mathbf{2}, \mathbf{2})$-partition I_1, I_2, C_1, C_2 then for all vertices $v \in C_1 \cup C_2$ condition (N4) is fulfilled and for all vertices $v \in I_1 \cup I_2$ condition (N3) is fulfilled. An immediate consequence is that if G has a vertex $v \in V$ with $N(v) \notin (\mathbf{1}, \mathbf{2})$ and $\bar{N}(v) \notin (\mathbf{2}, \mathbf{1})$ then $G \notin (\mathbf{2}, \mathbf{2})$ holds.

Assume now that G has no such vertex, i.e. every vertex $v \in V$ fulfills at least one of the conditions (N3), (N4). Assume furthermore that G has a $(\mathbf{2}, \mathbf{2})$-partition I_1, I_2, C_1, C_2. If $N(v) \notin (\mathbf{1}, \mathbf{2})$ then necessarily v belongs to the clique part $C_1 \cup C_2$, and if $\bar{N}(v) \notin (\mathbf{2}, \mathbf{1})$ then necessarily v belongs to the bipartite part $I_1 \cup I_2$ of that $(\mathbf{2}, \mathbf{2})$-partition of G.

Thus for vertices which fulfill exactly one of the conditions (N3), (N4) it is determined to which part of the $(2,2)$-partition they belong (if there is such a partition). This means that the decision of '$G \in (2,2)$?' can be started by first assigning:

(A3) $C_1^S \cup C_2^S := \{v: N(v) \notin (1,2) \text{ and } \bar{N}(v) \in (2,1)\}$ and checking whether $C_1^S \cup C_2^S$ is co-bipartite (i.e. the union of two cliques).

(A4) $I_1^S \cup I_2^S := \{v: N(v) \in (1,2) \text{ and } \bar{N}(v) \notin (2,1)\}$ and checking whether $I_1^S \cup I_2^S$ is bipartite.

If $C_1^S \cup C_2^S$ is not co-bipartite or $I_1^S \cup I_2^S$ is not bipartite then evidently the correct answer is '$G \notin (2,2)$'. Otherwise continue by inserting the vertices for which both conditions (N3), (N4) are fulfilled. Let

$$R := \{v: N(v) \in (1,2) \text{ and } \bar{N}(v) \notin (2,1)\}.$$

If $R = \emptyset$ then $I_1^S, I_2^S, C_1^S, C_2^S$ is a $(2,2)$-partition of G. Otherwise we try an extension of the already existing bipartite $I_1^S \cup I_2^S$ and co-bipartite part $C_1^S \cup C_2^S$ to a $(2,2)$-partition by inserting R and checking whether the bipartite part remains bipartite and the co-bipartite part remains co-bipartite, respectively. Hereby the following observation is useful:

Proposition 2. *There is an extension of the independent sets I_1^S, I_2^S and the cliques C_1^S, C_2^S to a $(2,1)$-partition I_1, I_2, C_1, C_2 of G iff*

$$R \subseteq \bigcap_{v \in C_1^S} N(v) \cup \bigcap_{v \in C_2^S} N(v) \cup \bigcap_{v \in I_1^S} \bar{N}(v) \cup \bigcap_{v \in I_2^S} \bar{N}(v) \qquad (**)$$

holds.

Proof. '$\Rightarrow$': If I_1, I_2, C_1, C_2 is a $(2,2)$-partition of G with $C_j^S \subseteq C$ and $I_j^S \subseteq I_j$, $j \in \{1,2\}$, then also $R \subseteq I_1 \cup I_2 \cup C_1 \cup C_2$. Let $R = R_{C_1} \cup R_{C_2} \cup R_{I_1} \cup R_{I_2}$ with $R_{C_j} = R \cap C_j$, $R_{I_j} = R \cap I_j$, $j \in \{1,2\}$. Then $C_j = C_j^S \cup R_{C_j}$, $j \in \{1,2\}$, are cliques and thus $R_{C_j} \subseteq \bigcap_{v \in C_j^S} N(v)$, $j \in \{1,2\}$. Also $I_j = I_j^S \cup R_{I_j}$, $j \in \{1,2\}$, are independent sets and thus $R_{I_j} \subseteq \bigcap_{v \in I_j^S} \bar{N}(v)$, $j \in \{1,2\}$, holds. Thus $(**)$ is fulfilled.

'$\Leftarrow$': Let $(**)$ be fulfilled and let $R_{C_1} = R \cap \bigcap_{v \in C_1^S} N(v)$, $R_{C_2} = (R \setminus R_{C_1}) \cap \bigcap_{v \in C_2^S} N(v)$. Let $R_C = R_{C_1} \cup R_{C_2}$. Let $R_{I_1} = (R \setminus R_C) \cap \bigcap_{v \in I_1^S} \bar{N}(v)$, $R_{I_2} = (R \setminus (R_C \cup R_{I_1})) \cap \bigcap_{v \in I_2^S} \bar{N}(v)$. Then $R = R_C \cup R_{I_1} \cup R_{I_2}$ is a partition of R with the property: $I_j^S \cup R_{I_j}$, $j \in \{1,2\}$, are independent sets and $C_j^S \cup R_{C_j}$, $j \in \{1,2\}$, are cliques. Thus $I_j^S \cup R_{I_j}$ and $C_j^S \cup R_{C_j}$, $j \in \{1,2\}$, form an extension of $I_1^S, I_2^S, C_1^S, C_2^S$, and this extension is a $(2,2)$-partition of G. □

This is the basis for the following simple recognition algorithm for $(2,2)$:

Algorithm 2

Input: A graph $G = (V, E)$
Output: A decision of '$G \in (2,2)$?' and a $(2,2)$-partition I_1, I_2, C_1, C_2 if $G \in (2,2)$.
(0) $I_1 := \emptyset$; $I_2 := \emptyset$; $C_1 := \emptyset$; $C_2 := \emptyset$
(1) **for all** $v \in V$ check whether $N(v) \in (1,2)$, $\bar{N}(v) \in (2,1)$;

(2) **if** *there is a vertex $v \in V$ with $N(v) \notin (1,2)$ and $\bar{N}(v) \notin (2,1)$* **then**
(3) $G \notin (2,2)$. *STOP*
    else
      begin
(4) $C_1^S \cup C_2^S := \{v : N(v) \notin (1,2) \text{ and } \bar{N}(v) \in (2,1)\}$;
(5)   **if** $C_1^S \cup C_2^S$ *is not co-bipartite* **then** $G \notin (2,2)$. *STOP*
(6) $I_1^S \cup I_2^S := \{v : N(v) \in (1,2) \text{ and } \bar{N}(v) \notin (2,1)\}$;
(7)   **if** $I_1^S \cup I_2^S$ *is not bipartite* **then** $G \notin (2,2)$. *STOP*
(8) $R := \{v : N(v) \in (1,2) \text{ and } \bar{N}(v) \in (2,1)\}$;
(9)   **If** *R is empty* **then** *STOP with $G \in (2,2)$ and the $(2,2)$-partition*
(10) $I_1 := I_1^S,\ I_2 := I_2^S,\ C_1 := C_1^S,\ C_2 := C_2^S$
    else
(11)   **if** $R \subseteq \bigcap_{v \in C_1^S} N(v) \cup \bigcap_{v \in C_2^S} N(v) \cup \bigcap_{v \in I_1^S} \bar{N}(v) \cup \bigcap_{v \in I_2^S} \bar{N}(v)$ **then**
        extend $I_1^S, I_2^S, C_1^S, C_2^S$ to a $(2,2)$-partition
        (as described in the proof of Proposition 2)
(12)   **else** $G \notin (2,2)$. *STOP*
   end;

Theorem 2. *Algorithm 2 checks in $O(n^4)$ steps whether a graph G is in $(2,2)$.*

Proof. *Correctness*: Up to step (10) the correctness of the algorithm immediately follows from the preceding arguments. The correctness of the rest follows from Proposition 2.

Time bound: We do not discuss the trivial steps. (1) can be carried out in $O(n^4)$ steps since for each vertex $v \in V$ it can be tested in $O(n^3)$ steps whether v fulfills the conditions (N3), (N4). (5) and (7) can be checked easily in time $O(n^2)$. (11) can be done in $O(n^2)$ steps. $\square$

4. Structure properties of graphs in $(2,1)$

We do not have a structure characterization for the classes $(2,1)$, $(1,2)$ or $(2,2)$. Because of the rich neighborhood properties given in such graphs it could be promising to search for such characterizations. Perhaps the following lemma could be helpful for that purpose.

Lemma 1. *Let $G = (V,E)$ be a graph that is not a split graph and let $i \in \mathcal{N}$, $i \geq 2$, be such that*
  (i) *for all $v \in V$, $N(v)$ induces a split graph, and*
  (ii) *for all $v \in V$, $\bar{N}(v)$ induces a K_i-free graph.*
Then $\omega(G) \leq 2i - 2$.

Corollary 2. *Let $G = (V, E)$ be a graph that is not a split graph and*

(i) *for all $v \in V$, $N(v) \in (\mathbf{1}, \mathbf{1})$, and*

(ii) *for all $v \in V$, $\bar{N}(v) \in (\mathbf{2}, \mathbf{0})$.*

Then $\omega(G) \leqslant 4$.

Proof of Lemma 1. Assume to the contrary that $\omega(G) > 2i - 2$. Let $C \subseteq V$ be a maximum clique in G, $|C| \geqslant 2i - 1$, with the additional property that for C the number of edges in $V \backslash C$ is minimum among all maximum cliques of G. (Note that there is at least one edge in $V \backslash C$ since G is no split graph.) We first show the following claims.

(1) For all $x \in V \backslash C$ $|N(x) \cap C| \geqslant |C| - i + 1$ since otherwise $\bar{N}(x)$ contains a K_i.

(2) For all $x, y \in V \backslash C$, $N(x) \cap N(y) \cap C \neq \emptyset$ since otherwise $|C| \geqslant |N(x) \cap C| + |N(y) \cap C| \geqslant |C| - i + 1 + |C| - i + 1 = 2|C| - 2i + 2$, i.e. $2i - 2 \geqslant |C|$ — a contradiction to $|C| \geqslant 2i - 1$.

(3) For all $x, y \in V \backslash C$ with $xy \in E$ we have $C \backslash N(x) \subseteq C \backslash N(y)$ or $C \backslash N(y) \subseteq C \backslash N(x)$: Assume that there is a $c_x \in C \cap (\bar{N}(x) \backslash \bar{N}(y))$ and there is a $c_y \in C \cap (\bar{N}(y) \backslash \bar{N}(x))$. Then for $z \in N(x) \cap N(y) \cap C$ the neighborhood $N(z)$ contains an induced C_4 $\{c_x, c_y, x, y\}$ — a contradiction to (i).

Now we discuss three cases for edges $xy \in E$, $x, y \in V \backslash C$: Because of the maximality of C it is clear that $|N(u) \cap C| \leqslant |C| - 1$ for all $u \in V \backslash C$.

(4.1) $|N(x) \cap C| = |N(y) \cap C| = |C| - 1$. Let $C \backslash N(x) = \{c_x\}$, $C \backslash N(y) = \{c_y\}$. From claim (3) it follows that $c_x = c_y$. But then $(C \backslash \{c_x\}) \cup \{x, y\}$ is a larger clique than C — a contradiction.

(4.2) $|N(x) \cap C| \leqslant |C| - 2$ and $|N(y) \cap C| \leqslant |C| - 2$. Then because of claim (3) $\bar{N}(x) \cap \bar{N}(y) \cap C$ contains at least two elements: Let $c_1, c_2 \in \bar{N}(x) \cap \bar{N}(y) \cap C$. Let $z \in N(x) \cap N(y) \cap C$ according to claim (2). Then $N(z)$ contains an induced $2K_2$ $\{c_1, c_2\}, \{x, y\}$ — a contradiction to (i).

(4.3) $|N(x) \cap C| \leqslant |C| - 2$ and $|N(y) \cap C| = |C| - 1$. Let $C \backslash N(y) = \{c_y\}$. Because of claim (3) $c_y \in C \backslash N(x)$. Let $c_y = c_1^x$ for $C \backslash N(x) = \{c_1^x, \ldots, c_j^x\}$, $j \geqslant 2$. We first show that c_y has no neighbors outside C: Assume to the contrary that there is a vertex $u \in V \backslash C$ with $c_y u \in E$.

(4.3.1) $uy \notin E$: If $uy \in E$ then claim (3) applies to the edge uy and $c_y \in C \backslash N(u)$, i.e. $c_y u \notin E$ — a contradiction. Thus $uy \notin E$.

(4.3.2) $ux \in E$: Note first that $N(u) \cap N(x) \cap N(y) \cap C \neq \emptyset$. Because of claim (2) $N(u) \cap N(x) \cap C \neq \emptyset$ holds. Since c_y is the only non-neighbor of y in C and c_y is not a neighbor of x, i.e. $c_y \in C \backslash N(x)$, any vertex of $N(x) \cap N(u) \cap C$ is also a neighbor of y. Hence $N(u) \cap N(x) \cap N(y) \cap C \neq \emptyset$ holds.

If $ux \notin E$ then for any vertex $z \in N(u) \cap N(x) \cap N(y) \cap C$ $N(z)$ contains an induced $2K_2$ $\{c_y, u\}, \{x, y\}$ — a contradiction.

(4.3.3) Now for $z \in N(x) \cap N(u) \cap N(y) \cap C$ $N(z)$ contains an induced C_5: Since $ux \in E$ claim (3) applies to this edge. Assume that $c_1^u = c_2^x$ is an element with $c_1^u \in \bar{N}(u) \cap \bar{N}(x) \cap C$. Then $\{c_1^x, u, x, y, C_2^x\}$ induces a C_5 in $N(z)$.

This means that there is no vertex $u \in V \backslash C$ with $c_y u \in E$. But then $C' = (C \backslash \{c_y\}) \cup \{y\}$ is again a clique of maximum size for which there are less edges in

$V \setminus C'$ than in $V \setminus C$ since $c_y = c_1^x$—thus $c_y x \notin E$ and there are no other edges $c_y u$, $u \in V \setminus C'$. Thus the number of edges in $V \setminus C'$ is at least by 1 smaller than the number of edges in $V \setminus C$ — a contradiction.

Thus there is no edge in $V \setminus C$ but G was assumed to be no split graph—a contradiction. Therefore the clique size of G is bounded by $2i - 2$. □

Note that the lemma is also correct in the special case $i = 1$: If (ii) is fulfilled for $i = 1$ then for all $v \in V$ $\bar{N}(v)$ is empty and therefore G is a clique (which is a split graph). Thus there are no graphs which fulfill the suppositions of Lemma 1 for $i = 1$.

Remark. Lemma 1 can be interpreted also as a property related with dominating cliques: A clique C of the graph $G = (V, E)$ *dominates* G iff every vertex v of $V \setminus C$ has a neighbor in C. Then conditions (ii) of Lemma 1 means that every clique of G of size i dominates G.

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