Kronecker products and the RSK correspondence
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\textbf{A B S T R A C T}

A matrix $M$ with nonnegative integer entries is \textit{minimal} if the nonincreasing sequence of its entries (called $\pi$-sequence) is minimal, in the dominance order of partitions, among all nonincreasing sequences of entries of matrices with nonnegative integers that have the same 1-marginals as $A$.

The starting point for this work is an identity that relates the number of minimal matrices that have fixed 1-marginals and $\pi$-sequence to a linear combination of Kronecker coefficients. In this paper we provide a bijection that realizes combinatorially this identity. From this bijection we obtain an algorithm that to each minimal matrix associates a minimal component, with respect to the dominance order, in a Kronecker product, and a combinatorial description of the corresponding Kronecker coefficient in terms of minimal matrices and tableau insertion. Our bijection follows from a generalization of the dual RSK correspondence to 3-dimensional binary matrices, which we state and prove. With the same tools we also obtain a generalization of the RSK correspondence to 3-dimensional integer matrices.

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1. Introduction

Let $\lambda$, $\mu$, $\nu$ be partitions of a positive integer $m$ and let $\chi^{\lambda}$, $\chi^{\mu}$, $\chi^{\nu}$ be their corresponding complex irreducible characters of the symmetric group $S_m$. The \textit{Kronecker coefficient} $k(\lambda, \mu, \nu)$ is the multiplicity $\langle \chi^{\lambda} \otimes \chi^{\mu}, \chi^{\nu} \rangle$ of $\chi^{\nu}$ in the Kronecker product $\chi^{\lambda} \otimes \chi^{\mu}$ of $\chi^{\lambda}$ and $\chi^{\mu}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product of characters. Via the Frobenius map, $k(\lambda, \mu, \nu)$ is equal to the multiplicity of the Schur function $s_{\nu}$ in the internal product of Schur functions $s_{\lambda} \ast s_{\mu}$, namely

$$k(\lambda, \mu, \nu) = \langle s_{\lambda} \ast s_{\mu}, s_{\nu} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of symmetric functions.

It is a long-standing problem to give a combinatorial or geometric description of Kronecker coefficients. However, the multiplicities of extremal (minimal or maximal) components of $\chi^{\lambda} \otimes \chi^{\mu}$ with respect to the dominance order of partitions can be easily described combinatorially (Lemma 3.2). In general, the farther away a component in a Kronecker product is from the extremal components, the harder it is to compute. Therefore it is natural to try to determine extremal components in a Kronecker product. These components were investigated for the first time in [23], where a connection between minimal components and discrete tomography was discovered. There it was shown that the existence of a minimal matrix with row sum vector $\alpha$, column sum vector $\mu$ and $\pi$-sequence $\nu$ (see Section 6 for the definitions) imply the vanishing of $k(\alpha, \beta, \gamma)$ for all $\alpha \succ \lambda$, $\beta \succ \mu$ and $\gamma \prec \nu$. It was also shown that if there exists a minimal matrix with row sum vector $\lambda$, column sum vector $\mu$ and $\pi$-sequence $\nu$, then the number $m_{\nu}(\lambda, \mu)$ of all such matrices satisfies identity (18), namely

$$m_{\nu}(\lambda, \mu) = \sum_{\alpha \succ \lambda, \beta \succ \mu} K_{\alpha \lambda} K_{\beta \mu} k(\alpha, \beta, \nu).$$

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(Here $K_{\alpha\beta}$ is a Kostka number, that is, the number of semistandard Young tableaux of shape $\alpha$ and content $\lambda$.) Thus any minimal matrix on the left contributes, up to a constant, to some Kronecker coefficient $k(\alpha, \beta, \nu)$ on the right. And in this situation, $\chi^\nu$ is a minimal component of $\chi^\alpha \otimes \chi^\beta$. However, the knowledge of $m_{\nu}(\lambda, \mu)$ is clearly not enough to determine the values of the Kronecker coefficients on the right or even to decide which of them are positive. This paper grew out of the attempt to understand combinatorially the contributions of the minimal matrices on the left side to the Kronecker coefficients on the right side.

The main results of this paper are a bijective proof of identity (18) and a combinatorial description for the Kronecker coefficients appearing in its right side (Theorem 6.6). In fact, we were seeking a bijection that was an extension of mapping (20), see Remark 5.5. While looking for it, we discovered, by interpreting integral matrices as binary 3-dimensional matrices (Proposition 6.3), that (18) is in fact a particular case of identity (13). Combinatorially, this identity can be thought of as a sort of extension of the dual RSK correspondence to 3-dimensional binary matrices. This is stated in Theorem 4.2. Although it was not evident at first, we found a surprisingly simple bijective proof of identity (13); even nicer, the bijection is a combination of the dual RSK correspondence with Thomas’ proof of the Littlewood–Richardson rule [20], which is not so well known nowadays. The combinatorial description for the Kronecker coefficients in Theorem 6.6 is given in terms of minimal matrices and the insertion algorithm.

The paper is organized as follows: In Section 2 we motivate, using character theory, the correspondences presented in Section 4. In Section 3 we introduce certain pairs of sets of Littlewood–Richardson multitableaux which are needed in Section 4. We also show that they describe combinatorially the multiplicities of extremal components in Kronecker products (Lemma 3.2). Section 4 contains a bijection between 3-dimensional binary matrices and certain triples of tableaux (Theorem 4.2) and its proof. This bijection provides a combinatorial realization of identity (13), and therefore, as explained above, of identity (18). It can be seen as a generalization of the dual RSK correspondence to 3-dimensional binary matrices. With no extra effort we state and prove a similar result (Theorem 4.1) for 3-dimensional integer matrices, which can be seen as a generalization of the RSK correspondence to 3-dimensional integer matrices. In Section 5 we provide a detailed example of how the correspondence from Theorem 4.2 is defined. We start Section 6 by recalling the definition of minimal matrix and some related results. Then we show how Theorem 4.2 yields a combinatorial realization of identity (18). This establishes an explicit link between minimal matrices and some Kronecker coefficients. As a consequence we obtain the second of our main results (Theorem 6.6); a new combinatorial description of the multiplicities of some minimal components in Kronecker products in terms of minimal matrices and the insertion algorithm. This is illustrated at the end of the section with an example. Finally, Section 7 contains, for the benefit of the reader, a brief summary of how some notions from discrete tomography apply to Kronecker products.

2. Matrices and RSK correspondences

In this section we motivate, using character theory, the correspondences presented in Section 4. We start with a known formula that relates the number of integral matrices with prescribed 1-marginals to certain inner products of characters (2). This formula yields a second one for which the RSK-correspondence is a combinatorial realization (3). A similar approach can be carried out for binary matrices with prescribed 1-marginals and the dual RSK-correspondence (Eqs. (4) and (5)).

In a similar way, but now working with 3-dimensional matrices with prescribed 1-marginals we obtain formulas (6)–(9), which suggest generalizations of the RSK and the dual RSK correspondences to dimension 3.

Let $\phi^\nu = \text{Ind}^{S_n}_{S_\nu} (1)$ denote the character of $S_n$ induced from the trivial character $1$, of the Young subgroup $S_\nu$ associated to $\nu$. Throughout this paper we will make frequent use of Young’s rule, which expresses $\phi^\nu$ as a linear combination of irreducible characters, namely

$$\phi^\nu = \sum_{\gamma \vdash n} K_{\gamma\nu} \chi^\gamma.$$  

We use the notation $\alpha \triangleright \beta$ to indicate that $\alpha$ is bigger or equal than $\beta$ in the dominance order of partitions, and $\alpha \trianglerightneq \beta$ to indicate that $\alpha \triangleright \beta$ and $\alpha \neq \beta$. This partial order is of interest to us because $K_{\alpha\beta} > 0$ if and only if $\alpha \triangleright \beta$.

Given a matrix $A = (a_{ij})$ of size $p \times q$ we denote by $\text{row}(A)$ the row sum vector of $A$ and by $\text{col}(A)$ the column sum vector of $A$, that is, $\text{row}(A) = (r_1, \ldots, r_p)$, where $r_i = \sum_j a_{ij}$ and $\text{col}(M) = (c_1, \ldots, c_q)$, where $c_j = \sum_i a_{ij}$. The vectors $\text{row}(A)$ and $\text{col}(A)$ are also called the 1-marginals of $A$. Given $\lambda$, $\mu$ compositions of $n$, that is, vectors with nonnegative integer coordinates whose sum is $n$, we denote by $M(\lambda, \mu)$ the set of all matrices $A = (a_{ij})$ with nonnegative integer entries and 1-marginals $\lambda$, $\mu$, and by $m(\lambda, \mu)$ its cardinality. It is well known that $m(\lambda, \mu)$ can be described as an inner product involving permutation characters and the trivial character (see [4, Thm. 15], [18, Cor. 3.1], [5, Thm. 1], [12, 6.1.9] or [19, 7.9.1]):

$$m(\lambda, \mu) = \langle \phi^\lambda \otimes \phi^\mu, \chi^{(n)} \rangle.$$  

If we expand $\phi^\lambda$ and $\phi^\mu$ as a linear combination of irreducible characters by Young’s rule (1), we obtain the following identity

$$m(\lambda, \mu) = \sum_{\sigma \triangleright \lambda, \beta \triangleright \mu} K_{\sigma\lambda} K_{\beta\mu} \langle \chi^\sigma \otimes \chi^\beta, \chi^{(n)} \rangle = \sum_{\sigma \triangleright \lambda, \mu} K_{\sigma\lambda} K_{\sigma\mu}.$$  

The RSK correspondence is a combinatorial realization of this identity.
Similarly, let $M^*(\lambda, \mu)$ denote the set of all binary matrices (matrices whose entries are either zeros or ones) in $M(\lambda, \mu)$ and let $m^*(\lambda, \mu)$ denote its cardinality. It is also known (see [4, Thm.16], [18, Cor. 7.1], [5, Thm. 2] or [12, 6.1.9]) that
\[
m^*(\lambda, \mu) = \langle \phi^\lambda \otimes \phi^\mu, \chi^{(1)} \rangle.
\] (4)

Applying Young's rule (1) we obtain the identity:
\[
m^*(\lambda, \mu) = \sum_{\sigma|\lambda \geq \mu} K_{\sigma \lambda} K_{\sigma' \mu}.
\] (5)

The dual RSK correspondence is a combinatorial realization of this identity. Let us observe that identities (2) and (4) can also be written in terms of inner products of symmetric functions.

There are similar results for $n$-dimensional matrices due to Snapper. We will be concerned only with the 3-dimensional case. Let $\lambda = (\lambda_1, \ldots, \lambda_p)$, $\mu = (\mu_1, \ldots, \mu_q)$, $\nu = (\nu_1, \ldots, \nu_r)$ be compositions of $n$. We denote by $M(\lambda, \mu, \nu)$ the set of all 3-dimensional matrices with nonnegative integer entries and 1-marginals (plane sums) $\lambda$, $\mu$, $\nu$ namely, matrices $A = (a_{ijk})$ such that their entries satisfy
\[
a_{ijk} = \lambda_i, \quad a_{ijk} = \mu_j \quad \text{and} \quad a_{ijk} = \nu_k,
\]
for all $i \in [p]$, $j \in [q]$ and $k \in [r]$ (here $[n] = \{1, 2, \ldots, n\}$). Let $m(\lambda, \mu, \nu)$ denote the cardinality of $M(\lambda, \mu, \nu)$. Snapper showed (see Theorem 3.1 in [18]) that
\[
m(\lambda, \mu, \nu) = \langle \phi^\lambda \otimes \phi^\mu \otimes \phi^\nu, \chi^{(0)} \rangle.
\]

Applying Young's rule (1) we get
\[
m(\lambda, \mu, \nu) = \sum_{\alpha \geq \lambda, \beta \geq \mu, \gamma \geq \nu} K_{\alpha \lambda} K_{\beta \mu} K_{\gamma \nu} k(\alpha, \beta, \gamma).
\] (6)

It is natural to ask for a one-to-one correspondence between 3-dimensional matrices and some triples of semistandard tableaux that generalizes the RSK correspondence. Due to the nature of Kronecker coefficients, formula (6) shows that this is not possible. This formula also shows that an appropriate correspondence (neither injective nor surjective) that associates to a given 3-dimensional matrix a triple of semistandard tableaux would yield a combinatorial description of Kronecker coefficients.

Similarly, let $M^*(\lambda, \mu, \nu)$ denote the set of all 3-dimensional binary matrices contained in $M(\lambda, \mu, \nu)$ and let $m^*(\lambda, \mu, \nu)$ denote its cardinality. Snapper also showed (see Theorem 7.1 in [18]) that
\[
m^*(\lambda, \mu, \nu) = \langle \phi^\lambda \otimes \phi^\mu \otimes \phi^\nu, \chi^{(1)} \rangle.
\]

and, by Young's rule (1), we have
\[
m^*(\lambda, \mu, \nu) = \sum_{\alpha \geq \lambda, \beta \geq \mu, \gamma \geq \nu} K_{\alpha \lambda} K_{\beta \mu} K_{\gamma \nu} k(\alpha, \beta, \gamma).
\] (7)

Formulas (6) and (7) give us a hint of how generalizations of the RSK correspondence and its dual to 3-dimensional matrices should be. Let us stress that the ultimate goal is not to extend the RSK correspondence or its dual to dimension 3 for its own sake, but to use such an extension to obtain combinatorial descriptions of Kronecker coefficients.

Caselli [2, Section 4] has found some properties such generalizations of the RSK correspondence and its dual should satisfy in order to yield combinatorial descriptions of Kronecker coefficients.

A variation of these ideas leads to a more modest but more realistic approach: We apply Young's rule to only two factors. By doing so, we get formulas in which all terms have a combinatorial description (see Lemma 3.1):
\[
m(\lambda, \mu, \nu) = \sum_{\alpha \geq \lambda, \beta \geq \mu} K_{\alpha \lambda} K_{\beta \mu} \langle \chi^\alpha \otimes \chi^\beta \otimes \phi^\nu, \chi^{(0)} \rangle
\] (8)

and
\[
m^*(\lambda, \mu, \nu) = \sum_{\alpha \geq \lambda, \beta \geq \mu} K_{\alpha \lambda} K_{\beta \mu} \langle \chi^\alpha \otimes \chi^\beta \otimes \phi^\nu, \chi^{(1)} \rangle.
\] (9)

Our results in Section 4 will give combinatorial proofs of formulas (8) and (9).

3. Littlewood–Richardson multitableaux

In this section we introduce two kinds of sets of pairs of Littlewood–Richardson multitableaux. They will be used in the combinatorial realizations of formulas (8) and (9). They can also be viewed as combinatorial approximations of Kronecker
coefficients (Eqs. (10) and (11)). In addition we deal with extremal (minimal and maximal) components \( \chi^v \) of Kronecker products \( \chi^\alpha \otimes \chi^\beta \) with respect to the dominance order of partitions and observe that their Kronecker coefficients are combinatorially described by those sets of pairs (Lemma 3.2).

For the undefined terms we refer the reader to [9,16,19].

Let \( \alpha \) be a partition of \( n \) and \( v = (v_1, \ldots, v_r) \) be a composition of \( n \), then a sequence \( T = (T_1, \ldots, T_r) \) of tableaux is called a Littlewood–Richardson multitableau of shape \( \alpha \) and type \( v \) if there exists a sequence of partitions \( \alpha(0), \alpha(1), \ldots, \alpha(r) \) such that

\[
\emptyset = \alpha(0) \subseteq \alpha(1) \subseteq \cdots \subseteq \alpha(r) = \alpha
\]

and \( T_i \) is a Littlewood–Richardson tableau of shape \( \alpha(i)/\alpha(i-1) \) and size \( v_i \) (the number of squares of \( T_i \) is \( v_i \)) for all \( i \in [r] \). If each \( T_i \) has content \( \rho(i) \), then we say that \( T \) has content \((\rho(1), \ldots, \rho(r)) \). Note that, since \( T_i \) is a Littlewood–Richardson tableau, \( \rho(i) \) is a partition of \( v_i \). See Section 5 for an example.

Given partitions \( \alpha \) and \( \beta \) of \( n \) and a composition \( v \) of \( n \), we denote by \( LR(\alpha, \beta; v) \) the set of all pairs \((T, S)\) of Littlewood–Richardson multitableaux of shape \((\alpha, \beta)\) and type \( v \) such that \( S \) and \( T \) have the same content and by \( lr(\alpha, \beta; v) \) its cardinality. Similarly, let \( LR^*(\alpha, \beta; v) \) denote the set of all pairs \((T, S)\) of Littlewood–Richardson multitableaux of shape \((\alpha, \beta)\), type \( v \) and conjugate content, that is, if \( T \) has content \((\rho(1), \ldots, \rho(r)) \), then \( S \) has content \((\rho(1)', \ldots, \rho(r)') \) and by \( lr^*(\alpha, \beta; v) \) its cardinality. Here \( \rho' \) denotes the partition conjugate to \( \rho \). We have

**Lemma 3.1.** Let \( \alpha \) and \( \beta \) be partitions of \( n \) and let \( v \) be a composition of \( n \), then

(1) \( lr(\alpha, \beta; v) = \langle \chi^\alpha \otimes \chi^\beta \otimes \phi^v, \chi^{(0)} \rangle \).

(2) \( lr^*(\alpha, \beta; v) = \langle \chi^\alpha \otimes \chi^\beta \otimes \phi^v, \chi^{(1v)} \rangle \).

The proof of this lemma follows from Frobenius reciprocity and the Littlewood–Richardson rule. Part (1) of the lemma appears implicitly in [11, 2.9.17] and explicitly in [22, 29]. Part (2) is similar and appears in an equivalent form in identity (8) in [23].

A component \( \chi^v \) of \( \chi^\alpha \otimes \chi^\beta \) is called maximal if for all \( \gamma \succ v \) one has \( k(\alpha, \beta, \gamma) = 0 \), and it is called minimal if for all \( \gamma \prec v \) one has \( k(\alpha, \beta, \gamma) = 0 \). Minimal components were studied for the first time in [23]. Since conjugation is an order–reversing involution in the set of partitions of \( n \) under the dominance order (see [12, 6.1.18]) and since Kronecker coefficients satisfy the symmetry \( k(\alpha, \beta, \gamma') = k(\alpha, \beta', \gamma) \), the study of maximal components can be reduced to the study of minimal components. An algorithm for computing extremal components in the lexicographic order of partitions is given in [3].

It follows directly from Young’s rule (1) that

\[
lr(\alpha, \beta; v) = \sum_{\gamma \preceq v} K_{\gamma,v} k(\alpha, \beta, \gamma) \tag{10}
\]

and

\[
lr^*(\alpha, \beta; v') = \sum_{\gamma \preceq v} K_{\gamma,v'} k(\alpha, \beta, \gamma). \tag{11}
\]

**Lemma 3.2.** Let \( \chi^v \) be a component of \( \chi^\alpha \otimes \chi^\beta \), then

(1) \( \chi^v \) is a maximal component of \( \chi^\alpha \otimes \chi^\beta \) if and only if \( k(\alpha, \beta, \gamma) = \lr(\alpha, \beta; v) \).

(2) \( \chi^v \) is a minimal component of \( \chi^\alpha \otimes \chi^\beta \) if and only if \( k(\alpha, \beta, \gamma) = \lr^*(\alpha, \beta; v') \).

The proof of this lemma is straightforward. It follows from (10) and (11). Part (2) is already implicit in Corollary 3.3.2 from [23].

Since \( \lr(\alpha, \beta; v) \) is defined combinatorially, we can think of it, because of identity (10), as a combinatorial approximation of \( k(\alpha, \beta, v) \); by Lemma 3.2 both numbers coincide when \( \chi^v \) is a maximal component of \( \chi^\alpha \otimes \chi^\beta \); when \( v \) is smaller than a maximal component of \( \chi^\alpha \otimes \chi^\beta \), the farther is \( v \) from it, the bigger is the difference between \( \lr(\alpha, \beta; v) \) and \( k(\alpha, \beta, v) \). A similar remark applies for \( \lr^*(\alpha, \beta; v') \).

4. Combinatorial realizations

In this section we give an explicit bijection (Theorem 4.1) for the identity

\[
m(\lambda, \mu, v) = \sum_{\alpha = \lambda, \beta = \mu} K_{\alpha,\beta} \lr(\alpha, \beta; v), \tag{12}
\]

which, by Lemma 3.1, is equivalent to (8). We also give a dual bijection (Theorem 4.2) for the identity

\[
m^*(\lambda, \mu, v) = \sum_{\alpha = \lambda, \beta = \mu} K_{\alpha,\beta} \lr^*(\alpha, \beta; v), \tag{13}
\]

which, by Lemma 3.1, is equivalent to (9).
Let $\lambda$, $\mu$, $\nu$ be compositions of $n$. For any partition $\alpha$ of $n$ let $K_{\alpha \beta}$ denote the set of all semistandard tableaux of shape $\alpha$ and content $\lambda$. Our results are

**Theorem 4.1.** There is a one-to-one correspondence between the set $M(\lambda, \mu, \nu)$ of 3-dimensional matrices with nonnegative integer coefficients that have 1-marginals $\lambda$, $\mu$, $\nu$ and the set of triples $\prod_{\alpha \beta \mu} K_{\alpha \lambda} \times K_{\beta \mu} \times LR(\alpha, \beta; \nu)$.

**Theorem 4.2.** There is a one-to-one correspondence between the set $M^*(\lambda, \mu, \nu)$ of 3-dimensional binary matrices that have 1-marginals $\lambda$, $\mu$, $\nu$ and the set of triples $\prod_{\alpha \beta \mu} K_{\alpha \lambda} \times K_{\beta \mu} \times LR^*(\alpha, \beta; \nu)$.

The correspondences of the previous theorems will be given as compositions of three bijections. The first one is tautological, the second is given by a correspondence between matrices and pairs of tableaux, such as the RSK or the dual RSK correspondence, applied simultaneously several times, and the third is a consequence of a bijection given by Thomas in [20] for his proof of the Littlewood–Richardson rule. These bijections have already been presented without proofs in [26].

Theorem 4.1 follows directly from the first, the second and the third bijections given below, while Theorem 4.2 follows from the first, the second and the third dual bijections. Note that when $\nu = (n)$ any matrix $A \in M(\lambda, \mu, \nu)$ has exactly one horizontal slice and our correspondence is just the usual RSK correspondence, or the dual one if $A$ is a binary matrix. In this case the set $LR(\alpha, \beta; \nu)$ is empty when $\alpha \neq \beta$, and it has one element when $\alpha = \beta$. Similarly, $LR^*(\alpha, \beta; \nu)$ is empty when $\alpha \neq \beta$, and it has one element when $\alpha = \beta$.

In the statement of the bijections we use the following notation: If $T$ is a semistandard tableau, $sh(T)$ denotes its shape, $cont(T)$ its content and $|T|$ its size.

We also let $p, q, r$ denote the number of parts of $\lambda$, $\mu$, $\nu$, respectively.

**First bijection.** There is a one-to-one correspondence between the set of matrices $M(\lambda, \mu, \nu)$ and the set of $r$-tuples $(A_1, \ldots, A_r)$ of matrices with nonnegative integer coefficients of size $p \times q$ such that

$$\sum_{k=1}^{r} \text{row}(A_k) = \lambda, \quad \sum_{k=1}^{r} \text{col}(A_k) = \mu, \quad \text{sum of the entries of } A_k \text{ equals } v_k, \quad \text{for all } k \in [r].$$ (14)

To construct this bijection we split $A \in M(\lambda, \mu, \nu)$ into its level matrices $A^{(k)} = \left(a_{ij}^{(k)}\right)$, $k \in [r]$, where $a_{ij}^{(k)} = a_{ij}$. Hence

$$A \mapsto (A^{(1)}, \ldots, A^{(r)})$$

is the desired bijection.

**First dual bijection.** There is a one-to-one correspondence between the set of binary matrices $M^*(\lambda, \mu, \nu)$ and the set of $r$-tuples $(A_1, \ldots, A_r)$ of binary matrices satisfying (14). This correspondence is the restriction of the first bijection to $M^*(\lambda, \mu, \nu)$.

**Second bijection.** There is a one-to-one correspondence between the set of $r$-tuples $(A_1, \ldots, A_r)$ of matrices with nonnegative integer coefficients of size $p \times q$ satisfying (14) and the set of pairs $((P_1, \ldots, P_r), (Q_1, \ldots, Q_r))$ of $r$-tuples of semistandard tableaux such that

$$\sum_{k=1}^{r} \text{cont}(Q_k) = \lambda, \quad \sum_{k=1}^{r} \text{cont}(P_k) = \mu, \quad \text{sh}(P_k) = \text{sh}(Q_k) \text{ and } |\text{sh}(P_k)| = v_k, \quad \text{for all } k \in [r].$$ (15)

In order to establish this bijection we choose any one-to-one correspondence between matrices $M$ with nonnegative integer coefficients and pairs $(P, Q)$ of semistandard tableau of the same shape such that $\text{cont}(P) = \text{col}(M)$ and $\text{cont}(Q) = \text{row}(M)$. Examples of such correspondences are the RSK correspondence [13], [9, 4.1], [16, 4.8] and [19, 7.11], and the Burge correspondence [9, p. 198]. The bijection is as follows: For any $r$-tuple $(A_1, \ldots, A_r)$ of matrices satisfying (14), let $(P_k, Q_k)$ be the pair associated to $A_k$ under the chosen correspondence, then

$$(A_1, \ldots, A_r) \mapsto ((P_1, \ldots, P_r), (Q_1, \ldots, Q_r))$$

is the desired bijection.

**Second dual bijection.** There is a one-to-one correspondence between the set of $r$-tuples $(A_1, \ldots, A_r)$ of binary matrices of size $p \times q$ satisfying (14) and the set of pairs $((P_1, \ldots, P_r), (Q_1, \ldots, Q_r))$ of $r$-tuples of semistandard tableaux such that

$$\sum_{k=1}^{r} \text{cont}(Q_k) = \lambda, \quad \sum_{k=1}^{r} \text{cont}(P_k) = \mu, \quad \text{sh}(P_k) = \text{sh}(Q_k)' \text{ and } |\text{sh}(P_k)| = v_k, \quad \text{for all } k \in [r].$$ (16)

In order to establish this bijection we choose any one-to-one correspondence between binary matrices $M$ and pairs $(P, Q)$ of semistandard tableaux of conjugate shape such that $\text{cont}(P) = \text{col}(M)$ and $\text{cont}(Q) = \text{row}(M)$. Examples of such correspondences are the dual RSK correspondence [13], [9, p.203], [16, 4.8] and [19, 7.14], and the dual of the Burge correspondence [9, p. 205]. The construction of this bijection is analogous to the one of the second bijection.

The two remaining bijections are based on the following result due to Thomas (see the corollary in page 29 from [20]). There he stated it for $r = 2$, but the generalization for arbitrary $r$ is straightforward. We present his result in a slightly different form.
Theorem 4.3. There is a one-to-one correspondence between the set of all \( r \)-tuples \((P_1, \ldots, P_r)\) of semistandard tableaux and the set of pairs \((P, S)\) such that \(P\) is a semistandard tableau and \(S\) is a Littlewood–Richardson multitableau of shape \(sh(P)\). Moreover, under this correspondence

\[
\text{cont}(P) = \sum_{k=1}^{r} \text{cont}(P_k) \quad \text{and} \quad \text{cont}(S) = (sh(P_1), \ldots, sh(P_r)).
\]

Third bijection. There is a one-to-one correspondence between the set of pairs of \( r \)-tuples \((P_1, \ldots, P_r), (Q_1, \ldots, Q_r)\) of semistandard tableaux satisfying (15) and the set \([\alpha_1, \ldots, \alpha_r] \times [\beta_1, \ldots, \beta_r] \times LR(\alpha, \beta; v)\).

The third bijection is as follows: Let \((P_1, \ldots, P_r), (Q_1, \ldots, Q_r)\) be a pair of \( r \)-tuples satisfying (15), and let \((P, S)\), respectively \((Q, T)\), be the pair corresponding to \((P_1, \ldots, P_r)\), respectively \((Q_1, \ldots, Q_r)\), under the bijection of Theorem 4.3. Hence

\[(P_1, \ldots, P_r), (Q_1, \ldots, Q_r) \mapsto (Q, P, (T, S))\]

is the desired bijection.

Third dual bijection. There is a one-to-one correspondence between the set of pairs of \( r \)-tuples \((P_1, \ldots, P_r), (Q_1, \ldots, Q_r)\) of semistandard tableaux satisfying (16) and the set \([\alpha_1, \ldots, \alpha_r] \times [\beta_1, \ldots, \beta_r] \times LR^*(\alpha, \beta; v)\).

This bijection is constructed in a similar way as the third.

Remark 4.4. The correspondence in Theorem 4.1 satisfies, when we use the RSK correspondence in the second bijection of the construction, an obvious symmetry, which is inherited from the symmetry of the RSK correspondence, namely if \(A = (a_{ik})\) of size \( p \times q \times r \) corresponds to \((Q, P, (T, S))\), then its transpose \(A^T = (a_{kj})\) of size \( q \times p \times r \) corresponds to \((P, Q, (S, T))\). Also the symmetry theorem given in [9, p. 205] is inherited by the construction given in the proof of Theorem 4.2.

5. An example

In this section we explain how Thomas’ bijection is defined and give an example of the correspondence in Theorem 4.2.

For the definition of column insertion we refer the reader to [9,16,19]. Let \( x \rightarrow T \) denote the result of column inserting \( x \) in a semistandard tableau \( T \). For any partition \( \gamma \), we denote by \( C(\gamma) \) the unique semistandard tableau of shape \( \gamma \) and content \( \nu \). Besides, given a semistandard tableau \( T \), let \( w_{col}(T) \) denote the column word of \( T \), that is, the word obtained from \( T \) by reading its entries from bottom to top (in English notation), in successive columns, starting in the left column and moving to the right. For example \( w_{col}(3(2, 1)) = 321211 \).

Thomas’ bijection is as follows: Let \((P_1, \ldots, P_r)\) be an \( r \)-tuple of semistandard tableau, and let \( \gamma(k) = sh(P_k), k \in [r] \). The pair \((P, S)\) associated to \((P_1, \ldots, P_r)\) is constructed as follows. Let \( P^{(1)} = P_1 \) and \( S_1 = C(\gamma(1)) \). We define \( P^{(k+1)} \) and \( S_{k+1} \) inductively: Let \( w_{col}(P_{k+1}) = v_m \cdots v_1 \) and \( w_{col}(C(\gamma(k+1))) = u_m \cdots u_1 \). The tableau \( P^{(k+1)} \) is obtained by column inserting \( v_1, \ldots, v_m \) in \( P^{(k)} \), that is,

\[ p^{(k+1)} = v_m \rightarrow (\cdots \rightarrow v_1 \rightarrow p^{(k)}) \]

and \( S_{k+1} \) is the tableau obtained by placing \( u_1, \ldots, u_m \) successively in the new boxes. Let \( P = P^{(r)} \) and \( S = (S_1, \ldots, S_r) \). Thus \( P \) is a semistandard tableau, \( S \) is a Littlewood–Richardson multitableau, \( sh(P) = sh(S) \), \( \text{cont}(S) = (\gamma(1), \ldots, \gamma(r)) \), and \( \text{cont}(P) = \sum_{k=1}^{r} \text{cont}(P_k) \). Note that \( P = P_1 \cdots P_r \), the product of tableaux as defined in Fulton’s book [9].

We conclude this section with an illustration of the bijection described in Theorem 4.2. We use the dual RSK correspondence in the second bijection of the construction. Let \( A \) be the following 3-dimensional matrix of zeros and ones of size \( 4 \times 5 \times 3 \).

\[
\begin{bmatrix}
0 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

It has 1-marginals \( \lambda = (9, 7, 5, 4), \mu = (7, 6, 5, 4, 3) \) and \( v = (10, 8, 7) \). The triple \((Q, P, (T, S))\) corresponding to \( A \) is constructed as follows: To each of the three level matrices corresponds, under the dual RSK correspondence, a pair of semistandard tableaux of conjugate shape.
6. Minimal matrices and Kronecker products

Minimal matrices were introduced in [21] to characterize 3-dimensional binary matrices that are uniquely determined by its 1-marginals. They were used in [23] as a tool to produce minimal components in Kronecker products. In this section we go a step further towards an understanding of the relation between minimal matrices and Kronecker products. Our main result (Theorem 6.6) is an algorithm that, out of a list of minimal matrices, computes several Kronecker coefficients. We start by recalling some definitions and results.

For any matrix $A$ with nonnegative integer entries, we denote by $\pi(A)$ the weakly decreasing sequence of its entries and call it a $\pi$-sequence. Let $\lambda, \mu, \nu$ be three partitions of some integer $n$. We denote by $M_\nu(\lambda, \mu)$ the subset of $M(\lambda, \mu)$ formed by all matrices $A$ with $\pi(A) = \nu$, and by $m_\nu(\lambda, \mu)$ its cardinality. A matrix $A$ in $M(\lambda, \mu)$ is called minimal (see [21]) if there is no other matrix $B$ in $M(\lambda, \mu)$ such that $\pi(B) \prec \pi(A)$. Note that if $A \in M_\nu(\lambda, \mu)$ is minimal, then all matrices in $M_\nu(\lambda, \mu)$ are minimal. We say that $\nu$ is minimal for $(\lambda, \mu)$ if there is a minimal matrix in $M_\nu(\lambda, \mu)$.

Example 6.1. Let

$$A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

then $A$, $B$, $C$ and $D$ have the same 1-marginals $\lambda = \mu = (3^2)$, $\pi(A) = \pi(D) = (3^2)$ and $\pi(B) = \pi(C) = (2^2, 1^2)$. The set $M(\lambda, \mu)$ is equal to $\{A, B, C, D\}$, thus $B$ and $C$ are minimal, $A$ and $D$ are not, and $(2^2, 1^2)$ is minimal for $(\lambda, \mu)$. 

\((P_1, Q_1) = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & 4 & 4 \\ 4 & 5 \end{pmatrix}, \quad (P_2, Q_2) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 3 \\ 4 & 5 \\ 3 & 3 \end{pmatrix}, \quad (P_3, Q_3) = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 4 \\ 3 & 5 \end{pmatrix}.$$ 

Hence $(Q, T)$ and $(P, S)$ are the pairs associated to $(Q_1, Q_2, Q_3)$ and $(P_1, P_2, P_3)$, respectively, under the correspondence given in the proof of Theorem 4.3. The pair $(Q, P)$ of semistandard tableaux is

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 \\
3 & 3 & 3 & 3 & 4 & 4 \\
4 & 4 \\
\end{pmatrix},
\]

and the pair $(T, S)$ of Littlewood-Richardson multitableaux is

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 2 & 2 & 3 \\
3 & 3 \\
\end{pmatrix}.
\]
The following theorem establishes a connection between minimal matrices and multiplicities of minimal components in Kronecker products.

**Theorem 6.2.** If \( v \) is minimal for \((\lambda, \mu)\), then

1. \( k(\alpha, \beta, \gamma) = 0 \) for all \( \alpha \geq \lambda, \beta \geq \mu, \gamma < v \).
2. \( k(\alpha, \beta, v) = \text{lr}^*(\alpha; \beta; v') \) for all \( \alpha \geq \lambda, \beta \geq \mu \).

In particular, for any pair of partitions \((\alpha, \beta)\) such that \( \alpha \geq \lambda \) and \( \beta \geq \mu \) we have that \( \chi^v \) is a minimal component of \( \chi^\alpha \otimes \chi^\beta \) if and only if \( \text{lr}^*(\alpha; \beta; v') \) is positive.

Note that 6.2(1) is Proposition 3.2 in [23], and 6.2(2) follows from (11) and 6.2(1). The last remark follows from Lemma 3.2.

To each matrix \( A \) we associate a 3-dimensional matrix \( \overline{A} \) by

\[
a_{ijk} = \begin{cases} 1 & \text{if } k \leq a_{ij}, \\ 0 & \text{otherwise}. \end{cases}
\]

The correspondence \( A \mapsto \overline{A} \) defines an injective map

\[
G_{\lambda, \mu, v} : M_v(\lambda, \mu) \longrightarrow M^*(\lambda, \mu, v').
\]  

(17)

We have the following characterization of minimality.

**Proposition 6.3** ([28, Thm. 13]). Let \( \lambda, \mu, v \) be partitions of \( n \), then \( v \) is minimal for \( (\lambda, \mu) \) if and only if \( G_{\lambda, \mu, v} \) is bijective.

The proof given in [28] is combinatorial. A different proof follows immediately from Proposition 3.1 in [23] and the fact that the map \( G_{\lambda, \mu, v} \) is injective. Therefore, from this proposition, identity (9) and Lemmas 3.1 and 3.2, we obtain

**Corollary 6.4** ([23, Cor. 3.3.2]). Let \( v \) be a minimal for \((\lambda, \mu)\), then

\[
m_v(\lambda, \mu) = \sum_{\alpha \geq \lambda, \beta \geq \mu} K_{\alpha \lambda} K_{\beta \mu} \text{lr}^*(\alpha; \beta; v') = \sum_{\alpha \geq \lambda, \beta \geq \mu} K_{\alpha \lambda} K_{\beta \mu} k(\alpha, \beta, v).
\]  

(18)

Let

\[
\Phi^* : M^*(\lambda, \mu, v') \longrightarrow \prod_{\alpha \geq \lambda, \beta \geq \mu} K_{\alpha \lambda} \times K_{\beta \mu} \times \text{lr}^*(\alpha; \beta; v')
\]

denote the bijection that we get from Theorem 4.2, when we apply in each level the dual RSK-correspondence. Hence the composition

\[
\Phi^* \circ G_{\lambda, \mu, v} : M_v(\lambda, \mu) \longrightarrow \prod_{\alpha \geq \lambda, \beta \geq \mu} K_{\alpha \lambda} \times K_{\beta \mu} \times \text{lr}^*(\alpha; \beta; v')
\]  

(19)

is injective.

**Remark 6.5.** (1) Let \( P_v(\lambda, \mu) \) denote the set of plane partitions in \( M_v(\lambda, \mu) \). There is an injective map

\[
P_v(\lambda, \mu) \longrightarrow \text{lr}^*(\lambda, \mu; v'),
\]  

(20)

which was defined in the proof of Theorem 3.4 in [23]. It is straightforward to verify that (19) is an extension of (20). Note that, in the image \((P, Q, (S, T))\) under \( \Phi^* \circ G_{\lambda, \mu, v} \) of a plane partition, the two semistandard tableaux \( P, Q \) are the unique tableaux whose shape is equal to its content.

(2) The starting point of this paper was the attempt to find an extension of injection (20) to \( M_v(\lambda, \mu) \). We managed to extend it, not only to \( M_v(\lambda, \mu) \), but to \( M^*(\lambda, \mu, v') \), thus getting Theorem 4.2.

(3) Let us note that Manivel [14, Prop. 3.1] showed that \( |P_v(\lambda, \mu)| \leq k(\lambda, \mu, v) \) for all \( \lambda, \mu, v \), thus getting a better approximation than (20). However, this is still, in general, a weak lower bound on \( k(\lambda, \mu, v) \). This approximation is much better when \( v \) is minimal for \((\lambda, \mu)\) (see Remark 4.4 in [23]).

Note that if \( v \) is minimal for \((\lambda, \mu)\), the map (19) is a bijection and provides a combinatorial realization of (18). Thus, if \( v \) is minimal for \((\lambda, \mu)\) and we have the list of all elements in \( M_v(\lambda, \mu) \), we can compute several Kronecker coefficients. In fact, we get a combinatorial description of these coefficients: Let \( f, g, h \) denote the components of \( \Phi^* \circ G_{\lambda, \mu, v} \), that is, for any \( A \in M_v(\lambda, \mu) \) we have \( \Phi^* \circ G_{\lambda, \mu, v}(A) = (f(A), g(A), h(A)) \). Thus

**Theorem 6.6.** Suppose that \( v \) is minimal for \((\lambda, \mu)\). Let \( P \) be a semistandard tableau of shape \( \alpha \) and content \( \lambda \), and \( Q \) be a semistandard tableau of shape \( \beta \) and content \( \mu \), then

\[
k(\alpha, \beta, v) = \#\{ A \in M_v(\lambda, \mu) : f(A) = P \text{ and } g(A) = Q \}.
\]

Moreover, if \( k(\alpha, \beta, v) > 0 \), then \( \chi^v \) is a minimal component of \( \chi^\alpha \otimes \chi^\beta \).
We conclude this section with an illustration of Theorem 6.6.

**Example 6.7.** Let \( \lambda = (6, 6) \) and \( \mu = (3, 3, 3, 3) \). Thus there are six minimal matrices in \( M(\lambda, \mu) \) (see Theorem 1.1 and Lemma 4.1 in [25]), namely

\[
A = \begin{bmatrix}
2 & 2 & 1 & 1 \\
1 & 1 & 2 & 2 \\
\end{bmatrix}, \quad
B = \begin{bmatrix}
1 & 2 & 2 & 1 \\
2 & 1 & 1 & 2 \\
\end{bmatrix}, \quad
C = \begin{bmatrix}
2 & 1 & 2 & 1 \\
1 & 2 & 1 & 2 \\
\end{bmatrix}, \quad
D = \begin{bmatrix}
2 & 1 & 1 & 2 \\
1 & 2 & 2 & 1 \\
\end{bmatrix}, \quad
E = \begin{bmatrix}
1 & 2 & 1 & 2 \\
2 & 1 & 2 & 1 \\
\end{bmatrix}, \quad
F = \begin{bmatrix}
1 & 1 & 2 & 2 \\
2 & 2 & 1 & 1 \\
\end{bmatrix}.
\]

Let \( \nu = (2^4, 1^4) \) be the common \( \pi \)-sequence of the six matrices. After computing \( \Phi^* \circ G_{\lambda, \mu, \nu} \) for each matrix we get

\[
\begin{align*}
f(A) &= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 2 \\
\end{bmatrix} \quad \text{and} \quad g(A) = \begin{bmatrix}
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3 \\
4 & 4 & 4 \\
\end{bmatrix}, \\
\end{align*}
\]

\[
\begin{align*}
f(B) &= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 2 \\
2 & 2 & 2 & 2 & 2 \\
\end{bmatrix} \quad \text{and} \quad g(B) = \begin{bmatrix}
1 & 1 & 1 & 2 \\
2 & 2 & 3 \\
3 & 3 & 4 \\
4 & 4 \\
\end{bmatrix}, \\
\end{align*}
\]

\[
\begin{align*}
f(C) &= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 2 \\
2 & 2 & 2 & 2 & 2 \\
\end{bmatrix} \quad \text{and} \quad g(C) = \begin{bmatrix}
1 & 1 & 1 & 3 \\
2 & 2 & 2 \\
3 & 3 & 4 \\
4 & 4 \\
\end{bmatrix}, \\
\end{align*}
\]

\[
\begin{align*}
f(D) &= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 2 \\
2 & 2 & 2 & 2 & 2 \\
\end{bmatrix} \quad \text{and} \quad g(D) = \begin{bmatrix}
1 & 1 & 1 & 4 \\
2 & 2 & 2 \\
3 & 3 & 3 \\
4 & 4 \\
\end{bmatrix}, \\
\end{align*}
\]

\[
\begin{align*}
f(E) &= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
\end{bmatrix} \quad \text{and} \quad g(E) = \begin{bmatrix}
1 & 1 & 1 & 2 \\
2 & 2 & 3 & 4 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}, \\
\end{align*}
\]

\[
\begin{align*}
f(F) &= \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 \\
\end{bmatrix} \quad \text{and} \quad g(F) = \begin{bmatrix}
1 & 1 & 1 & 3 \\
2 & 2 & 2 & 4 \\
3 & 3 \\
4 & 4 \\
\end{bmatrix}. \\
\end{align*}
\]

Let \( \alpha = \text{sh}(f(B)) = (7, 5) \), \( \beta = \text{sh}(g(B)) = (4, 3, 3, 2) \), \( \gamma = \text{sh}(f(A)) = (8, 4) \), \( \delta = \text{sh}(g(E)) = (4, 4, 2, 2) \). Thus we obtain from Theorem 6.6 that \( k(\gamma, \mu, \nu) = 1 \), \( k(\alpha, \beta, \nu) = 1 \) and \( k(\lambda, \delta, \nu) = 1 \), and that \( \chi^\pi \) is a minimal component of \( \chi^\gamma \otimes \chi^\mu \), \( \chi^\alpha \otimes \chi^\beta \) and \( \chi^\lambda \otimes \chi^\delta \). The remaining Kronecker coefficients in (18) are all zero. For example \( k(\lambda, \mu, \nu) = 0 \).
7. Discrete tomography and Kronecker products

In this brief section we show, for the benefit of the interested reader, how some notions from discrete tomography apply to Kronecker products.

Let $\lambda, \mu, \nu$ be partitions of some integer $n$. A matrix $X \in M^n(\lambda, \mu, \nu)$ is called a matrix of uniqueness if $m^\pi(\lambda, \mu, \nu) = 1$, that is, if $X$ is the only binary matrix with $1$-entries $\lambda, \mu, \nu$. This notion appears in discrete tomography (see [1,7,8,10,28]) and is of interest to Kronecker products because, if $m^\pi(\lambda, \mu, \nu) = 1$ all Kronecker coefficients but one vanish in Eq. (7).

There is a combinatorial characterization of uniqueness that is useful in some instances. We need a definition in order to explain it: A matrix $A \in M_n(\lambda, \mu)$ is called $\pi$-unique if it is the only matrix in $M(\lambda, \mu)$ with $\pi$-sequence $\nu$, that is, if $m_\nu(\lambda, \mu) = 1$. It was shown in [21] (see also Theorem 11 in [28]) that a matrix $X \in M^n(\lambda, \mu, \nu')$ is a matrix of uniqueness if and only if there is a matrix $A \in M_n(\lambda, \mu)$ that is minimal and $\pi$-unique and such that $G_{\pi,\nu'}(A) = X$ (see (17) for the definition of $G_{\pi,\nu'}$). Therefore, the existence of a matrix $A \in M_n(\lambda, \mu)$ that is minimal and $\pi$-unique implies that $m^\pi(\lambda, \mu, \nu') = 1$. For example, let $A$ be the $p \times q$ matrix such that all its entries are equal to $r$, then $A \in M_n(\lambda, \mu)$ where $\lambda = ((qr)^l)$, $\mu = (pr)^l$ and $\nu = (r^p)$. It is very easy to see that $A$ is minimal and $\pi$-unique. Another family of matrices that are minimal and $\pi$-unique appears in [21, p. 446]. In this case $\lambda, \mu$ and $\nu$ are hooks.

Corollary 4.2 in [23] summarizes all consequences of uniqueness to Kronecker coefficients. It can be reformulated in the following way:

**Theorem 7.1.** Let $\lambda, \mu, \nu$ be partitions of $n$ and let $A \in M_n(\lambda, \mu)$. If $A$ is minimal and $\pi$-unique, then $\chi^\nu$ is a minimal component of $\chi^\lambda \otimes \chi^\mu$, $k(\lambda, \mu, \nu) = 1$ and $k(\alpha, \beta, \gamma) = 0$ for all other triples $(\alpha, \beta, \gamma)$ such that $\alpha \geq \lambda$, $\beta \geq \mu$ and $\gamma \geq \nu$.

There is still another useful tool to determine uniqueness of a matrix. A notion of additivity for 3-dimensional binary matrices was introduced in [7] and was shown to be a sufficient condition for a matrix in $M^n(\lambda, \mu, \nu)$ to be a matrix of uniqueness. This notion was later translated to a version of additivity for integer matrices (Theorem 1 in [24]: a matrix $A = (a_{ijk})$ of size $p \times q$ with nonnegative integer entries is called additive if there exist real numbers $x_1, \ldots, x_p$ and $y_1, \ldots, y_q$ such that the condition

$$a_{ijk} > a_{k,l} \implies x_i + y_j > x_k + y_l$$

holds for all $i, j, k, l$. Later, the obvious extension of additivity from integer to real matrices was studied from a geometric point of view in [15]. Additivity for binary matrices seems to have been motivated by a related notion for binary relations (see [6] and the references therein).

The next result appears as Theorem 6.1 in [24] and Corollary 6.2 in [27]. A geometric proof can be found in [15].

**Theorem 7.2.** Any additive matrix with nonnegative integer entries is minimal and $\pi$-unique.

In particular each additive matrix in $M_n(\lambda, \mu)$ yields a minimal component $\chi^\nu$ of $\chi^\lambda \otimes \chi^\mu$ with multiplicity 1.

**Remark 7.3.** Minimal matrices of size $2 \times q$ were classified in [25]. Any plane partition of size $2 \times q$ is additive (see Proposition 4.1 in [24] and Lemma 8 in [28] for a shorter proof). There is no general known result for minimal or additive matrices of size $3 \times q$. However, a complete set of obstructions for additivity – the so-called arrow diagrams – was given in [17]. There, it was also shown (Theorem 4.1) that no finite subset of such obstructions is enough to determine additivity of an arbitrary integer matrix.

If $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ are nonnegative integers then, by the very definition of additivity, the matrix $A = (a_i + b_j)_{i \in [p], j \in [q]}$ is additive. Other examples of minimal or additive matrices appear in [23,24,28].

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