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Shannon multiresolution analysis on the Heisenberg group ☆

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A R T I C L E I N F O

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A B S T R A C T

We present a notion of frame multiresolution analysis on the Heisenberg group, abbreviated by FMRA, and study its properties. Using the irreducible representations of this group, we shall define a sinc-type function which is our starting point for obtaining the scaling function. Further, we shall give a concrete example of a wavelet FMRA on the Heisenberg group which is analogous to the Shannon MRA on $\mathbb{R}$.

1. Introduction

Multiresolution analysis (MRA) is an important mathematical tool since it provides a natural framework for understanding and constructing discrete wavelet systems (wavelet frames). The theory of a frame multiresolution analysis, for instance on $\mathbb{R}$, and some of its properties are studied in [1].

There are different approaches to construct wavelet frames on the Heisenberg group $\mathbb{H}$ (e.g. discretization of a continuous wavelet [11]). In the present work, we shall define and present a frame multiresolution analysis FMRA on $\mathbb{H}$, which will imply the existence of a normalized tight wavelet frame (n.t. frame) on this group. More precisely, in Theorem 4.16 we shall show that:

> *There exists a band-limited function $\psi \in L^2(\mathbb{H})$ and a lattice Γ in $\mathbb{H}$ such that the discrete wavelet system $\{L_{2^{-j}\gamma} D_{2^{-j}} \psi\}_{j,\gamma}$ forms a n.t. frame of $L^2(\mathbb{H})$.*

Accordingly, any function in $L^2(\mathbb{H})$ can be expanded in this wavelet frame with associated wavelet coefficients.

A standard way to construct a wavelet frame by the multiresolution analysis technique is by starting with a scaling function, i.e., a function which is refinable. In contrast to the standard approach, our starting point here is not a scaling function. Rather, we first construct a sinc-type function on $\mathbb{H}$ which is band-limited, self-adjoint, and has additional properties as in Theorem 4.5. The existence of a sinc-type function with the desired properties implies the existence of a scaling and wavelet function on $\mathbb{H}$.

A different notion of multiresolution analysis on stratified Lie groups was obtained by Lemarié [15]. There, in fact, the left-translations of the scaling function under a discrete set constitute an unconditional basis for the central scaling space.

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As an example, he constructed a MRA, which arises from a generalized spline-surface space, and obtains a C^N wavelet orthonormal basis of spline wavelets on these groups. The wavelet orthonormal basis in this example is generated by finitely many wavelets.

Continuous wavelets on nilpotent Lie groups have been studied by many authors (e.g., see [5,10] and references therein). The existence of a Parseval frame on $\mathbb{H}$, which is a system of dilates and left-translations of a single wavelet, is proved in [4]. The existence of a continuous wavelet in closed subspaces of $L^2(\mathbb{H})$ was studied in [16]. (For the definition of continuous wavelet on $\mathbb{H}$, see for instance [17].) The authors in [16] do not study the discretization of the wavelet to obtain a wavelet frame or wavelet orthonormal basis.

The contributions of this work will be as follows: After the introduction and some notation and preliminaries in Section 2, in Section 3 we shall give a brief review of the group Fourier analysis on the Heisenberg group. The main results of this work are presented in Section 4. Here, we shall introduce the FMRA on $\mathbb{H}$, i.e., the concept of orthonormal basis will be replaced by frames. Then we present a concrete example of FMRA on $\mathbb{H}$, Shannon MRA, and hence we prove the existence of a scaling and wavelet function for the Heisenberg group.

Finally, we demonstrate the existence of a Shannon normalized tight frame on $\mathbb{H}$, i.e., existence of a *band-limited* function on $\mathbb{H}$ such that its translations under an appropriate lattice in $\mathbb{H}$ and its dilations with respect to the integer powers of a suitable automorphism of $\mathbb{H}$ yields a normalized tight frame for $L^2(\mathbb{H})$.

Some words about MRA: There are three things in MRA that mainly concern us: the density of the union, the triviality of the intersection of the nested sequence of closed subspaces, and the existence of refinable functions, i.e., functions which have an expansion in their scaling. The triviality of the intersection is derived from the other conditions of MRA. To obtain the density of the union, we have to generalize the concept of the *support* of the Fourier transform. The new concepts, such as *band-limited* in $L^2(\mathbb{H})$, arise in this generalization. As to refinability, it depends very much on the individual function ϕ, the so-called *scaling function*. An example of a scaling function is presented in this work.

Then, we create the Shannon-MRA, as a concrete example of a FMRA on the Heisenberg group, for which we prove the existence of a wavelet function. This wavelet function is related to a certain lattice of $\mathbb{H}$.

Although the central closed shift-invariant space in our multiresolution analysis is a Paley–Wiener space on the Heisenberg group, we do not study any sampling theorems here. For sampling theory in the general setting, see for instance [7,9,18] and the references therein.

Observe that we do not obtain any smoothness conditions on our wavelets here. A class of Schwartz wavelets in the general setting, i.e., stratified Lie groups, has already been constructed and studied in our earlier work [11]. Those wavelets did not arise from an MRA, and are only "nearly tight" (if sufficiently fine lattices are used).

2. Preliminaries and notations

We use the abbreviation ONB for orthonormal basis and the word projection for self-adjoint projection operator on a Hilbert space. We denote the space of Hilbert–Schmidt operators on $L^2(\mathbb{R})$ by $HS(L^2(\mathbb{R}))$. (For the facts we shall use about Hilbert–Schmidt operators and trace-class operators, see [8, Appendix 2] and [19, Sections 2 and 3].)

In the following, we outline some notation and results concerning direct integrals. For further information on direct integrals, we refer the reader to [8, Section 7.4].

A family $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ of nonzero separable Hilbert spaces indexed by A will be called a field of Hilbert spaces over A. We assume A is a topological space with a Borel σ-algebra. A map f on A such that $f(\alpha)\in\mathcal{H}_\alpha$ for each $\alpha\in A$ will be called a *vector field* on A. We denote the inner product and norm on $\mathcal{H}_\alpha$ by $\langle\ ,\ \rangle_\alpha$ and $\|\cdot\|_\alpha$. A measurable field of Hilbert space over A is a field of Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ together with a countable family $\{e_j\}_1^\infty$ of vector fields with the following properties:

(a) the functions $\alpha\mapsto\langle e_j(\alpha), e_k(\alpha)\rangle_\alpha$ are measurable for all j,k,
(b) the linear span of $\{e_j(\alpha)\}_1^\infty$ is dense in $\mathcal{H}_\alpha$, for each α.

Given a measurable field of Hilbert spaces $\{\mathcal{H}_\alpha\}_{\alpha\in A}$, $\{e_j\}$ on A, a vector field f on A will be called **measurable** if the function $\alpha\to\langle f(\alpha), e_j(\alpha)\rangle_\alpha$ is measurable function on A, for each j. Finally, we are ready to define direct integrals. Suppose $\{\mathcal{H}_\alpha\}_{\alpha\in A}$, $\{e_j\}_1^\infty$ is a measurable field of Hilbert spaces over A, and suppose μ is a measure on A. The direct integral of the spaces $\{\mathcal{H}_\alpha\}_{\alpha\in A}$ with respect to μ is denoted by $\int_A^\oplus \mathcal{H}_\alpha\,d\mu(\alpha)$. This is the space of measurable vector fields f on A such that

$$\|f\|^2=\int_A \big\|f(\alpha)\big\|_\alpha^2\,d\mu(\alpha)<\infty,$$

where two vector fields agreeing almost everywhere are identified. Then it easily follows that $\int^\oplus \mathcal{H}_\alpha\,d\mu(\alpha)$ is a Hilbert space with the inner product

$$\langle f,g\rangle=\int_A \big\langle f(\alpha), g(\alpha)\big\rangle_\alpha\,d\mu(\alpha).$$

In case of a constant field, that is, $\mathcal{H}_\alpha = \mathcal{H}$ for all $\alpha \in A$, $\int^{\oplus} \mathcal{H}_\alpha \, d\mu(\alpha) = L^2(A, \mu, \mathcal{H})$, all the measurable functions $f : A \to \mathcal{H}$ defined on the measure space (A, μ) with values in $\mathcal{H}$ such that

$$\|f\|^2 = \int_A \|f(\alpha)\|^2 \, d\mu(\alpha) < \infty.$$

Here $\mathcal{H}$ is considered as a Borel space with the Borel σ-algebra of the norm topology. We will be taking $\mathcal{H} = L^2(\mathbb{R})$.

2.1. Heisenberg group

The Heisenberg group $\mathbb{H}$ is a Lie group with underlying manifold $\mathbb{R}^3$. We denote points in $\mathbb{H}$ by (p, q, t) with $p, q, t \in \mathbb{R}$, and define the group operation by

$$(p_1, q_1, t_1) * (p_2, q_2, t_2) = \left(p_1 + p_2, q_1 + q_2, t_1 + t_2 + \frac{1}{2}(p_1 q_2 - q_1 p_2)\right). \tag{1}$$

It is straightforward to verify that this is a group operation, with the origin $0 = (0, 0, 0)$ as the identity element. Note that the inverse of (p, q, t) is given by $(-p, -q, -t)$. We can identify both $\mathbb{H}$ and its Lie algebra $\mathfrak{h}$ with $\mathbb{R}^3$, with group operation given by (1) and Lie bracket given by

$$\big[(p_1, q_1, t_1), (p_2, q_2, t_2)\big] := (0, 0, p_1 q_2 - q_1 p_2).$$

The Haar measure on the Heisenberg group $\mathbb{H} = \mathbb{R}^3$ is the usual Lebesgue measure. More precisely, the Lie algebra $\mathfrak{h}$ of the Heisenberg group $\mathbb{H}$ has a basis $\{X, Y, T\}$, which we may think of as left invariant differential operators on $\mathbb{H}$; where $[X, Y] = T$ and all other brackets are zero, and where the exponential function $\exp : \mathfrak{h} \to \mathbb{H}$ is the identity, i.e.,

$$\exp(pX + qY + tT) = (p, q, t).$$

We define the action of $\mathfrak{h}$ on the space $C^\infty(\mathbb{H})$ via left invariant differential operators.

For two functions f and g, each in $L^1(\mathbb{H})$ or $L^2(\mathbb{H})$, the convolution of f and g is the function $f * g$ defined by

$$f * g(\omega) = \int_{\mathbb{H}} f(\nu) g\big(\nu^{-1}\omega\big) \, d\nu.$$

We note that, for any pair $f, g \in L^2(\mathbb{H})$, one has $f * \tilde{g} \in C_b(\mathbb{H})$, where $\tilde{g}(\omega) = \overline{g(\omega^{-1})}$. For more details about convolution of functions see for example [8, Proposition 2.39].

Definition 2.1. $f \in L^2(\mathbb{H})$ is called self-adjoint convolution idempotent if $f = \tilde{f} = f * f$.

The self-adjoint convolution idempotents and their support properties are studied in detail in [10, Section 2.5].

For properties of self-adjoint convolution idempotents in L^2 we refer the reader to [10, Section 2.5].

Our definition of a continuous, or a discrete, wavelet on $\mathbb{H}$, involves the one-parameter dilation group of $\mathbb{H}$, i.e., $H = (0, \infty)$, where any $a > 0$ defines an automorphism of $\mathbb{H}$, by

$$a(p, q, t) = \big(ap, aq, a^2 t\big) \quad \forall (p, q, t) \in \mathbb{H}. \tag{2}$$

(In the construction of a discrete wavelet, one takes a discrete version of the one-parameter group. Usually a dyadic discretization is considered.) Adapting the notation of dilation and translation operators on $L^2(\mathbb{R})$, for each $a > 0$, we define D_a to be the unitary operator on $L^2(\mathbb{H})$ given by

$$D_a f(p, q, t) = a^2 f(a(p, q, t)) = a^2 f\big(ap, aq, a^2 t\big) \quad \forall f \in L^2(\mathbb{H}),$$

and for any $\upsilon \in \mathbb{H}$, the left translation operator, L_ω is given by

$$L_\omega f(\upsilon) = f\big(\omega^{-1}\upsilon\big) \quad \forall \upsilon \in \mathbb{H}.$$

Using the dilation and translation operators, we can now define the *quasiregular representation* π of the semidirect product $G := \mathbb{H} \rtimes (0, \infty)$; it acts on $L^2(\mathbb{H})$ by

$$\big(\pi(\omega, a) f\big)(\upsilon) := L_\omega D_a f(\upsilon) = a^{-2} f\big(a^{-1}\big(\omega^{-1}\upsilon\big)\big),$$

for any $f \in L^2(\mathbb{H})$ and $(\omega, a) \in G$ and for all $\upsilon \in \mathbb{H}$.

2.2. Frames

We conclude this section with the definition of *frames* and some related notions, which will be used in the context of FMRA in this work. The concept of frames is a generalization of orthonormal bases, defined as follows:

Definition 2.2. A countable subset $\{e_n\}_{n\in I}$ of a Hilbert space $\mathcal{H}$ is said to be a *frame* of $\mathcal{H}$ if there exist two numbers $0 < a \leqslant b$ so that, for any $f \in \mathcal{H}$,

$$a\|f\|^2 \leqslant \sum_{n\in I}\left|\langle f, e_n\rangle\right|^2 \leqslant b\|f\|^2.$$

The positive numbers a and b are called *frame bounds*. Note that the frame bounds are not unique. The *optimal lower frame bound* is the supremum over all lower frame bounds, and the optimal upper frame bound is the infimum over all upper frame bounds. The optimal frame bounds are actually frame bounds. The frame is called a *tight frame* when one can take $a = b$ and a *normalized tight frame* when one can take $a = b = 1$.

Frames were introduced for the first time in [6]. See also [3] and [13] for more about frame theory.

3. Fourier analysis on the Heisenberg group

This section contains a brief review of Fourier analysis on the Heisenberg group $\mathbb{H}$. In order to study the Fourier analysis on this group, one has to study the irreducible representations of this group. The Heisenberg group is the best known example of a non-commutative nilpotent Lie group. The representation theory of $\mathbb{H}$ is simple and well understood. Using the fundamental theorem, due to *Stone and von Neumann*, we can give a complete classification of all the irreducible unitary representation of $\mathbb{H}$.

It is known that for the Heisenberg group there are two families of irreducible unitary representations, at least up to unitary equivalence. One family, giving all infinite-dimensional irreducible unitary representations, is parametrized by nonzero real numbers λ; the other family, giving all one-dimensional representations, is parametrized by $(b, \beta) \in \mathbb{R}\times\mathbb{R}$. We will see below that the one-dimensional representations give no contribution to the Plancherel formula and Fourier inversion transform, i.e., they form a set of representations that has zero Plancherel measure. Hence we will focus on the Schrödinger representation, defined next. For more about the representations of the Heisenberg group and the Plancherel theorem, we refer the interested reader to [12].

The infinite-dimensional irreducible unitary representations of the Heisenberg group may be realized on $L^2(\mathbb{R})$; there they are the called the *Schrödinger* representations. These are defined as follows. For each $\lambda \in \mathbb{R}^*(= \mathbb{R}\setminus\{0\})$ and for any $(p, q, t) \in \mathbb{H}$, the operator $\rho_\lambda(p, q, t)$ acts on $L^2(\mathbb{R})$ by

$$\rho_\lambda(p, q, t)\phi(x) = e^{i\lambda t}e^{i\lambda(px+\frac{1}{2}(pq))}\phi(x+q)$$

where $\phi \in L^2(\mathbb{R})$. It is easy to see that $\rho_\lambda(p, q, t)$ is a unitary operator satisfying the homomorphism property:

$$\rho_\lambda\big((p_1, q_1, t_1)(p_2, q_2, t_2)\big) = \rho_\lambda(p_1, q_1, t_1)\rho_\lambda(p_2, q_2, t_2).$$

Thus each ρ_λ is a strongly continuous unitary representation of $\mathbb{H}$. A theorem of Stone and von Neumann [8] says that up to unitary equivalence these are all the infinite-dimensional irreducible unitary representations of $\mathbb{H}$.

Recall that the dilation operator given by $a > 0$ is defined on $\mathbb{H}$ as follows:

$$a : (p, q, t) \to a(p, q, t) = \big(ap, aq, a^2t\big) \quad \forall(p, q, t) \in \mathbb{H}.$$

One then easily calculates that

$$\rho_\lambda\big(a^{-1}(p, q, t)\big) = D_{a^{-1}}\rho_{a^{-2}\lambda}(p, q, t)D_a \quad \forall(p, q, t) \in \mathbb{H}, \tag{3}$$

where $D_{a^{-1}} = D_a^*$.

3.1. Fourier transform on the Heisenberg group

Here we present a brief introduction to the group Fourier transform for functions on $\mathbb{H}$, (see [12]), and introduce the inversion and Plancherel theorems for the Fourier transform.

If $f \in L^1(\mathbb{H})$, we define the Fourier transform of f to be the measurable field of operators over $\widehat{\mathbb{H}}$ given by the weak operator integrals, as follows:

$$\widehat{f}(\lambda) = \int_{\mathbb{H}} f(\omega)\rho_\lambda(\omega)\,d\omega. \tag{4}$$

For simplicity, we write here $\widehat{f}(\lambda)$ instead of $\widehat{f}(\rho_\lambda)$. Note that the Fourier transform $\widehat{f}(\lambda)$ is an operator-valued function, which for any $\phi, \psi \in L^2(\mathbb{R})$ satisfies

$$\langle \widehat{f}(\lambda)\phi, \psi\rangle = \int_{\mathbb{H}} f(p,q,t)\langle \rho_\lambda(p,q,t)\phi, \psi\rangle\, dp\, dq\, dt,$$

by definition of the weak operator integral. The operator $\widehat{f}(\lambda)$ is bounded on $L^2(\mathbb{R})$ with the operator norm satisfying $\|\widehat{f}(\lambda)\| \leqslant \|f\|_1$. If $f \in L^1 \cap L^2(\mathbb{H})$, $\widehat{f}(\lambda)$ is actually a Hilbert–Schmidt operator and from the Plancherel theorem, the Fourier transform can be extended to a unitary map from $L^2(\mathbb{H})$ onto $L^2(\mathbb{R}^*, d\mu(\lambda), L^2(\mathbb{R}) \otimes L^2(\mathbb{R}))$, the space of functions on $\mathbb{R}^*$ taking values in $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ which are square integrable with respect to the Plancherel measure $d\mu(\lambda) = (2\pi)^{-2}|\lambda| d\lambda$. (We say a function g defined on $\mathbb{R}^*$ is square integrable, when it is a measurable vector field on $\mathbb{R}^*$ and $\int_{\mathbb{R}^*} \|g(\lambda)\|^2_{H.S}\, d\mu(\lambda) < \infty$.) The proof of the Plancherel theorem for the Heisenberg group may be found in [12]. For more general groups, see for example [8].

A simple computation shows that the basic properties of the Fourier transform remain valid for $f, g \in (L^1 \cap L^2)(\mathbb{H})$: More precisely,

(i) $\widehat{(af + bg)}(\lambda) = a\widehat{f}(\lambda) + b\widehat{g}(\lambda)$,
(ii) $\widehat{(f * g)}(\lambda) = \widehat{f}(\lambda)\widehat{g}(\lambda)$,
(iii) $\widehat{(L_\omega f)}(\lambda) = \rho_\lambda(\omega)\widehat{f}(\lambda)$, for $\omega \in \mathbb{H}$,
(iv) $\widehat{(\tilde{f})}(\lambda) = \widehat{f}(\lambda)^*$. (The superscript $*$ denotes adjoint.)

We conclude this section with a computation of the Fourier transform of $f(a\cdot)$ for any $f \in L^2(\mathbb{H})$.

Lemma 3.1. *For any $f \in L^2(\mathbb{H})$ is*

$$\widehat{f(a\cdot)}(\lambda) = a^{-4} D_{a^{-1}} \widehat{f(a^{-2}\lambda)} D_a.$$

Proof. From the definition of the Fourier transform (4), for $\lambda \neq 0$ we have

$$\begin{aligned}
\widehat{f(a\cdot)}(\lambda) &= \int_{\lambda} f\big(a(p,q,t)\big)\rho_\lambda(p,q,t)\, dp\, dq\, dt \\
&= \int_{\lambda} f\big(ap, aq, a^2 t\big)\rho_\lambda(p,q,t)\, dp\, dq\, dt \\
&= a^{-4}\int_{\lambda} f(p,q,t)\rho_\lambda\big(a^{-1}p, a^{-1}q, a^{-2}t\big)\, dp\, dq\, dt \\
&= a^{-4}\int_{\lambda} f(p,q,t)\big(\rho_\lambda\big(a^{-1}(p,q,t)\big)\big)\, dp\, dq\, dt,
\end{aligned}$$

Now inserting (3), we derive the following relation:

$$\begin{aligned}
\widehat{f(a\cdot)}(\lambda) &= a^{-4} D_{a^{-1}}\left(\int_{\lambda} f(p,q,t)\rho_{a^{-2}\lambda}(p,q,t)\, dp\, dq\, dt\right) D_a \\
&= a^{-4} D_{a^{-1}} \widehat{f(a^{-2}\lambda)} D_a,
\end{aligned} \tag{5}$$

as desired. □

3.2. Wavelet frames

For our purpose, in this work we will consider the wavelet frames which are produced from one function, as the generator of the wavelet frame, using a countable family of dilation and left translation operators. The generator function is usually called a "*discrete wavelet*."

Below we will give a concrete example of wavelet frames with respect to a very special lattice as the discrete translation set. Suppose Γ is a lattice in $\mathbb{H}$ and $a > 0$ refers to the automorphism $a : \omega \to a.\omega$ of $\mathbb{H}$. Suppose also $\mathcal{H}$ is a subspace of $L^2(\mathbb{H})$ and ψ is any function in $\mathcal{H}$. Then the discrete system $\{L_{a^{-j}\gamma} D_{a^{-j}}\psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ (assumed to be contained in $\mathcal{H}$) is called the *discrete wavelet system* generated by ψ, where $\{D_{a^{-j}}\}_{j\in\mathbb{Z}}$ is the class of discrete unitary dilation operators with respect to the positive number a obtained by $a^j : \omega \to a^j\omega$, and $\{L_\gamma\}_{\gamma\in\Gamma}$ is the class of left translation operators with regard to the lattice Γ. The discrete wavelet system $\{L_{a^{-j}\gamma} D_{a^{-j}} \psi\}_{j\in\mathbb{Z},\gamma\in\Gamma}$ is called a *(tight, normalized tight) wavelet frame* of $\mathcal{H}$ if it forms a (tight, normalized tight) frame for $\mathcal{H}$. More precisely, it is a frame if there exist positive numbers $0 < A \leqslant B < \infty$ such that for any $f \in \mathcal{H}$ we have

$$A\|f\|_2^2 \leqslant \sum_{j\in\mathbb{Z},\gamma\in\Gamma} \left|\langle f, \psi_{j,\gamma}\rangle\right|^2 \leqslant B\|f\|_2^2,$$

where $\psi_{j,\gamma} := L_{a^{-j}\gamma} D_{a^{-j}} \psi$.

Now, we are ready to present our main results concerning the multiresolution analysis.

4. Frame multiresolution analysis for $L^2(\mathbb{H})$

Analogous to the situation on $\mathbb{R}$, discrete wavelets in $L^2(\mathbb{H})$ are functions ψ with the property that their appropriate translates and dilates defined with respect to the Lie structure of the Heisenberg group can be used to approximate any L^2-function on $\mathbb{H}$. But here the special concept of multiresolution analysis needs to be appropriately adapted.

In this work we shall adapt the definition of MRA for $L^2(\mathbb{R})$ to one for $L^2(\mathbb{H})$, replacing the concept of orthonormal basis by frames and calling it *FMRA*. Since the triviality of the intersection is a direct consequence of the other conditions of the definition of an MRA, we prove this property immediately after we give the definition of an MRA.

We begin by properly interpreting the concept of MRA of $L^2(\mathbb{R})$. The shift-invariance of V_0, the central subspace in the definition of a MRA, can be interpreted as an invariance property with respect to the action of the discrete lattice subgroup $\mathbb{Z}$ of $\mathbb{R}$. The scaling operator a can be viewed as the action of some group automorphism of $\mathbb{R}$, with the property $a\mathbb{Z} \subset \mathbb{Z}$.

With this in mind, it is not difficult to conjecture the correct generalization of MRA to the Heisenberg group:

- First, a discrete subgroup Γ of $\mathbb{H}$ will play the same role in $\mathbb{H}$ as $\mathbb{Z}$ in $\mathbb{R}$. To say Γ is discrete means that the topology on Γ induced from $\mathbb{H}$ is the discrete topology.
- Since our setting is a non-abelian group, there are two kinds of translations: left translation $L := L_{\mathbb{H}}$ and right translation $R := R_{\mathbb{H}}$. We choose left translation here.

As our starting point, we need the following definition:

Definition 4.1. Suppose Ω is a subset of $\mathbb{H}$ and $\mathcal{H}$ is a subspace of $L^2(\mathbb{H})$. We say $\mathcal{H}$ is left shift-invariant under Ω, if for any $\omega \in \Omega$ we have $L_\omega\mathcal{H} \subseteq \mathcal{H}$.

After this preparation, we can give a definition of FMRA for $L^2(\mathbb{H})$ related to an automorphism of $\mathbb{H}$ given by $a > 0$ (see (2)) and a lattice Γ in $\mathbb{H}$.

Definition 4.2. We say that a sequence of closed subspaces $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{H})$ forms a *FMRA* of $L^2(\mathbb{H})$, associated to an automorphism $a \in \mathrm{Aut}(\mathbb{H})$ and a lattice Γ in $\mathbb{H}$, if the following conditions are satisfied:

(1) $V_j \subseteq V_{j+1} \quad \forall j \in \mathbb{Z}$,
(2) $\overline{\bigcup V_j} = L^2(\mathbb{H})$,
(3) $\bigcap V_j = \{0\}$,
(4) $f \in V_j \Leftrightarrow f(a\cdot) \in V_{j+1}$,
(5) V_0 is left shift-invariant under Γ, and consequently V_j is left shift-invariant under $a^{-j}\Gamma$, and,
(6) there exists a function $\phi \in V_0$, called the *scaling function*, or generator of the FMRA, such that the set $L_\Gamma(\phi)$ constitutes a normalized tight frame for V_0.

Remark 4.3.

(a) Observe that property (4) in Definition 4.2 implies that

$$f \in V_j \Leftrightarrow f\left(a^{-j}\cdot\right) \in V_0. \tag{6}$$

It follows that an MRA is essentially completely determined by the closed subspace V_0. But from property (6), V_0 is the closure of the linear span of the Γ-translations of the scaling function ϕ. Thus the starting point of the construction of MRA is the existence of the scaling function ϕ. Therefore, it is especially important to give some conditions under which an initial function ϕ generates an MRA.

(b) Eq. (6) implies that if $f \in V_j$, then $f(\gamma^{-1}(a^j\omega)) \in V_0$ for all $\gamma \in \Gamma$. Finally property (6) in Definition 4.2 and Eq. (6) imply that the system $\{L_{a^{-j}\gamma} D_{a^{-j}}\phi\}_{\gamma\in\Gamma}$ is a normalized tight frame V_j for all $j \in \mathbb{Z}$, where $\forall\gamma \in \Gamma$, $\forall\omega \in \mathbb{H}$, $L_{a^{-j}\gamma} D_{a^{-j}}\phi(\omega) = a^{j/2}\phi(\gamma^{-1}(a^j\omega))$.

(c) Here, for the scaling function we do not impose any regularity or decay condition on ϕ. In our case to make the argument simple and general, we require only that $\phi \in L^2(\mathbb{H})$.
(d) In analogy with $L^2(\mathbb{R})$, we say V_0 is refinable if $D_{a^{-1}}(V_0) \subseteq V_0$. Thus condition (1) in Definition 4.2 is equivalent to saying that V_0 is refinable. Thus, the basic question concerning a FMRA is whether the scaling function exists. We shall see in Theorem 4.12 that such scaling functions do exist. We will enter into details later for a very special case.
(e) To have a sequence of nested closed subspaces, we must find a refinable function like ϕ in V_0. It is already known by Boor, DeVore and Ron in [2] for the real case that the refinability of ϕ is not enough to generate an MRA. Hence we need other requirements. We will consider this in detail later.
(f) The basic property of multiresolution analysis is that whenever a collection of closed subspaces satisfies properties (1)–(6) in Definition 4.2, then there exists a basis $\{L_{a^{-j}\gamma} D_{a^{-j}}\psi;\ j \in \mathbb{Z},\ \gamma \in \Gamma\}$ of $L^2(\mathbb{H})$, such that for all $f \in L^2(\mathbb{H})$

$$P_{j+1} f = P_j f + \sum_{\gamma \in \Gamma} \langle f, L_{a^{-j}\gamma} D_{a^{-j}}\psi \rangle L_{a^{-j}\gamma} D_{a^{-j}}\psi,$$

where P_j is the orthogonal projection of $L^2(\mathbb{H})$ onto V_j.

The approach via the solution of the scaling equation, with methods of Lawton [14], leads to difficult analytical problems. Therefore we follow a new approach, which is based on the point of view of *Shannon multiresolution analysis*. This will allow us to derive the existence of a Shannon wavelet in $L^2(\mathbb{H})$.

In the setting of the real line, the canonical construction of wavelet bases starts with a multiresolution analysis $\{V_j\}_j$. In $L^2(\mathbb{R})$ one proves the existence of a wavelet $\psi \in W_0$, such that $\{L_k\psi,\ k \in \mathbb{Z}\}$ is an orthonormal basis for W_0. (L_k is the translation operator.)

Consequently the set $\{L_{2^{-j}k} D_{2^j}\psi\}_{k\in\mathbb{Z}}$, the set of dyadic dilations and translations of ψ, constitutes an orthonormal basis for W_j. (Dyadic refers here to the dilation D_2.) By the orthogonal decomposition $L^2(\mathbb{R}) = \bigoplus_{j\in\mathbb{Z}} W_j$, the wavelet system $\{L_{2^{-j}k} D_{2^j}\psi\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{R})$.

In our setting, we shall construct on $\mathbb{H}$ a Shannon-MRA as an example of a FMRA. In contrast of the case of $\mathbb{R}$, the construction of the scaling function is not our starting point for obtaining a FMRA, but rather, first we intend to construct a special function which implies the existence of the scaling function in some closed subspace of $L^2(\mathbb{H})$. Furthermore, for the construction, we shall consider the automorphism $a = 2$ of $\mathbb{H}$ which is given by:

$$a(p,q,t) = \left(2p, 2q, 2^2 t\right) \quad \forall (p,q,t) \in \mathbb{H}.$$

4.1. An example: Shannon MRA for $L^2(\mathbb{H})$

As remarked before, we shall construct a generator function in some closed and shift-invariant subspace of $L^2(\mathbb{H})$, such that its translations and dilations yields a normalized tight frame of $L^2(\mathbb{H})$. For this reason, first we choose the dilation operator $D_a = D_2$ and try to associate a space V_0 which has similar properties as the Paley–Wiener space on $\mathbb{R}$. With this aim in mind, we start with the definition of a band-limited function on $\mathbb{H}$:

Definition 4.4. Suppose $\mathcal{I}$ is some bounded subset of $\mathbb{R}^*$ and S is a function in $L^2(\mathbb{H})$. We say S is $\mathcal{I}$-band-limited if $\widehat{S}(\lambda) = 0$ for all $\lambda \notin \mathcal{I}$.

In the next theorem we shall construct sinc-type function on the Heisenberg group which is our starting point for obtaining the scaling function:

Theorem 4.5. *Let d be any positive integer. There exists a self-adjoint convolution idempotent function S in $L^2(\mathbb{H})$ which is $\mathcal{I}$-band-limited for $\mathcal{I} = [-\frac{\pi}{2d}, \frac{\pi}{2d}] \setminus \{0\}$. Define $S_j = 2^{4j} S(2^j\cdot)$ for $j \in \mathbb{Z}$. Then S_j is $\mathcal{I}_j$-band-limited for $\mathcal{I}_j = [-\frac{2^{2j}\pi}{2d}, \frac{2^{2j}\pi}{2d}] \setminus \{0\}$ and the following consequences hold*:

(a) $S * S_j = S\ \forall j > 0$ *and* $S_j * S = S_j\ \forall j < 0$,
(b) $f * S_j \to 0$ *in* L^2*-norm as* $j \to -\infty\ \forall f \in L^2(\mathbb{H})$,
(c) $f * S_j \to f$ *in* L^2*-norm as* $j \to \infty\ \forall f \in L^2(\mathbb{H})$, *and,*
(d) $S_j = \widetilde{S}_j = S_j * S_j$.

Proof. Take $\mathcal{I}_0 := \mathcal{I}$. We intend to show that there exists a function S which is $\mathcal{I}_0$-band-limited and satisfies the assertion of our theorem. We start from the Fourier transform side, i.e., by constructing Hilbert–Schmidt operators $\widehat{S}(\lambda)$ associated to $\lambda \in \mathbb{R}^*$. For this purpose we choose an orthonormal basis $\{e_i\}_{i\in\mathbb{N}_0}$ in $L^2(\mathbb{R})$. For any $\lambda \neq 0$ define $e_i^\lambda = D_{|\lambda|^{-1/2}} e_i$. Observe that for any λ, $\{e_i^\lambda\}_i$ is an ONB of $L^2(\mathbb{R})$ since the dilation operators $D_{|\lambda|^{-1/2}}$ are unitary. Therefore $\{\{e_i^\lambda\}_i\}_\lambda$ is a measurable family of orthonormal bases in $L^2(\mathbb{R})$.

Let $\lambda \neq 0$ be such that $\lambda \in \mathcal{I}_0$. For $I_0 = \bigcup_k I_0^k$ where $I_0^k = [-\frac{\pi}{2^{2k+1}d}, -\frac{\pi}{2^{2k+3}d}) \cup (\frac{\pi}{2^{2k+3}d}, \frac{\pi}{2^{2k+1}d}]$, define the operator $\widehat{S}(\lambda)$ as follows:

$$\widehat{S}(\lambda) = \begin{cases} \sum_{i=0}^{2^{2k}} \left(e_i^{\frac{\lambda}{2\pi}} \otimes e_i^{\frac{\lambda}{2\pi}}\right) & \text{if } \lambda \in I_0^k, \text{ for some } k \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore for any $\lambda \in I_0^k$, the operator $\widehat{S}(\lambda)$ is a projection operator on the first $2^{2k}+1$ elements of the orthonormal basis $\{e_i^{\frac{\lambda}{2\pi}}\}_{i\in\mathbb{N}_0}$, where $e_i^{\frac{\lambda}{2\pi}} = D_{|\frac{\lambda}{2\pi}|^{-1/2}} e_i$. The definition of $\widehat{S}$ entails the following consequences:

(i) For $k \geqslant 0$ and $\lambda \in I_0^k$, $\|\widehat{S}(\lambda)\|_{H.S}^2 = 2^{2k}+1$,

(ii) $$\int_{|\lambda| \leqslant \frac{\pi}{2d}} \left\|\widehat{S}(\lambda)\right\|_{H.S}^2 \, d\mu(\lambda) = \sum_{k=0} \int_{I_0^k} \left(2^{2k}+1\right) d\mu(\lambda) < \infty,$$

where $d\mu(\lambda) = (2\pi)^{-2}|\lambda| d\lambda$, and

(iii) $\widehat{S}(\lambda) = \widehat{S}(\lambda)^* = \widehat{S}(\lambda) \circ \widehat{S}(\lambda)$ $\forall \lambda \neq 0$.

Observe that (ii) implies that the vector field $\{\widehat{S}(\lambda)\}_\lambda$ on $\mathbb{R}^*$ is contained in $\int_{\mathbb{R}^*}^{\oplus} L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \, d\mu(\lambda)$ and hence, by the surjectivity part of the Plancherel theorem, $\widehat{S}$ has a preimage S in $L^2(\mathbb{R})$ with Fourier transform $\widehat{S}$, given as above. Property (iii) shows that S is a self-adjoint convolution, idempotent by the convolution theorem.

Suppose $j \in \mathbb{Z}$ and $S_j := 2^{4j} S(2^j.)$. Using the equivalence of the representations ρ_λ and $\rho_{2^{-2j}\lambda}$, the relation (5) and the fact that $D_{2^j}^* = D_{2^{-j}}$ we obtain

$$\widehat{S_j}(\lambda) = D_{2^{-j}} \widehat{S}\left(2^{-2j}\lambda\right) D_{2^j}. \tag{7}$$

(7) implies $\widehat{S_j}(\lambda) = 0$ for any $|\lambda| > \frac{2^{2j}\pi}{2d}$, and hence the function S_j is $\mathcal{I}_j$-band-limited, where $\mathcal{I}_j = [-\frac{2^{2j}\pi}{2d}, 0) \cup (0, \frac{2^{2j}\pi}{2d}]$. As a consequence of (iii), the relation (7) shows that S_j is a self-adjoint and convolution idempotent, which proves (d).

To prove (a), suppose $j > 0$ and $\lambda \in \mathcal{I}_j$. Then $2^{-2j}\lambda \in \mathcal{I}_0$. Hence there exists a non-negative integer k_j such that

$$\frac{\pi}{2^{(2k_j+3)}d} < \left|2^{-2j}\lambda\right| \leqslant \frac{\pi}{2^{(k_j+1)}d},$$

or equivalently $\lambda \in I_0^{k_j}$.

For the case $k_j < j$, observe that $\widehat{S}(\lambda) = 0$. For the case $k_j \geqslant j$, from the definition of $\widehat{S}$ we have the following:

$$\widehat{S}(\lambda) = \sum_{i=0}^{2^{2(k_j-j)}} e_i^{\frac{\lambda}{2\pi}} \otimes e_i^{\frac{\lambda}{2\pi}} \quad \text{and}$$

$$\widehat{S}(2^{-2j}\lambda) = \sum_{i=0}^{2^{2k_j}} e_i^{\frac{\lambda}{2^{2j+1}\pi}} \otimes e_i^{\frac{\lambda}{2^{2j+1}\pi}}. \tag{8}$$

Recall that, from the definition of the family of orthonormal bases $\{e_i^\lambda\}_i$, $e_i^{\frac{\lambda}{2^{2j+1}\pi}}$ can be read as below:

$$e_i^{\frac{\lambda}{2^{2j+1}\pi}} = D_{|\frac{2^{-2j}\lambda}{2\pi}|^{-1/2}} e_i = D_{2^j}\left(D_{|\frac{\lambda}{2\pi}|^{-1/2}} e_i\right) = D_{2^j} e_i^{\frac{\lambda}{2\pi}}. \tag{9}$$

Plugging (9) into (8), we get

$$\widehat{S}\left(2^{-2j}\lambda\right) = \sum_{i=0}^{2^{2k_j}} \left(D_{2^j} e_i^{\frac{\lambda}{2\pi}}\right) \otimes \left(D_{2^j} e_i^{\frac{\lambda}{2\pi}}\right),$$

and hence

$$\widehat{S_j}(\lambda) = D_{2^{-j}} \widehat{S}\left(2^{-2j}\lambda\right) D_{2^j} = \sum_{i=0}^{2^{2k_j}} e_i^{\frac{\lambda}{2\pi}} \otimes e_i^{\frac{\lambda}{2\pi}}. \tag{10}$$

Observe that for any $\lambda \in \mathcal{I}_j$, the operator $\widehat{S}(\lambda)$ is a projection on the first $2^{2(k_j-j)}+1$ elements of the orthonormal basis $\{e_i^{\frac{\lambda}{2\pi}}\}$ for some suitable $k_j \geqslant j$, whereas $\widehat{S_j}(\lambda)$ is a projection on the first $2^{2k_j}+1$ elements of the same orthonormal basis. Hence we get

$$\widehat{S}(\lambda) \circ \widehat{S_j}(\lambda) = \widehat{S_j}(\lambda) \circ \widehat{S}(\lambda) = \sum_{i=0}^{2^{2(k_j-j)}} e_i^{\frac{\lambda}{2\pi}} \otimes e_i^{\frac{\lambda}{2\pi}} = \widehat{S}(\lambda), \tag{11}$$

which is a projection on the first $2^{2(k_j-j)}+1$ elements of the orthonormal basis $\{e_i^{\frac{\lambda}{2\pi}}\}$. For fixed $j>0$ since the relation (11) holds for any $\lambda\in\mathcal{I}_j$, so by applying the convolution and the Plancherel theorem respectively we obtain $S*S_j=S$, which proves the first hypothesis of (a).

Likewise for $j<0$, suppose $\lambda\in\mathcal{I}_j$. Then for some $k_j\in\mathbb{N}_0$, $\frac{\pi}{2^{2k_j+3}d}<|2^{-2j}\lambda|\leqslant\frac{\pi}{2^{2k_j+1}d}$. Analogous to the previous case, the operator $\widehat{S}(\lambda)$ is a projection on the first $2^{2(k_j-j)}+1$ elements of the orthonormal basis $\{e_i^{\frac{\lambda}{2\pi}}\}$ and $\widehat{S}_j(\lambda)$ is a projection on the first $2^{2k_j}+1$ elements of the same orthonormal basis. Thus

$$\widehat{S}(\lambda)\circ\widehat{S}_j(\lambda)=\widehat{S}_j(\lambda)\circ\widehat{S}(\lambda)=\sum_{i=0}^{2^{2k_j}}e_i^{\frac{\lambda}{2\pi}}\otimes e_i^{\frac{\lambda}{2\pi}}=\widehat{S}_j(\lambda). \tag{12}$$

Once again, applying the convolution and Plancherel theorems in the relation (12) yields $S*S_j=S_j$, and hence (a) is completely proved.

To prove (b), suppose $j\in\mathbb{Z}$ and $f\in L^2(\mathbb{H})$. Then $f*S_j\in L^2(\mathbb{H})$ by the structure and properties of the function S. Before we start to give a proof for this part, observe that, for any $\lambda\neq0$, since each $\widehat{S}(\lambda)$ is a projection, the operator $\widehat{S}(\lambda)$ is bounded and has operator norm less than or equal to 1. Hence for any $j\in\mathbb{Z}$ and $\lambda\neq0$ we have

$$\big\|\widehat{S}_j(\lambda)\big\|_\infty=\big\|D_{2^{-j}}\widehat{S}\big(2^{-2j}\lambda\big)D_{2^j}\big\|_\infty\leqslant1.$$

Using the inequality and applying the Plancherel and convolution theorems respectively we get the following:

$$\begin{aligned}\|f*S_j\|_2^2=\big\|\widehat{(f*S_j)}\big\|_{H.S}^2&=\int_{\mathbb{R}^*}\big\|\widehat{(f*S_j)}(\lambda)\big\|_2^2\,d\mu(\lambda) &(13)\\&=\int_{0<|4^{-j}\lambda|\leqslant\frac{\pi}{2d}}\big\|\widehat{f}(\lambda)\circ\widehat{S}_j(\lambda)\big\|_{H.S}^2\,d\mu(\lambda)\\&\leqslant\int_{0<|4^{-j}\lambda|\leqslant\frac{\pi}{2d}}\big\|\widehat{f}(\lambda)\big\|_{H.S}^2\big\|\widehat{S}_j(\lambda)\big\|_\infty^2\,d\mu(\lambda)\\&\leqslant\int_{0<|4^{-j}\lambda|\leqslant\frac{\pi}{2d}}\big\|\widehat{f}(\lambda)\big\|_{H.S}^2\,d\mu(\lambda)\\&=\int_{\mathbb{R}^*}\big\|\widehat{f}(\lambda)\big\|_{H.S}^2\chi_{\mathcal{I}_j}(\lambda)\,d\mu(\lambda), &(14)\end{aligned}$$

where χ denotes the characteristic function and $d\mu(\lambda)=(2\pi)^{-2}|\lambda|d\lambda$. If we take the limit of the right-hand side in (14), since $\int_\lambda\|\widehat{f}(\lambda)\|_{H.S}^2d\mu(\lambda)<\infty$, then by the dominated convergence theorem may pass the limit into the integral and hence

$$\lim_{j\to-\infty}\int_{\mathbb{R}^*}\big\|\widehat{f}(\lambda)\big\|_{H.S}^2\chi_{\mathcal{I}_j}\,d\mu(\lambda)=0.$$

The latter implies that the limit of the left-hand side in the relation (13) is also zero as $j\to-\infty$, i.e., $\lim_{j\to-\infty}\|f*S_j\|_2=0$, which proves (b).

In order to prove (c), suppose f is in $L^2(\mathbb{H})$. Recall that $\{e_i^{\frac{\lambda}{2\pi}}\}_{i=0}^\infty$ constitutes an orthonormal basis for $L^2(\mathbb{R})$ for any fixed λ. Therefore the identity operator I on $L^2(\mathbb{R})$ can be read as $I=\sum_{i=0}^\infty e_i^{\frac{\lambda}{2\pi}}\otimes e_i^{\frac{\lambda}{2\pi}}$, and hence the operator $\widehat{f}(\lambda)$ can be represented as

$$\widehat{f}(\lambda)=\sum_{i=0}^\infty\big(\widehat{f}(\lambda)e_i^{\frac{\lambda}{2\pi}}\big)\otimes e_i^{\frac{\lambda}{2\pi}}. \tag{15}$$

Therefore for any $j\in\mathbb{Z}$, according to the representation of $\widehat{f}(\lambda)$ in (15) and the representation of the operator $D_{2^{-j}}\widehat{S}(2^{-2j}\lambda)D_{2^j}$ in (10), for some $k_j\geqslant j$ we obtain the following:

$$\begin{aligned}\big\|\big[\widehat{f}(\lambda)\circ D_{2^{-j}}\widehat{S}\big(2^{-2j}\lambda\big)D_{2^j}\big]-\widehat{f}(\lambda)\big\|_{H.S}^2&=\Bigg\|\sum_{i=2^{2k_j}+1}^\infty\big(\widehat{f}(\lambda)e_i^{\frac{\lambda}{2\pi}}\big)\otimes e_i^{\frac{\lambda}{2\pi}}\Bigg\|_{H.S}^2\\&=\sum_{i=2^{2k_j}+1}^\infty\big\|\widehat{f}(\lambda)e_i^{\frac{\lambda}{2\pi}}\big\|_2^2. &(16)\end{aligned}$$

Letting $j \to \infty$ (hence $k_j \to \infty$), the right-hand side of (16) goes to zero. From the other side using the Plancherel theorem we have

$$\|f * S_j - f\|_2^2 = \int_{\mathbb{R}^*} \big\| \big[\widehat{f}(\lambda) \circ D_{2^{-j}} \widehat{S}\big(4^{-j}\lambda\big) D_{2^j} \big] - \widehat{f}(\lambda) \big\|_{H.S}^2 \, d\mu(\lambda)$$
$$= \int_{\mathbb{R}^*} \sum_{i=2^{2k_j}+1}^{\infty} \big\| \widehat{f}(\lambda) e_i^{\frac{\lambda}{2\pi}} \big\|_2^2 \, d\mu(\lambda). \tag{17}$$

As in the proof of (b), using the dominated convergence theorem in the relation (17) one gets:

$$\lim_{j\to\infty} \|f * S_j - f\|_2^2 = \int_{\mathbb{R}^*} \lim_{j\to\infty} \sum_{i=2^{2(k_j+j)}+1}^{\infty} \big\| \widehat{f}(\lambda) e_i^{\frac{\lambda}{2\pi}} \big\|_2^2 \, d\mu(\lambda) = 0,$$

as desired, which completes the proof of the theorem. □

Remark 4.6. In the previous theorem, one could for instance take the orthonormal basis of $\{\phi_n\}_{n\in\mathbb{N}_0}$ in $L^2(\mathbb{R})$, where ϕ_n are Hermite functions, and for any $\lambda \neq 0$, ϕ_n^λ are given by $\phi_n^\lambda(x) = D_{|\lambda|^{-1/2}}\phi_n = |\lambda|^{\frac{1}{4}}\phi_n(\sqrt{|\lambda|}x)$ for all $x \in \mathbb{R}$.

Now that we have constructed a function S as in above theorem, with the listed properties, the next step in the construction of an MRA via the function S will be the definition of a closed left invariant subspace of $L^2(\mathbb{H})$, V_0. Define $V_0 = L^2(\mathbb{H}) * S$, as the central subspace of an MRA. It is obvious that V_0 is closed and possesses the following additional properties:

1. V_0 is contained in the set of all bounded and continuous functions in $L^2(\mathbb{H})$. Hence V_0 is a proper subspace of $L^2(\mathbb{H})$. The boundedness of elements in V_0 is easy to see by the definition of convolution operator and Cauchy–Schwartz inequality:
$$\big|g * S(x)\big| \leqslant \|f\|_2 \|S\|_2 \quad \forall x \in \mathbb{H}\ g \in L^2(\mathbb{R}).$$
2. Since S is convolution idempotent then S behaves as an identity element in V_0 with respect to group convolution. More precisely, $f * S = f$ for any $f \in V_0$.
3. Suppose Γ is any lattice in $\mathbb{H}$. Then $L_\gamma(g * S) = L_\gamma g * S$ which shows V_0 is left shift-invariant under Γ.
4. An easy computation shows that $D_{2^j}(g * S) = D_{2^j} g * D_{2^j} S$ for any $g \in L^2(\mathbb{H})$ and $j \in \mathbb{Z}$.

Remark 4.7. Observe that not every space $L^2(\mathbb{H}) * S$ with $S = \tilde{S} = S * S$ gives rise to a normalized tight frame of the form $\{L_\gamma \phi\}_\gamma$ for some $\phi \in L^2(\mathbb{H}) * S$. As shall be seen later, this depends heavily on the *multiplicity function* associated to S, see Definition 4.10 and Theorem 4.11.

Recall that $L_{2^{-j}\gamma} D_{2^{-j}} S(\omega) = 2^{j/2} S(\gamma^{-1}(2^j\omega))$ $\forall j \in \mathbb{Z}$, $\gamma \in \Gamma$, $\omega \in \mathbb{H}$. Next define $V_1 = L^2(\mathbb{H}) * (2^4 S(2\cdot))$. V_1 is left-invariant under $2^{-1}\Gamma$ and is a closed subspace of $L^2(\mathbb{H})$ as well. The functions in V_1 are continuous bounded functions, and from (7) are $\mathcal{I}_1$-band-limited. With regard to the consequence (a) of Theorem 4.5, for any $f \in V_0$ we have

$$f = f * S = f * \big(S * 2^4 S(2\cdot)\big) = (f * S) * \big(2^4 S(2\cdot)\big).$$

The latter shows that the conclusion $V_0 \subseteq V_1$ holds. By continuing in this manner, we define $V_2 = L^2(\mathbb{H}) * (2^8 S(2^2\cdot))$ to be the closed subspace of functions which are $\mathcal{I}_2$-band-limited. Obviously, with a similar argument as above, one can easily prove that $V_1 \subseteq V_2$.

Similarly, one can define subspaces $V_3 \subseteq V_4 \subseteq \cdots$. On the other hand one may define negatively indexed subspaces. For example, we define

$$V_{-1} = L^2(\mathbb{H}) * 2^{-4} S\big(2^{-1}\cdot\big).$$

This space contains the functions which are $\mathcal{I}_{-1}$-band-limited and obviously $V_{-1} \subseteq V_0$. Again, one may continue in this way to construct the sequence of closed and left $(2^{-j}\Gamma)$-shift-invariant subspaces of $L^2(\mathbb{H})$:

$$\{0\} \subseteq \cdots \subseteq V_{-2} \subseteq V_{-1} \subseteq V_0 \subseteq V_1 \subseteq L^2(\mathbb{H}), \tag{18}$$

which are scaled versions of the central space V_0. Our next aim is to show that, in the sense of Definition 4.2, the sequence of closed subspaces $\{V_j\}$ forms a FMRA of $L^2(\mathbb{H})$. For this reason we must show that the all properties (1)–(6) in Definition 4.2 hold for the sequence $\{V_j\}$. But (18) proves the nested property of V_j's. The density and trivial intersection of V_j's are given in the next theorem:

Theorem 4.8. *$\{V_j\}_{j\in\mathbb{Z}}$ is dense in $L^2(\mathbb{H})$ and has trivial intersection.*

Proof. To show the density of $\{V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{H})$, i.e., $\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{H})$, suppose P_j denotes the projection operator of $L^2(\mathbb{H})$ onto V_j. Then P_j is given by

$$P_j : f \to f * 2^{4j} S\bigl(2^j\cdot\bigr). \tag{19}$$

Therefore the density of $\{V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{H})$ is equivalent to saying that for any f in $L^2(\mathbb{H})$ with $P_j f = 0$ $\forall j \in \mathbb{Z}$, we have $f = 0$. Theorem 4.5 (c) demonstrates this. More precisely

$$0 = P_j f = f * S_j \to f \quad \text{as } j \to \infty$$

which implies $f = 0$.

For the triviality of the intersection, observe that $f * 2^{4j} S(2^j\cdot) = f$ for any $f \in V_j$. Therefore for any $f \in \bigcap V_j$ we have $f * 2^{4j} S(2^j\cdot) = f$ for all j. Therefore (b) in Theorem 4.5 implies that $f = 0$, as desired. □

The other significant properties of V_j's are collected in the next remark:

Remark 4.9.

(1) Property (4) in Definition 4.2 is trivial from the construction of V_j's. This property enables us to pass up and down among the spaces V_j by scaling

$$f \in V_j \iff f\bigl(2^{k-j}\cdot\bigr) \in V_k.$$

(2) Generally, when V_0 is left shift-invariant under some lattice Γ, the spaces V_j are shift-invariant under $2^{-j}\Gamma$. We will return to this fact later and will show how one can choose an appropriate lattice Γ such that it allows the construction of a wavelet frame on $\mathbb{H}$.

(3) Observe that, by contrast to the multiresolution analysis on $\mathbb{R}$, condition (6) in Definition 4.2 requires the existence of some frame generator ϕ, not necessarily $\phi = S$. This is due to the fact that we did not assume any other conditions for the selection of the orthonormal basis $\{e_i^\lambda\}_i$ for the construction of the Hilbert–Schmidt operators $\widehat{S}(\lambda)$ (respectively S). This is one difference between our defined MRA of $L^2(\mathbb{H})$ and the one defined for $L^2(\mathbb{R})$. In the case of $\mathbb{R}$ the sinc function by which the subspaces V_j's are defined, generates an ONB for V_0 and hence for all V_j, under some other suitable discrete subgroups of $\mathbb{R}$. In our case on the Heisenberg group we shall show the existence of a function ϕ in V_0 such that its left translations under a suitable Γ form a normalized tight frame for V_0 and hence for all V_j under $2^{-j}\Gamma$.

As we briefly mentioned above, we shall show the existence of a function ϕ in V_0 such that property (6) in Definition 4.2 holds for V_0. We will observe below that this fact strongly depends on the structure of S and definition of V_0. To achieve this goal, we recall the following definition here:

Definition 4.10. Suppose $\mathcal{H}$ be a left-invariant subspace of $L^2(\mathbb{H})$ and P be the projection operator of $L^2(\mathbb{H})$ onto $\mathcal{H}$. There exists a unique associated projection field $(\widehat{P}_\lambda)_\lambda$ satisfying $\widehat{P(f)}(\lambda) = \widehat{f}(\lambda) \circ \widehat{P}_\lambda$ $\forall f \in L^2(\mathbb{H})$. The associated multiplicity function $m_{\mathcal{H}}$ is then defined by

$$m_{\mathcal{H}} : \mathbb{R}^* \to \mathbb{N}_0 \cup \{\infty\}; \qquad m_{\mathcal{H}}(\lambda) = \operatorname{rank}(\widehat{P}_\lambda).$$

$\mathcal{H}$ is called *band-limited* if the support of its associated multiplicity function $m_{\mathcal{H}}$, $\Sigma(\mathcal{H})$, is bounded in $\mathbb{R}^*$.

Following the notation of [10], the next theorem provides a characterization of closed left shift-invariant subspaces of $L^2(\mathbb{H})$ which admit a tight frame. However, before we state this theorem we need to introduce two numbers associated to a lattice Γ. The number $d(\Gamma)$ refers to a positive integer number d for which $\alpha(\Gamma_d) = \Gamma$ for some $\alpha \in \operatorname{Aut}(\mathbb{H})$, where Γ_d is a lattice in $\mathbb{H}$ and is defined by

$$\Gamma_d := \left\{\left(m, dk, l + \frac{1}{2}dmk\right):\ m, k, l \in \mathbb{Z}\right\}.$$

It is easy to check that Γ_d forms a group under the group operation (1). Observe that due to Theorem 6.2 in [10], such a strictly positive number d exists and is uniquely determined. As well, we define $r(\Gamma)$ be the unique positive real satisfying

$$\Gamma \cap Z(\mathbb{H}) = \bigl\{\bigl(0, 0, r(\Gamma)k\bigr):\ k \in \mathbb{Z}\bigr\},$$

where $Z(\mathbb{H})$ denotes the center of $\mathbb{H}$, $Z(\mathbb{H}) = \{0\} \times \{0\} \times \mathbb{R} \subset \mathbb{H}$. With the above notation we state the following theorem.

Theorem 4.11. *(See [10].) Suppose $\mathcal{H}$ is a left-invariant subspace of $L^2(\mathbb{H})$ and $m_{\mathcal{H}}$ is its associated multiplicity function. Then there exists a tight frame (hence normalized tight frame) of the form $\{L_\gamma\phi\}_{\gamma\in\Gamma}$ with an appropriate $\phi\in\mathcal{H}$ if and only if the inequality*

$$m_{\mathcal{H}}(2\pi\lambda)|2\pi\lambda| + m_{\mathcal{H}}\left(2\pi\lambda - \frac{1}{r(\Gamma)}\right)\left|2\pi\lambda - \frac{1}{r(\Gamma)}\right| \leqslant \frac{1}{d(\Gamma)r(\Gamma)} \tag{20}$$

holds for $m_{\mathcal{H}}$ almost everywhere.

From the inequality (20) it can be read off that $\mathcal{H}$ is band-limited. In fact, the support of $m_{\mathcal{H}}$ is contained in the interval $[-\frac{1}{d(\Gamma)r(\Gamma)}, \frac{1}{d(\Gamma)r(\Gamma)}]$ up to a set of measure zero.

Note that Theorem 6.4 in [10] refers to a different realization of the Schrödinger representations, hence we have the additional factor 2π in the relation (20).

Theorem 4.11 enables us to show the existence of a function ϕ in V_0 which provides a tight frame for V_0. Therefore as a consequence we have our next main result in this section:

Theorem 4.12. *There exists a normalized tight frame of the form $\{L_\gamma\phi\}_{\gamma\in\Gamma}$ for an appropriate $\phi\in V_0$ and a suitable lattice Γ in $\mathbb{H}$.*

Proof. For our purpose we pick a lattice with $r(\Gamma)=\frac{1}{2\pi}$ and $d(\Gamma)=d$. (Observe that it is possible due to Theorem 6.2 in [10] to select a lattice with the desired associated numbers r and d.) From the definition of V_0, $\{\widehat{S}(\lambda)\}_{\lambda\in\mathbb{R}^*}$ is the associated projection field of V_0 with the multiplicity function m_{V_0} which is given by

$$m_{V_0}(2\pi\lambda) = \operatorname{rank}\big(\widehat{S}(2\pi\lambda)\big) = \begin{cases} 2^{2k}+1 & \text{if } 2\pi\lambda\in I_0^k \text{ for some } k\in\mathbb{N}_0,\\ 0 & \text{elsewhere.}\end{cases}$$

One can easily prove that the inequality in (20) holds for m_{V_0}. By the construction of S in Theorem 4.5, $\widehat{S}(\lambda)=0$ for any $|\lambda|>\frac{\pi}{2d}$ which implies:

$$\Sigma(m_{V_0}) \subset \left[-\frac{\pi}{2d}, \frac{\pi}{2d}\right] \subset \left[-\frac{2\pi}{d}, \frac{2\pi}{d}\right] = \left[-\frac{1}{d(\Gamma)r(\Gamma)}, \frac{1}{d(\Gamma)r(\Gamma)}\right].$$

Therefore all the conditions of Theorem 4.11 hold for V_0. Hence there exists a function ϕ, so-called scaling function, such that for our selected lattice Γ, $L_\Gamma\phi$ forms a normalized tight frame for V_0. From this, property (6) of Definition 4.2 is satisfied. □

Corollary 4.13. *For any $j\in\mathbb{Z}$, $\{L_{2^{-j}\gamma}D_{2^{-j}}\phi\}_\gamma$ constitutes a normalized tight frame of V_j.*

As already mentioned, we have constructed the MRA for $L^2(\mathbb{H})$, with the aim of finding an associated discrete wavelet system in $L^2(\mathbb{H})$. More precisely, we want to construct a discrete wavelet system for $L^2(\mathbb{H})$ which is a normalized tight frame. We will study this in detail in the next section by considering a "scaling function" ϕ in V_0.

4.2. Existence of normalized tight wavelet frame for the Heisenberg group

It is natural to try to obtain one normalized tight frame (n.t. frame) for $L^2(\mathbb{H})$ by combining all the n.t. frames $\{L_{2^{-j}\gamma}D_{2^{-j}}\phi\}_{\gamma\in\Gamma}$ of V_j's. But although $V_j\subseteq V_{j+1}$, the n.t. frame for V_j is not necessarily contained in the n.t. frame $\{L_{2^{-(j+1)}\gamma}D_{2^{-(j+1)}}\phi\}_{\gamma\in\Gamma}$ of V_{j+1}. Therefore the union of all n.t. frames for V_j's does not necessarily constitute a n.t. frame for $L^2(\mathbb{H})$.

To find an n.t. frame for $L^2(\mathbb{H})$, we use the following standard approach. For every $j\in\mathbb{Z}$, use W_j to denote the orthogonal complement of V_j in V_{j+1}, i.e., $V_{j+1}=V_j\oplus W_j$, where the symbol $\oplus$ stands for direct sum of the subspaces. Suppose Q_j denotes the orthogonal projection of $L^2(\mathbb{H})$ onto W_j. Then $P_{j+1}=P_j+Q_j$ and evidently

$$V_j = \bigoplus_{k\leqslant j-1} W_k.$$

The most important thing remaining unchanged is that, the spaces W_j, $j\in\mathbb{Z}$, retain the scaling property from V_j. More precisely,

$$f\in W_j \iff f\big(2^{k-j}.\big)\in W_k. \tag{21}$$

Consequently we obtain the following orthogonal decomposition:

$$L^2(\mathbb{H}) = \bigoplus_{j\in\mathbb{Z}} W_j. \tag{22}$$

From this decomposition of $L^2(\mathbb{H})$, it follows that each $f \in L^2(\mathbb{H})$ has a representation $f = \sum_j Q_j f$, where $Q_j f \perp Q_k f$ for any pair of j, k, $j \neq k$.

Our goal is reduced to finding a n.t. frame for W_0. If we can find such a n.t. frame for W_0, then by the scaling property (21) and orthogonal decomposition of $L^2(\mathbb{H})$ into the W_j's in (22), we can easily get a n.t. frame for space $L^2(\mathbb{H})$. We explain this in detail, in the next lemma:

Lemma 4.14. *Suppose $\psi \in W_0$ and Γ is a lattice in $\mathbb{H}$ such that $\{L_\gamma \psi\}_{\gamma \in \Gamma}$ constitutes a n.t. frame of W_0. Then the wavelet system $\{L_{2^{-j}\gamma} D_{2^{-j}} \psi\}_{\gamma, j}$ is a n.t. frame of $L^2(\mathbb{H})$.*

Proof. Observe that this lemma is a consequence of the orthogonal decomposition of $L^2(\mathbb{H})$ into the W_j's. Suppose $f \in L^2(\mathbb{H})$. From (22), f can be written as $f = \sum_j Q_j(f)$. Therefore to prove that the system $\{L_{2^{-j}\gamma} D_{2^{-j}} \psi\}_{\gamma, j}$ forms a n.t. frame of $L^2(\mathbb{H})$, it is sufficient to show that for any j the system $\{L_{2^{-j}\gamma} D_{2^{-j}} \psi\}_\gamma$ is a n.t. frame of W_j. From the scaling property of the spaces W_j in (21) we have $Q_j(f)(2^{-j}.) \in W_0$. Take $Q_j(f) = f_j$. From the hypothesis of the lemma one has

$$\left\| f_j(2^{-j}.) \right\|^2 = \sum_{\gamma \in \Gamma} \left| \left\langle f_j\left(2^{-j}.\right), L_\gamma \psi \right\rangle \right|^2.$$

Replacing $2^{2j} D_{2^j} f_j(\cdot) = f_j(2^{-j}\cdot)$ in the above we obtain

$$\|f_j\|^2 = \|D_{2^j} f_j\|^2 = \sum_{\gamma \in \Gamma} \left| \langle D_{2^j} f_j, L_\gamma \psi \rangle \right|^2 = \sum_{\gamma \in \Gamma} \left| \langle f_j, L_{2^{-j}\gamma} D_{2^{-j}} \psi \rangle \right|^2. \tag{23}$$

Summing over j in (23) yields:

$$\|f\|^2 = \sum_{j \in \mathbb{Z}} \|f_j\|^2 = \sum_{j,\gamma} \left| \langle f_j, L_{2^{-j}\gamma} D_{2^{-j}} \psi \rangle \right|^2 = \sum_{j,\gamma} \left| \langle f, L_{2^{-j}\gamma} D_{2^{-j}} \psi \rangle \right|^2,$$

as desired. □

By Lemma 4.14 it remains to show that the space W_0 contains a function ψ generating a normalized tight frame of W_0.

Remark 4.15. By the definition of orthogonal projections P_1, P_0 in (19), for any $f \in L^2(\mathbb{H})$ we have

$$Q_0(f) = P_1(f) - P_0(f) = f * \left[\left(2^4 S(2.)\right) - S\right],$$

which implies that

$$W_0 = L^2(\mathbb{H}) * \left[\left(2^4 S(2.)\right) - S\right]. \tag{24}$$

Likewise for any j one can see that

$$W_j = L^2(\mathbb{H}) * \left[\left(2^{4j} S\left(2^j.\right)\right) - \left(2^{4(j-1)} S\left(2^{j-1}.\right)\right)\right] \quad \text{and}$$
$$Q_j(f) = f * \left[\left(2^{4j} S\left(2^j.\right)\right) - \left(2^{4(j-1)} S\left(2^{j-1}\right)\right)\right] \quad \forall f \in L^2(\mathbb{H}),$$

where Q_j, as earlier mentioned, is the projection operator of $L^2(\mathbb{H})$ onto W_j.

The representation of the space W_0 in (24) suggests that we can get a n.t. frame for W_0 by applying Theorem 4.11. We obtain this in the proof of the next theorem, which is the last main result of this work:

Theorem 4.16. *There exists a band-limited function $\psi \in L^2(\mathbb{H})$ and a lattice Γ in $\mathbb{H}$ such that the discrete wavelet system $\{L_{2^{-j}\gamma} D_{2^{-j}} \psi\}_{j,\gamma}$ forms a n.t. frame of $L^2(\mathbb{H})$.*

Proof. In order to prove the theorem, first we shall show that the space W_0 is band-limited and contains a function such that its left translations under a suitable lattice Γ forms a n.t. frame of W_0. Hence, the assertion of the theorem will follow from Lemma 4.14 and Theorem 4.16.

Due to the support of S, we have $\sum[(2^4 S(2.)) - S] \subset [-\frac{\pi}{d}, \frac{\pi}{d}]$, where Σ stands for the support on the Fourier transform side, and is applied for the function $(2^4 S(2.)) - S$. Hence W_0 is band-limited. To prove that the space W_0 contains a n.t. frame, observe that by Corollary 4.13 the set $\{L_{2^{-1}\gamma} D_{2^{-1}} \phi\}_{\gamma \in \Gamma}$ is a n.t. frame of V_1 for a suitable Γ. On the other hand, the projection of V_1 onto W_0, Q_0, is left invariant and hence for any $\gamma \in \Gamma$, we have $Q_0(L_{2^{-1}\gamma} D_{2^{-1}} \phi) = L_{2^{-1}\gamma}\left(Q_0(D_{2^{-1}} \phi)\right)$. Since the image of a n.t. frame under a left shift-invariant projection is again a n.t. frame of the image space, the set $\{Q_0(L_{2^{-1}\gamma} D_{2^{-1}} \phi)\}_\gamma = \{L_{2^{-1}\gamma}(Q_0(D_{2^{-1}} \phi))\}_\gamma$ constitutes a n.t. frame for W_0, as desired. □

We conclude our work with the following remark:

Remark 4.17. As mentioned earlier, in contrast to the case of $\mathbb{R}$, in the present work it is not required that the wavelet function ψ contained in W_0 be constructed through the so-called scaling function ϕ in V_0.

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