Effective descent for differential operators

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ABSTRACT

A theorem of N. Katz (1990) [Ka], p. 45, states that an irreducible differential operator L over a suitable differential field k, which has an isotypical decomposition over the algebraic closure of k, is a tensor product $L = M \otimes_k N$ of an absolutely irreducible operator M over k and an irreducible operator N over k having a finite differential Galois group. Using the existence of the tensor decomposition $L = M \otimes N$, an algorithm is given in É. Compoint and J.-A. Weil (2004) [C-W], which computes an absolutely irreducible factor F of L over a finite extension of k. Here, an algorithmic approach to finding M and N is given, based on the knowledge of F. This involves a subtle descent problem for differential operators which can be solved for explicit differential fields k which are C_1-fields.

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1. Introduction

C denotes an algebraically closed field of characteristic 0 and the differential field k is a finite extension of $(C(z), \partial = \frac{d}{dz})$. The algebraic closure of k will be written as $\bar{k}$. Let $L \in k[\partial]$ be a (monic) differential operator. The operator L is called irreducible if it does not factor over k and absolutely irreducible if it does not factor over $\bar{k}$. Here we are interested in the following special situation:

L is irreducible and L factors over $\bar{k}$ as a product $F_1 \ldots F_s$ of $s > 1$ equivalent (monic) absolutely irreducible operators.

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There are algorithms for factoring L over k, i.e., as element of $k[\partial]$ [H1,H2,vdP-S]. Algorithms for finding factors of order 1 in $\bar{k}[\partial]$ are proposed in [S-U,H-R-U-W,vdP-S]. An algorithm for finding factors of arbitrary order in $\bar{k}[\partial]$ is given in [C-W].

According to N. Katz [Ka], Proposition 2.7.2, p. 45, the assumption on L implies that L is a tensor product $M \otimes N$ of monic operators in $k[\partial]$ such that M is absolutely irreducible and the irreducible operator N has a finite differential Galois group (or equivalently all its solutions are algebraic over k). We will present a quick, down-to-earth proof of this in terms of differential modules over k. Further we note that the converse statement is immediate because N decomposes over $\bar{k}$ as a "direct sum" (least common left multiple) of operators $\partial - \frac{f'}{f}$ with $f \in \bar{k}^*$.

The absolute factorization algorithm in [C-W] uses the existence of this tensor decomposition to ensure its correctness but, in full generality, the problem of computing M and N is yet left open in there. For M or N of small order, methods for detecting and computing M and N are given in [C-W, N-vdP] and particularly in [H3] where additional references can be found. We will illustrate this (and propose another method) at the end of the paper.

The problem which we address to produce the operators $M, N \in k[\partial]$ by some decision procedure for M and N of arbitrary order. It follows from $L = M \otimes N$ that $F_1 \in \bar{k}[\partial]$ is equivalent to M (i.e., F_1 descends to k). Let $K_1 \supset k$ be the smallest Galois extension such that $F_1 \in K_1[\partial]$ (or equivalently K_1 is the field extension of k generated by all the coefficients of all F_i). One might think that the equivalence between F_1 and M, seen as elements of $K_1[\partial]$, takes place over K_1. However, in general, a (finite) extension $K' \supset K_1$ is needed for this equivalence. The title of this paper refers to this descent problem.[1]

Our method for finding K' is as follows. First the smallest extension $K \supset K_1$ is computed which guarantees that the factors $F_1, \ldots, F_s$ are equivalent over the field K. Using these equivalences, a certain 2-cocycle c for $Gal(K/k)$ with values in C^*, i.e. the obstruction for the descent of F_1, is computed. Since k is a C_1-field, the 2-cocycle c becomes trivial over a finite (cyclic) computable extension $K' \supset K$: we give a construction of K' in Section 3.1.2, particularly part (4) of Remark 3.4 which produces first order operators having the same obstruction to descent and for which the problem can be solved. Finally, once K' is found, the computation of M, N is easily completed.

In the sequel we will use differential modules because these are more natural for the problem. A translation in terms of differential operators is presented at the moment that actual algorithms are involved since the latter are frequently phrased in terms of differential operators.

The following notation is used. The trivial 1-dimensional differential module over a field K is Ke with $\partial e = 0$. This module will be denoted by $\mathbf{1}$ or $\mathbf{1}_K$. For a differential module A over K of dimension a one writes $\det A$ for the 1-dimensional module $\Lambda^a A$.

2. A version of Katz' theorem

In the proof we will use the following notion of *twist of a differential module*.

Let $Gal(K/k)$ be the Galois group of any Galois extension K of k (finite or infinite). Let A be a differential module over K and $\sigma \in Gal(K/k)$. The twist $^\sigma A$ is equal to A as additive group, has the same ∂ as A, but its structure as K-vector space is given by $\lambda * a := (\sigma^{-1}\lambda) \cdot a$ for $\lambda \in K$, $a \in A$.

The elements $\sigma \in Gal(K/k)$ act in a natural way on $K[\partial]$ by the formula $\sigma(\sum_n a_n \partial^n) = \sum_n \sigma(a_n)\partial^n$. If one presents A as $K[\partial]/K[\partial]F$ (with F monic), then $^\sigma A = K[\partial]/K[\partial]\sigma(F)$.

An isomorphism $\phi(\sigma): {}^\sigma A \to A$ can also be interpreted as a C-linear bijection $\Phi(\sigma) : A \to A$, commuting with ∂ and such that $\Phi(\sigma)(\lambda a) = \sigma(\lambda) \cdot \Phi(a)$. In other words $\Phi(\sigma)$ is σ-linear isomorphism.

Proposition 2.1. *Let L be a differential module over k. Suppose that*:

(1) *The field of constants C of k is algebraically closed, k has characteristic zero and k is a C_1-field.*

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[1] Techniques for *arithmetic* descent were proposed in [H-P], where the case of a differential field k with non-algebraically closed constant field is handled.

(2) *L is irreducible and $\bar{L} := \bar{k} \otimes_k L$ decomposes as a direct sum $\bigoplus_{i=1}^s A_i$ of isomorphic irreducible differential modules over $\bar{k}$.*

Then there are modules M, N over k such that $L \cong M \otimes_k N$, M is absolutely irreducible and the irreducible module N has a finite differential Galois group. The pair (M, N) is unique up to a change $(M \otimes_k D, N \otimes_k D^{-1})$ where D has dimension 1 and $D^{\otimes t}$ is the trivial module for some $t \geqslant 1$.

Proof. Write $A = A_1$ and let *Gal* denote the Galois group of $\bar{k}/k$. For any $\sigma \in Gal$, the twisted module $^\sigma A$ is a submodule of $^\sigma(\bar{k} \otimes_k L)$. As the latter module is isomorphic to $\bar{k} \otimes_k L$, there is a σ-linear isomorphism $\Phi(\sigma) : A \to A$.

This induces a 2-cocycle c for *Gal* with values in C^*, defined by $\Phi(\sigma\tau) = c(\sigma, \tau) \cdot \Phi(\sigma) \cdot \Phi(\tau)$. Since k is a C_1-field, the 2-cocycle is trivial ([Se], II-9, §3.2). After multiplying the isomorphisms $\{\Phi(\sigma)\}$ by suitable elements in C^*, one obtains descent data $\{\Phi(\sigma) \mid \sigma \in Gal\}$ satisfying the descent condition $\Phi(\sigma\tau) = \Phi(\sigma) \cdot \Phi(\tau)$ for all $\sigma, \tau \in Gal(\bar{k}/k)$. Define $M := \{a \in A \mid \Phi(\sigma)a = a$ for all $\sigma \in Gal\}$. It is easily verified that M is a differential module over k and that the canonical morphism $\overline{M} := \bar{k} \otimes_k M \to A$ is an isomorphism.

Consider now $\text{Hom}_\partial(\overline{M}, \bar{L})$. This is a vector space, isomorphic to C^s and provided with an action of *Gal*. Then $\bar{k} \otimes_C \text{Hom}_\partial(\overline{M}, \bar{L})$ is a trivial differential module over $\bar{k}$ provided with an action of *Gal*. It is a submodule of the differential module $\text{Hom}(\overline{M}, \bar{L})$. Taking invariants under *Gal* one obtains a differential module

$$N := \left(\bar{k} \otimes_C \text{Hom}_\partial(\overline{M}, \bar{L}) \right)^{Gal} \quad \text{over } k \text{ which is a submodule of } \text{Hom}(M, L).$$

The canonical morphism $\bar{k} \otimes N \to \bar{k} \otimes_C \text{Hom}_\partial(\overline{M}, \bar{L})$ is an isomorphism and thus the differential Galois group of N is finite.

The canonical morphism of differential modules $M \otimes_k \text{Hom}(M, L) \to L$, namely $m \otimes \ell \mapsto \ell(m)$, can be restricted to a morphism $f : M \otimes_k N \to L$. By construction the induced morphism $\overline{M} \otimes_{\bar{k}} \overline{N} \to \bar{L}$ is an isomorphism and thus so is f.

The descent data $\{\Phi(\sigma) \mid \sigma \in Gal\}$ for A are not unique. They can be changed into $\{h(\sigma)\Phi(\sigma) \mid \sigma \in Gal\}$ where $h : Gal \to C^*$ is any continuous homomorphism. We note that the image of h is a subgroup μ_t of the t-th roots of unity for some t. Consider the trivial differential module $\bar{k}e$ with $\partial e = 0$ and with *Gal* action given by $\sigma e = h(\sigma)e$ for all $\sigma \in Gal$. By taking the invariants under *Gal* one obtains a 1-dimensional module D over k such that the canonical morphism $\bar{k} \otimes D \to \bar{k}e$ is an isomorphism and respects the actions of *Gal*. Further $D^{\otimes t}$ is the trivial differential module $\mathbf{1}$.

The differential module obtained with the new descent data can be seen to be $M \otimes_k D$ and thus N will be changed into $N \otimes D^{-1}$. From this observation the last statement of the proposition follows. $\quad\square$

Remarks 2.2. The non-unicity of the pair (M, N) can be restricted by the condition that $\det N = \mathbf{1}$. Then only a change $(M \otimes D, N \otimes D^{-1})$ is possible with $D^{\otimes s} = \mathbf{1}$.

3. Algorithmic approach

First we make the relation between differential modules and differential operators explicit. Let A be a differential module and $a \in A$ a cyclic vector. One associates to this the monic differential operator $op(A, a) \in k[\partial]$ of minimal degree satisfying $op(A, a)a = 0$. The morphism $k[\partial] \to A$, which maps 1 to a, induces an isomorphism $k[\partial]/k[\partial]op(A, a) \to A$.

Let $B \subset A$ be a submodule. This yields a factorization $op(A, a) = \mathcal{L}\mathcal{R}$ with $\mathcal{R} \in k[\partial]$ is the monic operator of minimal degree such that $\mathcal{R}a \in B$. One observes that $\mathcal{R} = op(A/B, a + B)$.

Further $\mathcal{L} \in k[\partial]$ is the operator of minimal degree satisfying $\mathcal{L}b = 0$, where $b := \mathcal{R}a$. Clearly b is a cyclic vector for B and $\mathcal{L} = op(B, b)$.

Moreover, any factorization $op(A, a) = \mathcal{L}\mathcal{R}$ with monic $\mathcal{L}, \mathcal{R}$ corresponds in this way to a unique submodule $B \subset A$, namely $B = k[\partial]\mathcal{R}a$.

Let k' be an algebraic extension of k. Then the above bijection extends to a bijection between the (monic) factorizations of $op(A, a)$ in $k'[\partial]$ and the submodules of $k' \otimes_k A$.

As before, L denotes an irreducible differential module over k such that $\bar{k} \otimes L$ is a direct sum of $s > 1$ copies of an absolutely irreducible differential module.

Choose $\ell \in L, \ell \neq 0$. Since L is irreducible, ℓ is a cyclic vector. *We assume the knowledge of a factorization $op(L, \ell) = F.R$ with monic $F, R \in \bar{k}[\partial]$ and F absolutely irreducible*, given by [C-W]. Using this information we will describe the computation leading to a tensor product decomposition $L = M \otimes N$.

3.1. The special case $\dim M = 1$

Assume that the irreducible L is equal to $M \otimes_k N$ with $\dim M = 1$, $\dim N = s > 1$ and $\bar{k} \otimes_k N$ is trivial. Thus the Picard–Vessiot extension K^+ of N is a finite extension of k and can be considered as a subfield of $\bar{k}$. The (covariant) solution space V of N is equal to $\ker(\partial, K^+ \otimes_k N)$. The differential Galois group $G^+ = Gal(K^+/k)$ acts on V and there is a canonical isomorphism $K^+ \otimes_C V \to K^+ \otimes_k N$. Moreover, $\bar{M} := \bar{k} \otimes_k M$ is not a trivial module (equivalently, the differential Galois group of M is infinite and then equal to the multiplicative group $\mathbb{G}_m$).

There is a trivial way to produce a decomposition $L = M \otimes_k N$. Indeed, write $op(L, \ell) = (\partial^s + a_{s-1}\partial^{s-1} + \cdots + a_0)$. Then the tensor product decomposition $op(L, \ell) = (\partial + \frac{a_{s-1}}{s}) \otimes (\partial^s + b_{s-2}\partial^{s-2} + \cdots + b_0)$, for suitable elements $b_i \in k$, solves already the problem, since (as one easily sees) all the solutions of $\partial^{s-1} + b_{s-2}\partial^{s-2} + \cdots + b_0$ are in $\bar{k}$. However, the aim of this subsection is to describe in this easy situation an algorithm for obtaining the pair (M, N), up to a change $(M \otimes D, D^{-1} \otimes N)$, which works with small modifications for the general case.

After fixing a non-zero element $\ell \in L$, the module is represented by the monic operator $op(L, \ell)$. *The first step is to produce the smallest subfield $K \subset \bar{k}$ (containing k) such that $op(L, \ell)$ decomposes in $K[\partial]$ as a product $F_1 \cdots F_s$ of (monic) equivalent operators of degree 1.*

The (monic) left-hand factors $F = \partial + u \in \bar{k}[\partial]$ of $op(L, \ell)$ correspond to the 1-dimensional submodules of

$$\bar{k} \otimes L = M \otimes_k (\bar{k} \otimes_k N) = M \otimes_k (\bar{k} \otimes_C V)$$

and these are the $M \otimes_k (\bar{k} \otimes_C W)$ where W runs in the set of the 1-dimensional subspaces of V. The same can be done with $\bar{k}$ replaced by K^+. Therefore $u \in K^+$ and $K_0 := k(u) \subset K^+$. More precisely, let $St(W) \subset G^+$ be the stabilizer of W. This is the subgroup of G^+ leaving F invariant and thus $K_0 = (K^+)^{St(W)} \subset K^+$.

Now we suppose that a (monic) left-hand factor $F = \partial + u$ of $op(L, \ell)$ is known and explain how to obtain K from this. Let $K_1 \subset \bar{k}$ be the normal closure of K_0. Then $K_1 \otimes_k L$ contains a 1-dimensional submodule that we will call again D. The submodule $\sum_{\sigma \in Gal(K_1/k)} \sigma(D)$ of $K_1 \otimes_k L$ is invariant under the action of $Gal(K_1/k)$. Since L is irreducible, $\sum_{\sigma \in Gal(K_1/k)} \sigma(D) = K_1 \otimes_k L$ and it follows that $K_1 \otimes_k L$ is a direct sum of 1-dimensional submodules. As a consequence, $op(L, \ell)$ factors as $F_1 \cdots F_s$ in $K_1[\partial]$.

A priori, the factors $F_i = \partial + u_i \in K_1[\partial]$ need not be equivalent. For $i < j$ we consider a non-zero element $f_{ij} \in \bar{k}$ satisfying $\frac{f'_{ij}}{f_{ij}} = u_i - u_j$. Put $K = K_1(\{f_{ij}\})$. Then $K \supset k$ is the smallest field such that $op(L, \ell)$ factors as $F_1 \cdots F_s \in K[\partial]$ where the monic degree one factors F_i are equivalent.

Clearly $K \subset K^+$. From the condition that K is minimal such that $K \otimes_k N$ is a direct sum of isomorphic 1-dimensional submodules and the irreducibility of N, it follows easily that the center Z of $G^+ \subset \mathrm{GL}(V)$ is the finite cyclic group $(C^* \cdot id_V) \cap G^+$ and that $K = (K^+)^Z$.

Remarks 3.1.

(1) The field K and the above algorithm for K do not change if L is replaced by $D \otimes_k L$, where D is a 1-dimensional module satisfying $D^{\otimes t} = \mathbf{1}$ for some $t \geqslant 1$.

(2) The fields K_0 and K_1 depend on the given left-hand factor of degree 1 of $op(L, \ell)$. We illustrate this by an example where the differential Galois group $G^+ \subset GL(C^3)$ is generated by the matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \quad \text{with } a^6 = b^6 = c^6 = 1.$$

If this left-hand factor corresponds to Ce_1 (or Ce_2 or Ce_3), then its stabilizer is the subgroup of the diagonal matrices in G^+. Otherwise it is just the center Z. In the first case $K_1 \neq K$ and in the last one $K_1 = K$.

3.1.1. The 2-cocycle c and descent fields

Now we have arrived at the situation where $K \otimes_k L$ is a direct sum of s copies of a known 1-dimensional differential module $D = M \otimes_k E$ over K, where E is an, a priori, unknown 1-dimensional submodule of $K \otimes_k N$. We want to produce a field extension $K' \supset K$ such that $K' \otimes_K D$ descends to k.

Let D correspond to the operator $\partial - u$. Then, for $\sigma \in Gal(K/k)$, the operator $\partial - \sigma(u)$ corresponds to $^\sigma D$. There are elements $f_\sigma \in K^*$ such that $\sigma(u) - u = \frac{f'_\sigma}{f_\sigma}$. One obtains a 2-cocycle c for $Gal(K/k)$ with values in C^* by $f_{\sigma\tau} = c(\sigma, \tau) \cdot f_\sigma \cdot {}^\sigma f_\tau$ for all $\sigma, \tau \in Gal(K/k)$. The class of the 2-cocycle c in $H^2(Gal(K/k), C^*)$ is the *obstruction for the descent* of D (or, equivalently, for the descent of E).

Indeed, if D descends to k, then one can represent D by $\partial - u$ with $u \in k$. On the other hand, if the class of c is trivial, then after changing the $\{f_\sigma\}$ one has $f_{\sigma\tau} = f_\sigma \cdot {}^\sigma f_\tau$. By Hilbert 90, there exists $F \in K^*$ with $f_\sigma = \frac{\sigma F}{F}$ for all $\sigma \in Gal(K/k)$. Then $\partial - u$ is equivalent to $\partial - u + \frac{F'}{F}$. Further $u - \frac{F'}{F}$ lies in k, since it is invariant under $Gal(K/k)$.

The exact sequence $1 \to Z \to G^+ \xrightarrow{pr} Gal(K/k) \to 1$ induces a 2-cocycle with values in Z, in the following way. Let $\phi : Gal(K/k) \to G^+$ be a section, i.e., $pr \circ \phi(g) = g$ for all $g \in Gal(K/k)$. Then d, defined by $\phi(g_1 g_2) = d(g_1, g_2)\phi(g_1)\phi(g_2)$, is a 2-cocycle with values in Z. The class of d in $H^2(Gal(K/k), Z)$ is independent of the choice of the section ϕ. As before, Z is identified with a subgroup of C^*. Thus d induces an element of $H^2(Gal(K/k), C^*)$. We note that the homomorphism $H^2(Gal(K/k), Z) \to H^2(Gal(K/k), C^*)$ is, in general, not injective.

Lemma 3.2. *The cocycles c and d have the same image in $H^2(Gal(K/k), C^*)$. This image does not depend of the choice of N. Let $s = \dim N$. Then the image of c^s in $H^2(Gal(K/k), C^*)$ is trivial.*

Proof. Let D, as before, be given by $\partial - u$ with $u \in K$. Write $u = \frac{F'}{F}$ with $F \in (K^+)^*$. Let $\phi : G \to G^+$ be a section. Then $\sigma(u) - u = \frac{\phi(\sigma)F'}{\phi(\sigma)F} - \frac{F'}{F} = \frac{f'_\sigma}{f_\sigma}$ where $f_\sigma := \frac{\phi(\sigma)F}{F}$. One easily sees that f_σ is invariant under Z and thus $f_\sigma \in K^*$. The equality $f_{\sigma\tau} = c(\sigma, \tau)f_\sigma {}^\sigma f_\tau$ implies $\phi(\sigma\tau)F = c(\sigma, \tau)\phi(\sigma)\phi(\tau)F$. Hence $d(\sigma, \tau) = c(\sigma, \tau)$.

Replacing N by $(\partial - v) \otimes_k N$ with $v \in k$, induces the change of $\partial - u$ into $\partial - u - v$. Since $\sigma(u + v) - (u + v) = \sigma(u) - u$, the element c and its image in $H^2(Gal(K/k), C^*)$ are unchanged.

Suppose that N is chosen such that $\det N = \mathbf{1}$. The cocycle d has values in μ_s, since $\dim N = s$. Thus the image of d^s in $H^2(Gal(K/k), C^*)$ is trivial and the same holds for c^s. $\square$

Definition 3.3. Let (K, c) be a Galois extension K/k and c a 2-cocycle for $Gal(K/k)$ with values in C^* and such that c^s is a trivial 2-cocycle. A *descent field* for (K, c) is a Galois extension $K' \supset k$ containing K, such that the induced 2-cocycle c' for $Gal(K'/k)$, defined by $c'(g_1, g_2) = c(pr\,g_1, pr\,g_2)$, yields a trivial element in $H^2(Gal(K'/k), C^*)$. Here $pr : Gal(K'/k) \to Gal(K/k)$ denotes the natural map.

The assumption that k is a C_1-field implies the *existence of a descent field for every pair (K, c)*. Indeed, by [Se], II §3, the cohomological dimension of $Gal(\bar{k}/k)$ is 1 and therefore $H^2(Gal(\bar{k}/k), \mu_\infty) = 1$, where μ_∞ denotes the torsion subgroup of C^*. One has $H^2(Gal(K/k), C^*) = H^2(Gal(K/k), \mu_\infty)$ and

$\varinjlim H^2(Gal(K'/k), \mu_\infty) = H^2(Gal(\bar{k}/k), \mu_\infty) = \{1\}$, where the direct limit is taken over all Galois extensions $K' \supset k$, containing K. Hence for every class $\bar{c} \in H^2(Gal(K/k), C^*)$ there exists a Galois extension $K' \supset k$, containing K, such that the image of $\bar{c}$ in $H^2(Gal(K'/k), C^*)$ is 1.

Our contribution is now to produce a descent field for (K, c) by some algorithm.

3.1.2. A decision procedure constructing a descent field for (K, c)

Since k is a C_1-field, $H^2(Gal(K/k), K^*)$ is trivial ([Se], II-9, §3.2) and this implies the existence of elements $\{f_\sigma \mid \sigma \in Gal(K/k)\} \subset K^*$ satisfying $f_{\sigma\tau} = c(\sigma, \tau) \cdot f_\sigma \cdot {}^\sigma f_\tau$ for all $\sigma, \tau \in Gal(K/k)$.

Suppose that c is given in the form $f_{\sigma\tau} = c(\sigma, \tau) f_\sigma {}^\sigma f_\tau$ with $\{f_\sigma\} \subset K^*$. (This is trivially true for the present case $\dim M = 1$. For the general case, Remarks 3.4 part (4) describes a decision procedure producing suitable $\{f_\sigma\}$, a key step in the construction.) Then the following algorithm produces a descent field.

One has $\frac{f'_{\sigma\tau}}{f_{\sigma\tau}} = \frac{f'_\sigma}{f_\sigma} + \sigma\left(\frac{f'_\tau}{f_\tau}\right)$ and since $H^1(Gal(K/k), K) = 0$ there is an element $v \in K$ such that $\frac{f'_\sigma}{f_\sigma} = \sigma(v) - v$ for all $\sigma \in Gal(K/k)$; explicitly

$$v = \frac{-1}{[K:k]} \sum_{\tau \in Gal(K/k)} \frac{f'_\tau}{f_\tau}.$$

One observes that c is the obstruction for descent of the operator $\partial - v$.

Further $-mv = \frac{G'}{G}$ with $G = \prod_{\tau \in Gal(K/k)} f_\tau$ and $m = [K:k]$. Hence the field $K(\sqrt[m]{G})$ contains the Picard–Vessiot field of $\partial - v$ and is a descent field.

A less brutal way to compute a descent field is as follows. Since the cocycle c^s is trivial, there are computable elements $\{d(\sigma) \mid \sigma \in Gal(K/k)\} \subset C^*$ satisfying $d(\sigma\tau) = c(\sigma, \tau)^s d(\sigma) d(\tau)$. The elements $\{\frac{f_\sigma^s}{d(\sigma)}\}$ form a 1-cocycle. Since $H^1(Gal(K/k), K^*) = \{1\}$, one can effectively compute $F \in K^*$ such that $\frac{f_\sigma^s}{d(\sigma)} = \frac{\sigma F}{F}$ for all $\sigma \in Gal(K/k)$ (see [Se2], Chapitre X, §1, Prop. 2). One observes that $v - \frac{1}{s}\frac{F'}{F} \in k$ since it is invariant under $Gal(K/k)$. The field extension $K' = K(\sqrt[s]{F})$ has the property that $(\partial - v)$ is equivalent to $\partial - v + \frac{1}{s}\frac{F'}{F}$ over K'. Hence K' is a descent field.

Remarks 3.4.

(1) We note that the above algorithm proves, by considering 1-dimensional differential modules over K, the existence of a descent field only using that $H^2(Gal(K/k), K^*) = \{1\}$ (see Lemma A.1 for the general statement).

(2) Instead of assuming that the 2-cocycle c^s is trivial, we may consider a class $\bar{c} \in H^2(Gal(K/k), \mu_\infty)$, where $\mu_\infty \subset C^*$ denotes, as before, the group of the roots of unity. Any finite group G occurs as some $Gal(K/k)$. Therefore the group $H^2(Gal(K/k), \mu_\infty)$ is in general not trivial and the descent problem, i.e., finding an extension $K' \supset K$ such that the image of $\bar{c}$ in $H^2(Gal(K'/k), \mu_\infty)$ is 1, is non-trivial. However for a cyclic $Gal(K/k)$ one has $H^2(Gal(K/k), \mu_\infty) = \{1\}$ ([Se2], VIII, §4). In particular, non-trivial examples for the descent problem tend to be complicated.

(3) We ignore how to compute or characterize all minimal descent fields for a given pair (K, c).

(4) Computing elements $f_\sigma \in K^*$ satisfying $f_{\sigma\tau} = c(\sigma, \tau) \cdot f_\sigma \cdot {}^\sigma f_\tau$ appears to be far from trivial. A possible method, which uses explicitly the C_1-property of k, is the following. Starting with the 2-cocycle c, there is a well known construction (see [G-S]) of an algebra $A = \bigoplus_{\sigma \in G} K[\sigma]$, where $G = Gal(K/k)$, of dimension $m = \#G = [K:k]$ over K, defined by the rules:

$$[\sigma] \cdot \lambda = \sigma(\lambda) \cdot [\sigma] \quad \text{for } \lambda \in K, \ \sigma \in G,$$

$$[\sigma_1 \sigma_2] = c(\sigma_1, \sigma_2)[\sigma_1] \cdot [\sigma_2].$$

Then A is a central simple algebra with center k. Since k is a C_1-field there exists an isomorphism $I : A \to Matr(m, k)$. The latter algebra can be identified (by standard Galois theory) with the group

algebra $K[G] = \bigoplus K \cdot \sigma$. Suppose that we knew already elements $f_\sigma \in K^*$ satisfying $f_{\sigma\tau} = c(\sigma, \tau) \cdot f_\sigma \cdot {}^\sigma f_\tau$. Then $I_0 : A \mapsto K[G]$, given by $I_0(\sum \lambda_\sigma [\sigma]) = \sum \lambda_\sigma f_\sigma .\sigma$ is an isomorphism and is also K-linear. By the Skolem–Noether theorem, any isomorphism I of k-algebras has the form $I(\sum \lambda_\sigma [\sigma]) = x^{-1}\{\sum \lambda_\sigma f_\sigma .\sigma\}x$, where x is an invertible element of $Matr(m, k)$. Our aim is to compute a K-linear isomorphism and in that case x commutes with K and therefore belongs to K^*. Now I has the form $I(\sum \lambda_\sigma [\sigma]) = \sum \lambda_\sigma f_\sigma .\frac{\sigma(x)}{x}.\sigma$ and thus any K-linear isomorphism $A \to K[G]$ has the form $[\sigma] \mapsto g_\sigma \sigma$ for suitable elements $g_\sigma \in K^*$. It follows that $g_{\sigma\tau} = c(\sigma, \tau) \cdot g_\sigma \cdot {}^\sigma g_\tau$.

The computation of the isomorphism I uses the reduced norm *Norm* of A (see e.g. [F,Pi] or [R], Section 4, which adapts to C_1 fields). With respect to a basis of A over k, the reduced norm is a homogeneous form of degree m in m^2 variables. Again the C_1 property of k asserts that there are non-trivial solutions $a \in A$, $a \neq 0$ for $Norm(a) = 0$. An explicit calculation of such a is possible (but rather expensive). Applying this several times one obtains the isomorphism $I : A \to Matr(m, k)$.

Example 3.5. Consider the case where $K \supset k$ has degree 2 and c is a 2-cocycle for $G = \{1, \sigma\}$ with values in K^*.

One easily sees that the 2-cocycle can be given by $c(1, 1) = c(1, \sigma) = c(\sigma, 1) = 1$ and $c(\sigma, \sigma) = \alpha^{-1} \in k^*$. The K-linear morphism $\phi : A := K[1] \oplus K[\sigma] \to K[G] = K \oplus K\sigma$ should have the form $\phi([1]) = 1$, $\phi([\sigma]) = f\sigma$ with $f \in K^*$. We have to find f. Now the condition is $\alpha = f\sigma(f)$. Write $K = k \oplus kw$ with $w^2 \in k^*$ and write $f = a + bw$. Then we have to solve $a^2 - b^2 w^2 = \alpha$. Consider the equation $X_1^2 - X_2^2 w^2 - X_3^2 \alpha = 0$. By the C_1-property of k there is a solution $(x_1, x_2, x_3) \neq 0$. Now $x_3 \neq 0$, since $w^2 \in k^*$ is not a square. Then we can normalize to $x_3 = 1$ and the problem is solved.

(5) For dim $N = 2$ and M of any dimension we will give in Appendix A an easier algorithm for descent fields, not using the 2-cocycle c explicitly (and recall the former algorithms of [H3,C-W,N-vdP]).

(6) Let again a Galois extension $k \subset K$ with Galois group G and 2-cocycle $\bar{c} \in H^2(G, C^*)$ be given. The 2-cocycle class has finite order (dividing s) and corresponds to a short exact sequence $1 \to Z \to G^+ \to G \to 1$, where Z is a finite cyclic group, lying in the center of G^+. Suppose that the Galois extension $k \subset K^+$ with group G^+ is such that $(K^+)^Z = K$. Then the image of $\bar{c}$ in $H^2(G^+, C^*)$ is trivial and the descent condition holds for the field K^+. The C_1-property of the field k guarantees the existence of K^+, however there seems to be no explicit algorithm, based on the C_1-property, producing a K^+. Examples A.4 are based on this remark.

(7) Finally, we note that $H^2(Gal(K/k), \mu_s) = 1$ if $g.c.d.([K : k], s) = 1$. In that case there is no field extension needed for the descent.

3.2. Description of the algorithm for the general case

We will search for a decomposition $L = M \otimes N$ with det $N = 1$. The module $\bar{k} \otimes L$ can be written as $M \otimes_k (\bar{k} \otimes_k N) = M \otimes_k (\bar{k} \otimes_C V)$ where V is the solution space of N. The absolutely irreducible left-hand factors F of $op(L, \ell)$ correspond to the 1-dimensional subspaces W of V.

We suppose that at least one F is given. Let $K_0 \supset k$ denote the field extension generated by the coefficients of F. Let K_1 be the normal closure of K_0. For each $\sigma \in Gal(K_1/k)$ one considers the absolutely irreducible left-hand factor $\sigma(F)$. This factor is over $\bar{k}$ equivalent to F. We have to compute the field extension of K_1 needed for this equivalence.

An algorithm in terms of differential modules (which easily translates in terms of differential operators) is based upon the following lemma.

Lemma 3.6. *Let A be an irreducible differential module over $\bar{k}$. Then the differential module* $\mathrm{Hom}(A, A)$ *over $\bar{k}$ has only one 1-dimensional submodule, namely $\bar{k} \cdot id_A$.*

Proof. It is possible to prove this by using [N-vdP] and irreducible representations of semi-simple Lie algebras.

However, a more down-to-earth proof is the following. Let V be the solution space of A. This is a C-vector space of dimension equal to $a := \dim_{\bar{k}} A$, equipped with a faithful irreducible action of

the differential Galois group G of A. The group G is connected since $\bar{k}$ is algebraically closed. A 1-dimensional submodule of $\mathrm{Hom}(A, A)$ corresponds to a 1-dimensional subspace Cf of $\mathrm{Hom}_C(V, V)$, invariant under the action of G. There is a homomorphism $c : G \to C^*$ such that $gfg^{-1} = c(g) \cdot f$ holds for all $g \in G$. The kernel of f is G-invariant and is $\{0\}$ since the representation is irreducible. The action of G on $\mathrm{Hom}(\Lambda^a V, \Lambda^a V)$ is trivial. In particular, the isomorphism $\Lambda^a(f) : \Lambda^a V \to \Lambda^a V$ is invariant under G. Also $g(\Lambda^a(f))g^{-1} = c(g)^a \cdot \Lambda^a(f)$ and $c(g)^a = 1$. Since G is connected, $c(g) = 1$ for all $g \in G$. Thus f is G-invariant and is a multiple of id_V since the representation is irreducible. $\quad\square$

Corollary 3.7. *The differential module*

$$T(\sigma) := \mathrm{Hom}\big(K_1[\partial]/K_1[\partial]\sigma(F), K_1[\partial]/K_1[\partial]F\big)$$

over K_1 has a single 1-dimensional submodule $A(\sigma)$. Moreover, the Picard–Vessiot field of $A(\sigma)$ is a finite cyclic extension $K_1' \subset \bar{k}$ of K_1.

Proof. $S := \bar{k} \otimes_{K_1} T(\sigma)$ is isomorphic to $\mathrm{Hom}(\overline{M}, \overline{M})$, where $\overline{M} := \bar{k} \otimes_k M$. By Lemma 3.6, S has a unique 1-dimensional submodule, say, B. By uniqueness, B is invariant under the action of $Gal(\bar{k}/K_1)$ on S and has therefore the form $\bar{k} \otimes_{K_1} A(\sigma)$ for some submodule $A(\sigma)$ of $T(\sigma)$. The uniqueness of $A(\sigma)$ is clear.

Further $\bar{k} \otimes A(\sigma)$ is isomorphic to the trivial differential module $\bar{k} \cdot id_{\overline{M}}$. Thus the Picard–Vessiot field K_1' is a finite extension of K_1 and this extension is cyclic since $A(\sigma)$ has dimension 1. $\quad\square$

By factorization the 1-dimensional submodule $A(\sigma)$ can be obtained. The Picard–Vessiot field of $A(\sigma)$ is a finite cyclic extension $K_1' \subset \bar{k}$ of K_1. Then $\ker(\partial, K_1' \otimes T(\sigma))$ has dimension 1 over C and a generator $\phi(\sigma)$ of this kernel is an isomorphism $\phi(\sigma) : K_1'[\partial]/K_1'[\partial]\sigma(F) \to K_1'[\partial]/K_1'[\partial]F$.

The field $K \subset \bar{k}$ is the compositum of the Picard–Vessiot fields of all $A(\sigma)$. We note that K is the field called "*stabilisateur*" in [C-W]. The isomorphisms $\phi(\sigma) : K[\partial]/K[\partial]\sigma(F) \to K[\partial]/K[\partial]F$ are now also known, they are K-rational solutions of the modules $K \otimes_{K_1} A(\sigma)$. The 2-cocycle c for $Gal(K/k)$ with values in C^* has the property that c^s is trivial by the assumption that $\det N = \mathbf{1}$. Then, as in Section 3.1, one can construct a cyclic extension $K' \supset K$ such that the module $K'[\partial]/K'[\partial]F$ descends to k. The result is called M.

The module N is obtained by computing the unique irreducible direct summand of $M^* \otimes_k L$ having dimension s. Indeed, this direct summand of $M^* \otimes_k L = M^* \otimes_k M \otimes_k N = \mathrm{Hom}(M, M) \otimes_k N$ is $(k \cdot id_M) \otimes_k N \cong N$.

3.2.1. An example for the construction of a descent field
Consider the irreducible operator

$$L = \partial^4 + \frac{(-4 + 8z)}{z^2 - 1}\partial^3 + \frac{(53z^2 - 40z - 1)}{4(z^2 - 1)^2}\partial^2 + \frac{(5z^3 - z^2 - 13z - 3)}{2(z^2 - 1)^3}\partial + \frac{61z^2 + 64z + 67}{16(z^2 - 1)^4}.$$

The algorithm of [C-W] produces the following absolutely irreducible right-hand factor

$$L_1 = \partial^2 + \frac{(3u + 4z^2 - 26z + 24)}{2(z^2 - 1)(4z - 5)}\partial + \frac{(-3z - 6)u + 45z^2 - 40z - 13}{4(z^2 - 1)^2(4z - 5)},$$

where $u^2 = z^2 - 1$. It is isomorphic to its conjugate L_2 over $K = k(\Phi)$ with $\Phi^4 - 2z\Phi^2 + 1 = 0$ (or $\Phi = \sqrt{z - u}$). Explicitly, there exists $S \in K[\partial]$ such that $L_2.R = S.L_1$ with

$$R = \frac{(1 - 2z + 2u)}{\Phi(4z - 5)}\big(2(z^2 - 1)\partial - z - 2\big),$$

i.e. R maps a solution of L_1 to a solution of L_2.

Let $G = Gal(K/k)$, acting via $\sigma_1(\Phi) = \Phi$, $\sigma_2(\Phi) = -\Phi$, $\sigma_3(\Phi) = 1/\Phi$, $\sigma_4(\Phi) = -1/\Phi$. The 2-co-cycle c is given by $c(\sigma_2, \sigma_3) = c(\sigma_2, \sigma_4) = c(\sigma_4, \sigma_3) = c(\sigma_4, \sigma_4) = -1$ (and $c(\sigma_i, \sigma_j) = 1$ otherwise).

Remark 3.4 part (4) (see also Example A.3 in Appendix A) produce the elements f_σ which are, respectively, $1, 1, \Phi, \Phi$; hence the construction from Section 3.2.1 shows that $K' = K(\sqrt{\Phi})$ is a descent field. An anihilating operator for $\sqrt{\Phi}$ is

$$N = \partial^2 + \frac{z}{z^2 - 1}\partial - 1/16\big(z^2 - 1\big)^{-1}$$

(one could find it by writing K' as a $k[\partial]$-module and decomposing it).

Decomposing $L \otimes N^\star$ (over k), we then obtain

$$M = \partial^2 - \frac{2}{z - 1}\partial + \frac{35z + 37}{16(z + 1)(z - 1)^2}$$

and L_1 is isomorphic over K' to M. At the end of Appendix A, we give alternative (easier) methods to handle such small order examples.

Acknowledgments

Appendix A

The following lemma makes the relation between 2-cocycles and descent for one-dimensional differential modules more explicit. We present some examples and present an algorithm producing descent fields for the case $\dim N = 2$.

Lemma A.1. *Let k be a differential field having the properties*: *the field of constants C is algebraically closed and has characteristic 0; k is a C_1-field.*

Let K/k be a Galois extension (finite or infinite). The collection $H(K)$ of the (isomorphy classes of the) 1-dimensional differential modules A over K, satisfying $^\sigma A \cong A$ for all $\sigma \in Gal(K/k)$, forms a group with respect to the operation tensor product. Let $h(K) \subset H(K)$ denote the subgroup consisting of the modules of the form $K \otimes_k B$, where B is a 1-dimensional differential module over k.

There is a canonical isomorphism $H(K)/h(K) \to H^2(Gal(K/k), C^*)$.

Proof. The first statement is obvious. Let the differential module Ke with $\partial e = ue$ lie in $H(K)$. Then for any $\sigma \in Gal(K/k)$ there is an element $f_\sigma \in K^*$ such that $\sigma(u) - u = \frac{f'_\sigma}{f_\sigma}$. Define the 2-cocycle c by $f_{\sigma\tau} = c(\sigma, \tau) \cdot f_\sigma \cdot {}^\sigma f_\tau$ for all $\sigma, \tau \in Gal(K/k)$. Replacing the f_σ by $d(\sigma)f_\sigma$, with $d(\sigma) \in C^*$, changes the 2-cocycle into an equivalent one. Tensoring Ke with an element of $h(K)$ changes u into $u + v$ with $v \in k$ and this does not effect the f_σ. Thus the above construction defines a homomorphism $H(K)/h(K) \to H^2(Gal(K/k), C^*)$.

This homomorphism is injective since the triviality of the 2-cocycle class $\bar{c}$ implies that $f_{\sigma\tau} = f_\sigma \cdot {}^\sigma f_\tau$. Since $H^1(Gal(K/k), K^*) = \{1\}$ there is an $F \in K^*$ with $f_\sigma = \frac{\sigma F}{F}$ for all σ. Thus $\sigma(u) - u = \sigma(\frac{F'}{F}) - \frac{F'}{F}$ and $\bar{u} := u - \frac{F'}{F}$ is invariant under $Gal(K/k)$ and belongs to k. Now $Ke = K \cdot \bar{e}$ with $\bar{e} := F^{-1}e$ and $\partial\bar{e} = \bar{u}\bar{e}$. Thus Ke belongs to $h(K)$.

The homomorphism is surjective. Indeed, consider a 2-cocycle c for $Gal(K/k)$ with values in C^*. Since $H^2(Gal(K/k), K^*) = \{1\}$ there are elements $f_\sigma \in K^*$ such that $f_{\sigma\tau} = c(\sigma, \tau) \cdot f_\sigma \cdot {}^\sigma f_\tau$ for all $\sigma, \tau \in Gal(K/k)$. Then $\frac{f'_{\sigma\tau}}{f_{\sigma\tau}} = \frac{f'_\sigma}{f_\sigma} + \sigma(\frac{f'_\tau}{f_\tau})$ and since $H^1(Gal(K/k), K) = \{0\}$ there is an element $u \in K$ such that $\sigma(u) - u = \frac{f'_\sigma}{f_\sigma}$ for all $\sigma \in Gal(K/k)$. Thus the class of c is the image of the module Ke with $\partial e = ue$, belonging to $H(K)$. $\square$

Example A.2. Let $k = C(z)$ and $K = C(t)$ with $t^2 = z$. Let σ be the non-trivial element in $Gal(K/k)$. The module Ke with $\partial e = ue$ belongs to $h(K)$ if and only if

$$u = w + \frac{1}{2t} \sum_{\alpha \neq 0} \frac{d_\alpha \sqrt{\alpha}}{z - \alpha} \quad \text{with all } d_\alpha \in \mathbb{Z} \text{ and } w \in k.$$

(We note that $\sqrt{\alpha}$ denotes an arbitrary choice of a square root for $\alpha \in C^*$.) This follows from the computation: if $K^* \ni F = t^{n_0} \prod_{\beta \neq 0} (t - \beta)^{n_\beta}$, then

$$\frac{F'}{F} = \frac{n_0}{2t^2} + \sum_{\beta^2 \neq 0} \frac{n_\beta + n_{-\beta}}{2(t^2 - \beta^2)} + \frac{1}{2t} \sum_{\beta^2 \neq 0} \frac{n_\beta \beta - n_{-\beta}\beta}{t^2 - \beta^2}.$$

Consider now Ke with $\partial e = ue$, belonging to $H(K)$. Write $u = a + \frac{1}{2t}b$ with $a, b \in k$. By assumption $\frac{-1}{t}b = \sigma(u) - u$ has the form $\frac{G'}{G}$ for some $G \in K^*$. Write $G = t^{m_0} \prod_{\beta \neq 0}(t - \beta)^{m_\beta}$, then, using the above formula, one finds that $m_0 = 0$ and $m_{-\beta} = -m_\beta$. Thus $b = \sum_{\alpha \neq 0} \frac{m_{\sqrt{\alpha}}\sqrt{\alpha}}{z - \alpha}$, where $\alpha = \beta^2$ (and some choice of $\sqrt{\alpha}$ is made). According to the above result, one has that Ke lies in $h(K)$. Therefore $H(K)/h(K) = \{0\}$. This is in accordance with $H^2(Gal(K/k), C^*) = \{1\}$ (since $Gal(K/k)$ is cyclic).

Example A.3. The group $D_2 \cong (\mathbb{Z}/2\mathbb{Z})^2$ has the property that $H^2(D_2, C^*)$ contains an element of order 2. In order to obtain this element for an obstruction to descent we consider the following differential fields

$$k = C(z) \subset K = C(t) \subset K' = C(s) \quad \text{with } z = t^2 + t^{-2}, \ t = s^2.$$

The group $Gal(K/k) = \{1, a, b, ab\} \cong D_2$ where the elements a, b are given by $a(t) = -t$ and $b(t) = t^{-1}$. One considers the differential module Ke with $\partial e = ue$ and $u = \frac{1}{4(t^2 - t^{-2})} = \frac{s'}{s}$. One observes that $a(u) - u = 0$ and $b(u) - u = \frac{(t^{-1})'}{t^{-1}}$. Thus we may take $f_1 = 1$, $f_a = 1$, $f_b = t^{-1}$, $f_{ab} = t^{-1}$. The corresponding 2-cocycle c has values in $\{\pm 1\}$ and $f_{ab} = c(a, b) \cdot f_a \cdot {}^a f_b$ holds with $c(a, b) = -1$. It is easily verified that $\bar{c} \in H^2(D_2, C^*)$ is not trivial. The module $K' \otimes_K e$ is trivial since $u = \frac{s'}{s}$ and therefore descends to k. In particular K' is a descent field for (K, c).

We note that for a finite group G, acting trivially on C^*, the cohomology group $H^2(G, C^*)$ is called the Schur multiplier of G. This group is well studied, see [Su].

Examples A.4. A construction of many examples of the type $L = M \otimes_k N$ under consideration, involving a non-trivial descent problem, is the following.

Suppose that G^+ is given as a finite irreducible subgroup of $GL(V)$ where $\dim_C V = n > 1$ and that the center Z of G^+ is non-trivial. Further, assume that a Galois extension $K^+ \supset k = C(z)$ with Galois group G^+ is given. Then $K := (K^+)^Z$ is a Galois extension of k with group $G := G^+/Z$.

Consider the differential module $K^+ \otimes_C V$ over K^+, defined by $\partial(f \otimes v) = f' \otimes v$ for $f \in K^+$, $v \in V$. This is a trivial differential module. The action of G^+ on $K^+ \otimes_C V$ is defined by $\sigma(f \otimes v) = \sigma(f) \otimes \sigma(v)$. This action commutes with ∂.

Define $N := (K^+ \otimes_C V)^{G^+}$. This is an irreducible differential module over k with Picard–Vessiot field K^+. The subfield K is the smallest field such that $K \otimes_k N$ is a direct sum of isomorphic copies of a 1-dimensional differential module D over K. In particular ${}^\sigma D \cong D$ for all $\sigma \in Gal(K/k)$. The 2-cocycle attached to D is non-trivial if there is no subgroup $H \subset G^+$ mapping bijectively to G. More precisely, K^+ is a smallest field over which the 2-cocycle becomes trivial if and only if no proper subgroup H of G^+ maps surjectively to G.

Take now any absolutely irreducible differential module M of dimension $m > 1$ over k. Then $L := M \otimes_k N$ has the required properties. For $\dim N = 2$ there is a rich choice of examples and there are similar explicit cases for $n = 3$, see [vdP-U].

Example A.5. Algorithms for the descent field for the case $\dim N = 2$.

Let $L = M \otimes N$ be given with M absolutely irreducible and an irreducible N with $\dim N = 2$, $\det N = \mathbf{1}$ and finite differential Galois group G^+. For this case, methods for finding N (hence the descent field) and M are proposed, e.g. in [H3,C-W,N-vdP] (and references therein).

Below is another nice method, adapted to our case. We have

$$G^+ \in \left\{ D_k^{SL_2}, A_4^{SL_2}, S_4^{SL_2}, A_5^{SL_2} \right\} \subset \mathrm{SL}(2, C)$$

with center $Z = \left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and there is no proper subgroup of G^+ mapping onto $G := G^+/Z \in \{D_k, A_4, S_4, A_5\}$. According to Lemma 3.2, the 2-cocycle class $\bar{c} = \bar{d} \in H^2(G, C^*)$ is non-trivial. The method of Section 3 provides the field K with $Gal(K/k) = G$, from the data $op(L, \ell)$ and an absolutely irreducible monic left-hand factor F of $op(L, \ell)$.

The (unknown) group G^+ is a subgroup of $\mathrm{SL}(V)$ with $\dim_C V = 2$. Write $W = sym^2 V$ and let $S \in sym^2(W)$ be a generator of the kernel of $sym^2(W) \to sym^4(V)$. Then S is a non-degenerate symmetric form of degree two. The homomorphism $\psi : \mathrm{SL}(V) \to \mathrm{SL}(W)$, defined by $A \mapsto A \otimes_s A$, has kernel $\left\{ \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$ and its image is $\{B \in \mathrm{SL}(W) \mid S \text{ is invariant under } B\}$. One observes that $\psi(G^+) = G$. Conversely, for any subgroup $G \subset \mathrm{SL}(W)$ preserving the form S, one has that $\psi^{-1}(G) = G^+$.

Let $a \in K$ be a general element, then the orbit Ga is a basis of K/k and the C-vector space with basis Ga is the regular representation of G. This vector space contains an irreducible representation W of G of dimension three. In each of the cases for G, there exists a (unique) non-degenerated symmetric form S on W which is invariant under G.

The unique monic differential operator $T_3 \in K[\partial]$ of degree 3 which is 0 on W belongs to $k[\partial]$ because W is invariant under G. This operator (or the corresponding differential module) is equivalent to the second symmetric power of an operator $T_2 \in k[\partial]$ (which can be found using [H3]). Let $\tilde{K}$ denote the Picard–Vessiot field of T_2. Then $[\tilde{K} : K] = 2$. Let $V \subset \tilde{K}$ denote the space of solutions of T_2. Then $W = \{v_1 v_2 \mid v_2, v_2 \in V\} = sym^2 V$. The differential Galois group of T_2 is $\psi^{-1}(G)$ and thus isomorphic to G^+. Hence $\tilde{K}$ is a descent field for (K, d) and then also for (K, c). Using this descent field one computes M and N.

Yet another observation (though less practical) is that J.J. Kovacic's fundamental algorithm for order 2 equations [Ko], could also be applied to $L = M \otimes N$. For example, the symmetric power $sym^{m+1}(L)$ contains an irreducible factor over k, which is projectively isomorphic to M, for $m = 2, 4, 6, 12$ for the cases $G^+ = D_k^{SL_2}, A_4^{SL_2}, S_4^{SL_2}, A_5^{SL_2}$. More refined factorization patterns may be established for each of these cases.

Example A.6. The referee's example. The operator

$$L_4 = \partial^4 + \frac{6z}{z^2 - 1} \partial^3 + \frac{1971z^2 - 947}{288(z^2 - 1)^2} \partial^2 + \frac{27z}{32(z^2 - 1)^2} \partial + \frac{9}{4096(z^2 - 1)^2}$$

has the absolutely irreducible *right*-hand factor

$$L_2 = \partial^2 + \frac{3z - \alpha + 1}{6(z^2 - 1)} \partial + \frac{3}{64(z^2 - 1)},$$

where α is a root of $T^4 + 12(z - 1)T^2 - 32(z - 1)T - 12(z - 1)^2 = 0$.

There are the following methods:

(1) By [C-W,H3]. $L_4 = M \otimes N$ with $\det M = \det N = \mathbf{1}$. The two factors of $\Lambda^2 L_4 = sym^2(M) \oplus sym^2(N)$ are easily computed and, using [H3], one finds M and N.

(2) By [N-vdP], Theorem 6.2. One computes $F \in sym^2(L_4)$ with $\partial F = 0$, $F \neq 0$ and a 2-dimensional isotropic subspace for F. From the last part of the proof of [N-vdP], Theorem 6.2 one reads off M and N.

(3) Example A.5 works here as follows. $K_0 = k(\alpha)$ and its normal closure K_1 has Galois group A_4 and $K = K_1$. The C-vector space W spanned by $A_4\alpha$ has dimension 3. The operator $T_3 = \partial^3 + a_2\partial^2 + a_1\partial + a_0 \in k[\partial]$ with solution space W is determined by the equation $T_3(\alpha) = 0$. This yields

$$T_3 = \partial^3 + \frac{3z-1}{(z+1)(z-1)}\partial^2 + \frac{27z+5}{36(z+1)^2(z-1)^2}\partial - \frac{9z+23}{36(z+1)^2(z-1)^3}.$$

The operator

$$T_2 = \partial^2 + \frac{3z-1}{3(z+1)(z-1)}\partial - \frac{3z-11}{48(z+1)(z-1)^2}$$

satisfies $sym^2(T_2) = T_3$ and its Picard–Vessiot field (an extension of k of degree 24) is a descent field. A minimum polynomial of an algebraic solution of T_2 is

$$P = Y^8 + \frac{1}{3}(z-1)Y^4 + \frac{4}{27}(z-1)Y^2 - \frac{1}{108}(z-1)^2.$$

It factors over $k(\alpha)$ as

$$\left(Y^2 + \frac{\alpha}{6}\right)\left(Y^6 - \frac{\alpha}{6}Y^4 + \frac{1}{3}\left(z - 1 + \frac{\alpha^2}{12}\right)Y^2 - \frac{\alpha^3}{216} - \frac{1}{18}(z-1)\alpha + \frac{4}{27}(z-1)\right),$$

which illustrates the fact that K^+ is obtained from K_1 by adjunction of a square root, here $\sqrt{-\frac{\alpha}{6}}$ (in fact, adjoining any solution of T_2 would do).

Continuation of our method (decomposing $L_4 \otimes T_2^*$ over k) then yields

$$M = \partial^2 - \frac{1}{3}\frac{(-1+3x)}{x^2-1}\partial + \frac{1}{192}\frac{189x^2 - 96x + 227}{(x^2-1)^2}$$

and we may check that

$$(M \otimes T_2).\left((x^2-1)\partial\right) = \left((x^2-1)\partial + 4x\right).L_4.$$

As solutions of both M and T_2 can be expressed in terms of special functions (e.g. using the methods of van Hoeij), this allows to solve L_4 in terms of algebraic and special functions.

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