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On the Subsections for Certain 2-Blocks

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In a previous paper [8], the author studied modular 2-blocks of finite groups with quaternion or quasi-dihedral defect groups, using some results of Brauer [4, 5]. To a certain point the analysis of 2-blocks with defect groups of maximal class is quite analogous, especially the determination of subsections for these blocks.

The present paper contains two almost independent sections. In the first we study a class of p-groups called "generalized Redei groups" and classify then for $p = 2$. The definition of these is directly inspired by the properties of the 2-groups of maximal class, which makes the analysis of the subsections possible in that case. Then in Section 2 we determine a complete set of representatives for the conjugacy classes of subsections for 2-blocks having a nonabelian generalized Redei defect group. In particular, this gives a fairly large lower bound on the number of ordinary characters in such a 2-block.

1. GENERALIZED REDEI GROUPS

DEFINITIONS. Let p be a prime integer and P be a finite p-group. We define a set $\mathscr{M}_P$ of subgroups of P as follows:

$$\mathscr{M}_P := \{Q \mid Q \subseteq P,\ Q \text{ is nonabelian } \textit{or } C_P(Q) = Q\}.$$

(If $C_P(Q) = Q$, then Q is a maximal abelian subgroup of P.)

P is called *generalized Redei* if

$$\forall Q \in \mathscr{M}_P : \begin{cases} 1. & |N_P(Q) : Q| \leqslant p, \\ 2. & C_P(Q) \subseteq Q. \end{cases}$$

We denote the set of generalized Redei groups by K_p (for a given prime p). By K_p' we denote the set of nonabelian elements in K_p.

P is called *Redei* if all proper subgroups of P are abelian, but P is nonabelian.

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EXAMPLES. Abelian p-groups and Redei groups are generalized Redei. $\mathbb{Z}_p \wr \mathbb{Z}_p \in K_p'$ for all primes p. Generalized quaternion groups, dihedral, and quasidihedral 2-groups belong to K_2'.

PROPOSITION 1.1. *Let P be a Redei group. Then one of the following hold*

(i) $P \ncong R_1(r, s) = \langle x, y \mid x^{p^r} = y^{p^s} = 1, y^{-1}xy = x^{1+p^{r-1}} \rangle, r \geqslant 2, s \geqslant 1.$

(ii) $P \ncong R_2(r, s) = \langle x, y \mid x^{p^r} = y^{p^s} = z^p = 1, [x, y] = z \rangle, r, s \geqslant 1, r + s \geqslant 3.$

(iii) $P \ncong Q_8$, *the quaternion group of order* 8.

This has been proved by Redei and is quoted in [7, p. 309]. We call the groups in (*i*) and (ii) for *Redei groups of type* (i) *and* (ii), respectively.

LEMMA 1.2. *Let $P \in K_p'$. Then P has an abelian subgroup of index p.*

Proof. Since P is noncyclic it has at least two maximal subgroups M_1 and M_2. Then $M_1 \cap M_2$ is a normal subgroup of P index p^2. By condition 1 $M_1 \cap M_2 \notin \mathscr{M}_P$. So $M_1 \cap M_2$ is abelian and not maximal abelian. It is therefore contained in a larger abelian subgroup, which then must have index p in P.

LEMMA 1.3. *If $P \in K_p$ and Q is a subgroup of P, then $Q \in K_p$.*

Proof. If Q is abelian, the result is true. By an inductive argument we may assume that $|P : Q| = p$ and that Q is nonabelian. Let $Q_1 \in \mathscr{M}_Q$. If Q_1 is nonabelian then $Q_1 \in \mathscr{M}_P$ and we are done. Suppose Q_1 is abelian and $C_Q(Q_1) = Q_1$. As $|P : Q| = p$ we have $|C_P(Q_1) : Q_1| = 1$ or p. If $C_P(Q_1) = Q_1$ then $Q_1 \in \mathscr{M}_P$, so suppose $|C_P(Q_1) : Q_1| = p$. Then $C_P(Q_1)$ is abelian and therefore $C_P(Q_1) \in \mathscr{M}_P$. Consequently, $|N_P(C_P(Q_1)) : C_P(Q_1)| = p$ by condition 1. Now $C_P(Q_1) \subseteq N_P(Q_1) \subseteq N_P(C_P(Q_1))$ so we have a diagram as shown below:

$$\begin{tikzcd}[column sep=small]
& N_P(C_P(Q_1)) \arrow[d, dash] & \\
& N_P(Q_1) \arrow[dl, dash] \arrow[dr, dash] & \\
C_P(Q_1) \arrow[dr, dash] & & N_Q(Q_1) \arrow[dl, dash] \\
& Q_1 &
\end{tikzcd}$$

If $|N_Q(Q_1) : Q_1| \geqslant p^2$, then $N_Q(Q_1) = N_P(C_P(Q_1))$, as by the above $|N_P(C_P(Q_1)) : Q_1| = p^2$. But $N_P(C_P(Q_1))$ contains $C_P(Q_1)$. This is a contradiction to the fact that $C_P(Q_1) \nsubseteq Q$.

PROPOSITION 1.4. *Let $P \in K_p'$. Then $|P : \Phi(P)| = p^2$, where $\Phi(P)$ is the Frattini subgroup of p. Therefore P is generated by two elements.*

Proof. Suppose the proposition is false and let P be a minimal counterexample. Then P is not Redei by Proposition 1.1, so P contains a maximal subgroup M, which is nonabelian. By induction hypothesis and the preceding lemma, $|M : \Phi(M)| = p^2$. Clearly $\Phi(M) \subseteq \Phi(P) \subseteq M \subseteq P$, so $|P : \Phi(P)| = p^3$ and $\Phi(P) = \Phi(M)$. Since every maximal subgroup of P lies in $\mathscr{M}_P$, every such subgroup contains $Z(P)$ by condition 2. Thus $Z(P) \subseteq \Phi(P)$. By Lemma 1.2 there exists an abelian subgroup A of P of index p in P. Let $x \in P - A$. As $P = \langle x, A\rangle$ we get that $Z(P) = C_A(x)$. Also $x \notin Z(P)$, so $C_P(x)$ contains $Z(P)$ properly. As $|P : A| = p$ we have $|C_P(x) : Z(P)| = |C_P(x) : C_A(x)| = p$. $C_P(x)$ is abelian, because it contains $Z(P)$ as a subgroup of index p, so $C_P(x) \in \mathscr{M}_P$. Let $Q = \langle \Phi(P), C_P(x)\rangle$. Then $|Q : \Phi(P)| = |C_P(x) : C_A(x)| = p$. Consider the diagram below:

$$\begin{tikzcd}
 & & P \arrow[dl, dash] \arrow[dr, dash, "p"] & \\
 & Q \arrow[dl, dash] \arrow[dr, dash, "p"'] & & A \arrow[dl, dash, "p^2"] \\
C_P(x) \arrow[dr, dash, "p"'] & & \Phi(P) \arrow[dl, dash] & \\
 & Z(P) = C_A(x) & &
\end{tikzcd}$$

We have that $|P : Q| = p^2$ and as $Q \supseteq \Phi(P) \supseteq [P, P]$, $Q \lhd P$. Consequently, Q is abelian by condition 1. As $C_P(x) \in \mathscr{M}_P$, we get $C_P(x) = Q$. Then $|P : C_P(x)| = |P : Q| = p^2$, a contradiction as $C_P(x) \in \mathscr{M}_P$.

LEMMA 1.5. *Let $p \in K_p'$ and let A be an abelian subgroup of index p. Moreover let $x \in P - A$.*

(i) *Then $x^p \in Z(P)$ and $C_P(x) = \langle x, C_A(x)\rangle = \langle x, Z(P)\rangle \in \mathscr{M}_P$. So in fact $C_P(y) \in \mathscr{M}_P$ for all $y \in P$.*

(ii) *The mapping φ_x: $a \to [a, x]$ is a homomorphism from A onto $[P, P]$. The kernel is $Z(P)$, so $A/Z(P) \simeq [P, P]$.*

Proof. (i) follows from general remarks in the proof of Proposition 1.4.

(ii) A is abelian. Using the commutator relations we get for all elements $a, b \in A$

$$\varphi_x(ab) = [ab, x] = [a, x]^b[b, x] = [a, x][b, x] = \varphi_x(a)\,\varphi_x(b),$$

so φ_x is a homomorphism. Clearly $\operatorname{Ker} \varphi_x = C_A(x) = Z(P)$. As $P = \langle x, A \rangle$ we see that every element in P can be written on the form ax^i, $a \in A$, $0 \leqslant i \leqslant p$. From the commutator relations it follows that any commutator element in P is a product of elements on the form $[a, x^i]$, $a \in A$, $0 \leqslant i < p$. However, $[a, x^i] = [a_1, x]$ for a suitable $a_1 \in A$, again by the commutator relations. This proves (ii).

LEMMA 1.6. *Let $P \in K_p'$ and $Z \subseteq Z(P)$. Let $\bar{P} = P/Z$. If $\bar{M} = M/Z \in \mathscr{M}_{\bar{P}}$, then $M \in \mathscr{M}_P$.*

Proof. Suppose $\bar{M} \in \mathscr{M}_{\bar{P}}$. If $\bar{M}$ is nonabelian, then M is nonabelian, so $M \in \mathscr{M}_P$. Suppose $\bar{M}$ is abelian. Then $C_{\bar{P}}(\bar{M}) = \bar{M}$ by definition of $\mathscr{M}_{\bar{P}}$, so $C_P(M) \subseteq M$, because $C_P(M)/Z \subseteq C_{\bar{P}}(\bar{M})$. Then either $C_P(M) = M$ and therefore M maximal abelian, or $C_P(M) \neq M$ and therefore M nonabelian. In both cases $M \in \mathscr{M}_P$.

PROPOSITION 1.7. *If $P \in K_p$ and $Z \subseteq Z(P)$, then $P/Z \in K_p$.*

Proof. Suppose false and let P be a minimal counterexample. Then we may assume $|Z| = p$. Let $\bar{P} = P/Z$. For $\bar{M} \in \mathscr{M}_P$ clearly $|N_{\bar{P}}(\bar{M}) : \bar{M}| \leqslant p$, because $N_P(M)/Z = N_{\bar{P}}(\bar{M})$ and $M \in \mathscr{M}_P$ by Lemma 1.6. So it must be condition 2 that is violated in $\bar{P}$. There exists a $\bar{M} = M/Z \in \mathscr{M}_{\bar{P}}$ such that $C_{\bar{P}}(\bar{M}) \not\subseteq \bar{M}$. Let $C_{\bar{P}}(\bar{M}) = N/Z$. Then $N \not\subseteq M$. We have that $\bar{M}$ is nonabelian. Otherwise $C_{\bar{P}}(\bar{M}) = \bar{M}$, as $\bar{M} \in \mathscr{M}_{\bar{P}}$, and then $N \subseteq M$, a contradiction. In particular M is nonabelian. By definition $[M, N] \subseteq Z \subseteq Z(P)$. Also $[M, N] \neq 1$. Otherwise again $N \subseteq C_P(M) \subseteq M$, as $M \in \mathscr{M}_P$. Consequently $[M, N] = Z$. Also we note that $M \not\subseteq N$. Otherwise $\bar{M} \subseteq C_{\bar{P}}(\bar{M}) = N/Z$. We have $C_P(M) \subseteq N \subseteq N_P(M)$. As $|N_P(M) : M| \leqslant p$ we have $N_P(M) = M \cdot N$. By minimality $P = M \cdot N$, so M is maximal in P. Also we have $N \lhd P$, because $[N, M] = Z \subseteq N$. For $x \in N - M$ we have $C_P(x) \in \mathscr{M}_P$ by Lemma 1.5(i). So if N is abelian then $N \in \mathscr{M}_P$. Thus in any case $N \in \mathscr{M}_P$, so by condition 1, $|P : N| = p$. Suppose $\bar{N}$ is abelian. Then $\bar{N} \subseteq C_{\bar{P}}(\bar{N})$, so $\bar{N} \subseteq Z(\bar{P}) \subseteq \bar{P}$. As $|\bar{P} : \bar{N}| = p$ this implies $\bar{P}$ to be abelian, a contradiction. Thus $\bar{N}$ and in particular N is nonabelian. Assume that there exists an $x \in N - M$, such that $[x, M] \neq 1$. Then consider the homomorphism φ from M onto Z defined by $\varphi(m) = [m, x]$ for all $m \in M$. The kernel is $C_M(x)$. As $x \notin M$, also $x \notin \Phi(P)$, so $x \notin A$, where A is an abelian subgroup of P of index p. By Lemma 1.5(i) we conclude that $C_M(x) = Z(P)$. But then $M/Z(P) \cong Z$, so $|M : Z(P)| = p$, which implies that M is abelian, a contradiction. Consequently, M is centralized by all elements in $N - M$. However, N is generated by these elements. (Take $x_1 \in N - \Phi(P)$ and $x_2 \in \Phi(P) - \Phi(N)$. Then $x_1x_2 \in N - \Phi(P)$ and $x_1 \not\equiv x_1x_2 \bmod \Phi(N)$. By Proposition 1.4 we get $N = \langle x_1, x_1x_2 \rangle$.) Again we have a contradiction.

COROLLARY. *If $P \in K_p$ and $N \lhd P$, then $P/N \in K_p$.*

This follows by a straightforward inductive argument.

LEMMA 1.8. *Let $P \in K_p'$. The following statements are equivalent:*

(i) *P is a Redei group.*

(ii) *P has more than one abelian subgroup of index p.*

(iii) *$|P : Z(P)| = p^2$, i.e., $Z(P) = \Phi(P)$.*

Proof. (i) $\Rightarrow$ (ii) is trivial by definition.

(ii) $\Rightarrow$ (iii): Let A_1 and A_2 be 2 different abelian subgroups of index p in P. Let $Z = A_1 \cap A_2$. Then as $A_i \subseteq C_G(Z)$, $i = 1, 2$, and $P = \langle A_1, A_2 \rangle$, we get $Z \subseteq Z(P)$. As $|P : Z| = p^2$ and $|P : Z(P)| > p$, (iii) follows.

(iii) $\Rightarrow$ (i): Let M be any maximal subgroup of P. Then M contains $\Phi(P) = Z(P)$ as a subgroup of index p. Therefore M is abelian.

Remark. From (1.8), (i) $\Leftrightarrow$ (iii) and from Lemma 1.5(ii) it follows that P is Redei if and only if $|[P, P]| = p$.

DEFINITION. $P \in K_p$ is called *strongly generalized Redei*, if $[P, P]$ is cyclic. The set of strongly generalized Redei-groups is denoted SK_p. The subset of nonabelian groups is denoted SK_p'. Clearly Redei groups belong to SK_p'. However, for p *odd*, $\mathbb{Z}_p \wr \mathbb{Z}_p \in K_p' - SK_p'$.

PROPOSITION 1.9. *$K_2 = SK_2$, i.e., for $P \in K_2$, $[P, P]$ is cyclic.*

Proof. Suppose the proposition is false and let P be a minimal counterexample. So P is non-Redei. Let $z \in Z(P) - 1$. By Proposition 1.7 $\bar{P} = P/\langle z \rangle \in K_2$. Moreover, $\bar{P}$ is nonabelian by the above remark. By induction hypothesis, $[\bar{P}, \bar{P}] = [P, P] \cdot \langle z \rangle / \langle z \rangle \cong [P, P]/([P, P] \cap \langle z \rangle)$ is cyclic. So we get $[P, P] \cap \langle z \rangle \neq 1$. In particular

$$\Omega_1(Z(P)) \subseteq [P, P]. \tag{$*$}$$

Let $z \in Z(P) \cap [P, P]$ be of order 2. Then $[P, P]/\langle z \rangle$ is cyclic, so $[P, P] = \langle z_1 \rangle \times \langle z \rangle$, as $[P, P]$ is noncyclic. Suppose that $|z_1| > 2$. Let z_2 be the involution in $\langle z_1 \rangle$. Then $\langle z_2 \rangle$ char $[P, P]$ char P, so $z_2 \in Z(P) \cap [P, P]$. But $P/\langle z_2 \rangle$ has then a noncyclic commutator subgroup, a contradiction. Thus

$$[P, P] \text{ is elementary abelian of order 4.} \tag{$**$}$$

By ($*$) we conclude that either $Z(P)$ is cyclic or $Z(P) \supseteq [P, P]$. In the last case, P is of class 2, and we get a contradiction using [1, Lemma 1.2]. So $Z(P)$ is cyclic. By Lemma 1.5(ii) above and ($**$), $|P : Z(P)| = 8$. As $Z(P)$ is cyclic, $P/Z(P)$ is nonabelian. Moreover, if A is the abelian subgroup of index 2 in P, then $A/Z(P) \cong [P, P]$ is elementary abelian.

Now $P/Z(P)$ is nonabelian of order 8 and contains an elementary abelian subgroup of order 4. Therefore $P/Z(P)$ is dihedral. Let $M/Z(P)$ be the cyclic subgroup of index 2 in $P/Z(P)$. Then $M \neq A$ and M is abelian. By Lemma 1.8 P is Redei, a contradiction.

Remark to proof. Only in the last few lines above is it used, that the prime is 2. For odd p, $P/Z(P)$ can be nonabelian and of exponent p.

LEMMA 1.10. *Let $P \in SK_p$, p an arbitrary prime. Then P is of class 2 if and only if P is Redei.*

Proof. We need only show that P of class 2 implies P Redei. Suppose not and let P be a minimal counterexample. Then all proper nonabelian subgroups of P are Redei. Let M be a nonabelian subgroup of P of index p. M is Redei, so $|M : Z(M)| = p^2$. However, from Lemma 1.5(i) it follows that $Z(M) = Z(P)$, so $|P : Z(P)| = p^3$. As P is of class 2, $P/Z(P)$ is abelian. If A is the abelian subgroup of P of index p, then $A/Z(P) \cong [P, P]$ is cyclic of order p^2. Therefore $P/Z(P)$ is abelian of type (p^2, p). This shows, that $P/Z(P)$ has more than one cyclic subgroup of order p^2, so P has more than one abelian subgroup of index p. Lemma 1.8 gives a contradiction.

PROPOSITION 1.11. *Let $P \in SK_p'$. Then P is Redei or metacyclic.*

Proof. Again let P be a minimal counterexample. By the remark after Lemma 1.8 we have $|[P, P]| > p$. Take $z \in \Phi(P') \cap Z(P)$ of order p. By a result of Blackburn (see [7, p. 336]) $P/\langle z\rangle = \bar{P}$ is not metacyclic. So by induction it must be Redei. So $|[\bar{P}, \bar{P}]| = p$. As $[\bar{P}, \bar{P}] = [P, P]/\langle z\rangle$ we get $|[P, P]| = p^2$. By Proposition 1.4(ii) $|P : Z(P)| = p^3$. $P/Z(P)$ is nonabelian, as P is not of class 2. If A is abelian, $|P : A| = p$, then $A/Z(P) \cong [P, P]$ is cyclic of order p^2. So $P/Z(P)$ has a cyclic subgroup of index p. For p odd it has more than one such subgroup, so P is Redei by Lemma 1.8, a contradiction. The same holds if $P/Z(P)$ is quaternion of order 8. So $P/Z(P)$ is dihedral of order 8.

Pick $x \in P - A$ and $a \in A$, such that $\langle [x, a]\rangle = [P, P]$, and let $P_x = \langle x, [a, x]\rangle$. Clearly $P_x \lhd P$. Also P_x is nonabelian, since otherwise P would be of class 2. $P_x \neq P$, as P_x is metacyclic. So $|P : P_x| = 2$. We conclude that $P_x \cap A = \Phi(P)$ has at most two generators. However, by a result of Blackburn [2, Theorem 3.2], A must have at least three generators. If not, all subgroups of P would be metacyclic. Therefore $A = \Phi(P) \times \langle z_1\rangle$, where $|z_1| = 2$, by a straightforward argument. Now $[\Phi(P), x] = [P_x, P]$ has order 2. Also $[z_1, x]$ has order 2. Therefore $[A, x] = [P, P]$ is of exponent 2, a contradiction.

This proposition, of course, makes it almost straightforward to actually determine explicitly SK_p' for any prime p. This is done in Proposition 1.12. Since $K_2 = SK_2$ by Proposition 1.9, the 2-groups listed in Proposition 1.12 are all possible nonabelian generalized Redei 2-groups.

PROPOSITION 1.12. *Let $P \in SK_p'$.*

(i) *For p odd, P is Redei.*

(ii) *For $p = 2$, P is Redei or of one of the following types:*

$$P \ncong P_1(r, s) = \langle x, y \mid x^{2^r} = y^{2^s} = 1, y^{-1}xy = x^{-1}\rangle, \quad r \geqslant 3, s \geqslant 1$$

$$P \ncong P_2(r, s) = \langle x, y \mid x^{2^r} = y^{2^s} = 1, y^{-1}xy = x^{-1+2^{r-1}}\rangle, \quad r \geqslant 3, s \geqslant 1$$

$$P \ncong P_3(r, s) = \langle x, y \mid x^{2^{r-1}} = y^{2^{s-1}} = z, z^2 = 1, y^{-1}xy = x^{-1}\rangle, \quad r, s \geqslant 2, r + s \geqslant 5$$

$$P \ncong P_4(r, s) = \langle x, y \mid x^{2^{r-1}} = y^{2^{s-1}} = z, z^2 = 1, y^{-1}xy = x^{-1+2^{r-1}}\rangle, \quad r \geqslant 3, s \geqslant 3.$$

Proof. Suppose that P is not Redei. By the preceding proposition, P is metacyclic. By [7, Satz III, 11.2]

$$P = \langle x, y \mid x^{p^r} = 1, y^{p^s} = x^{p^t}, y^{-1}xy = x^k\rangle,$$

where $t \geqslant 0$, $k^{p^s} \equiv 1 \pmod{p^r}$ and $p^t(k-1) \equiv 0 \pmod{p^r}$.

Let A be the abelian subgroup of index p in P. Then either $x \notin A$ or $y \notin A$. Suppose that $x \notin A$. Then $x^p \in Z(P)$ by Lemma 1.5(i). This means that $x^p = y^{-1}x^py = x^{kp}$, so $k \equiv 1 \pmod{p^{r-1}}$. Then also $k^p \equiv 1 \pmod{p^r}$, whence $y^p \in Z(P)$. This implies that $|P : Z(P)| = |P : \langle x^p, y^p\rangle| = p^2$, so P is Redei by Lemma 1.8, a contradiction. Consequently $x \in A$, so $y \notin A$. Thus $y^p \in Z(P)$. This implies that for p odd $k = 1 + up^{r-1}$ for $0 \leqslant u < p - 1$. Again $x^p \in Z(P)$, a contradiction. This proves (i). Let $p = 2$. Then we may choose k as -1, $-1 + 2^{r-1}$ or $1 + 2^{r-1}$. In the last case P is Redei. In the first two cases only $x^{2^{r-1}}$ is centralized by y, so $t = r$ or $r - 1$. In each of the cases, $t = r$ and $t = r - 1$, we have two possibilities for k, giving the four cases listed in (ii).

Finally, a rather special result, which is needed in Section 2.

LEMMA 1.13. *Let $P \in K_2'$. Then* Aut(P) *is a 2-group, unless P is Redei of type* (ii) *with $r = s$ or P is quaternion of order* 8. *In these cases* Aut(P) *contains an automorphism of order* 3, *which centralizes* $[P, P]$ *and acts fixed point free on* $P/[P, P]$.

Proof. By Proposition 1.4, $|P : \Phi(P)| = 4$, so by a result of P. Hall [7, Satz III, 3.19], $|\mathrm{Aut}(P)|_{2'} = 1$ or 3. Suppose that the last case occur, $\varphi \in \mathrm{Aut}(P)$, $|\varphi| = 3$. φ permutes the three maximal subgroups of P cyclically, so in particular they are all abelian, i.e., P is Redei. Suppose that P is not quaternion of order 8. If P is Redei of type (i), the maximal subgroups of P are abelian of type $(2^r, 2^{s-1})$, $(2^{r-1}, 2^s)$ and $(2^{\max(r,s)}, 2^{\min(r,s)-1})$. Then we must have $r = s$. Thus $P/[P, P]$ is abelian of type $(2^{r-1}, 2^r)$ and $|[P, P]| = 2$. But then $\mathrm{Aut}(P)$ is a 2-group, because $\mathrm{Aut}(P/[P, P])$ and $\mathrm{Aut}([P, P])$ are 2-groups, a contradiction. So P is Redei of type (ii). Again we see, that $r = s$. Then we can define φ by $\varphi(x) = y$, $\varphi(y) = x^{-1}y^{-1}$. $P/[P, P]$ is abelian of type $(2^r, 2^r)$, so any automorphism on $P/[P, P]$ of order 3 must be fixed point free.

It should be mentioned that all generalized Redei p-groups have been classified by D. Rocke.

2. The Subsections

Let G be a finite group and B be a 2-block of G having a nonabelian defect group D, which is a generalized Redei 2-group. Then it is possible to list a set S of representatives for the conjugacy classes of subsections for B using [5, 6C]. The determination of S is somewhat more difficult than in [4, 8], mainly because some special cases have to be considered. We also calculate $|S|$ in all cases, since $|S|$ is a lower bound for the number $k(B)$ of ordinary irreducible characters in B. In most cases there is a natural one-to-one correspondance between the set of conjugacy classes of D and the set S. The author would like to conjecture, that in many cases $|S| = k(B)$, and therefore $l(B) = 1$, i.e., B has only one modular irreducible character.

The notation is as in [4, 8].

Let us note the following properties:

Lemma 2.1. *Let $D \in K_2'$. Then*

(i) *D has an abelian subgroup A of index 2.*

(ii) *Any subgroup of D belongs to K_2.*

(iii) *$|D : \Phi(D)| = 4$, where $\Phi(D)$ is the Frattini subgroup.*

(iv) *If $u \in D - A$, then $u^2 \in Z(D)$ and $C_D(u) = \langle u, Z(D)\rangle$. So for any $u \in D - Z(D)$ $C_D(u)$ is abelian and in fact $C_D(u) \in \mathscr{M}_D$.*

(v) *D has three conjugacy classes of maximal abelian subgroups.*

Lemma 2.2. *The structure of the maximal abelian subgroups of elements in K_2':*

$R_1(r, s)$: $(2^r, 2^{s-1})$, $(2^{r-1}, 2^s)$, $(2^{\max(r,s)}, 2^{\min(r,s)-1})$, *center*: $(2^{r-1}, 2^{s-1})$.

$R_2(r, s)$: $(2^r, 2^{s-1}, 2)$, $(2^r, 2^{s-1}, 2)$, $(2^{r-1}, 2^s, 2)$, *center*: $(2^{r-1}, 2^{s-1}, 2)$.

$P_1(r, s)$: $(2^r, 2^{s-1})$, $(2, 2^s)$, $(2, 2^s)$, *center*: $(2, 2^{s-1})$.

$P_2(r, s)$: $(2^r, 2^{s-1})$, $(2, 2^s)$, $(2, 2^s)$ *for* $s > 1$, (4) *for* $s = 1$, *center*: $(2, 2^{s-1})$.

$P_3(r, s)$: $(2^{\max(r,s-1)}, 2^{\min(r,s-1)-1})$, (2^s), (2^s), *center*: (2^{s-1}).

$P_4(r, s)$: *As* $P_3(r, s)$.

We assume in the rest of this section that B is a 2-block for the finite group G with a defect group D of order 2^n which belongs to K_2' and that b is a root of B in $DC_G(D)$.

LEMMA 2.3. *Any double chain for B is special. In particular,*

$$\mathcal{O}\!\ell(D, b) = \mathcal{O}\!\ell_0(D, b).$$

Proof. This follows from [8, (1.8)] and the definition of a generalized Redei group, since $\mathcal{O}\!\ell(D, b) \subseteq \mathcal{M}_D$.

LEMMA 2.4. *$\mathcal{O}\!\ell(D, b) = \mathcal{M}_D$, except possibly when $D \simeq P_3(r, 2)$ (D generalized quaternion) or $D \simeq P_2(r, 1)$ (D quasi-dihedral). Then $\mathcal{O}\!\ell(D, b)$ need not contain the self-centralizing cyclic subgroups of order* 4.

Proof. If D is a Redei group this follows from [8, (1.7)]. So suppose $D \simeq P_i(r, s)$. Then any nonabelian subgroup of D has a 2-group as an automorphism group, except if D has a quaternion subgroup of order 8, by Lemma 1.13. This happens only for $P_2(r, 1)$ and $P_3(r, 2)$. Let $\mathcal{R}$ be the set of subgroups of D which are nonabelian and not quaternion of order 8. $\mathcal{R}$ satisfies the conditions of [8, (1.9)]. Consequently $\mathcal{R} \subseteq \mathcal{O}\!\ell_0(D, b)$ and $T(b_R) = N_D(R)\, C_G(R)$ for all $R \in \mathcal{R}$.

Let $R \in \mathcal{R}$ and let Q be a subgroup of R of index 2, such that $C_D(Q) \subseteq Q$. We show that $Q \in \mathcal{O}\!\ell(D, b)$, which will finish the proof. By [3, (3D)] it will suffice to show

$$N_D(R)\, C_G(R) \cap RC_G(Q) = RC_G(R).$$

This follows from an elementary consideration using that $C_R(Q) \subseteq Q$. The exceptional cases in Lemma 2.4 are described in [8, (2.4)].

LEMMA 2.5. *Assume that* $\mathrm{Aut}(D)$ *is a 2-group. Then if $U, V \in \mathcal{O}\!\ell(D, b)$ we have $U \approx V$ if and only if U and V are conjugate in D. Here $\approx$ denotes strong conjugacy.*

Proof. Let

$$(D, b) = (D_0, b_0), (D_1, b_1), \ldots, (D_r, b_r) = (U, b_U)$$

be the double chain from D to U. We may assume $r \geqslant 1$. Since the above double chain is special by Lemma 2.3, U and V are conjugate in the complex $T = T(b_{r-1}) \cdots T(b_0)$. We show that

$$\text{For } 0 \leqslant i \leqslant r-1\colon T(b_i) = D_{i-1}C_G(D_i) = N_D(D_i)\, C_G(D_i);\ (D_{-1} = D_0). \quad (*)$$

If $\mathrm{Aut}(D_i)$ is a 2-group, this is easily verified, using that the chain is special and that $T(b_i)/C_G(D_i)$ is isomorphic to a subgroup of $\mathrm{Aut}(D_i)$.

If $\mathrm{Aut}(D_i)$ is not a 2-group, then $i = r - 1$ and D_i is quaternion of order 8. However, then the double chain from D to D_{r-1} does not stop, so by (2.6) condition $(*)$ is fulfilled.

From $(*)$ we get immediately that $T = C_G(D_{r-1})D$, so $U \sim_D V$.

The proof of the next result is similar to [8, (2.6)] and we omit it.

LEMMA 2.6. *Suppose that* $\mathrm{Aut}(D)$ *is not a 2-group, i.e.,* $D \cong R_2(r, r)$ *or* D *quaternion of order* 8. *Then* $|T(b) : DC_G(D)|$ *is* 1 *or* 3, *and the following conditions are equivalent*:

(i) $|T(b) : DC_G(D)| = 3$.

(ii) *All three maximal subgroups of* D *are strongly conjugate.*

(iii) *Two maximal subgroups are strongly conjugate.*

Next a general lemma.

LEMMA 2.7. *Let* A *be an abelian* 2-*group of type* $(2^{i_1}, \dots, 2^{i_k})$. *Then* $\mathrm{Aut}(A)$ *is a* 2-*group if and only if* $i_1, \dots, i_k$ *are all different.*

Proof. A homocyclic 2-group of type $(2^i, 2^i)$, $i \geqslant 1$ has an automorphism of order 3, so the only-if part is true. Suppose $i_1, \dots, i_k$ are all different and that α is an $2'$-automorphism of A. By [6, Theorem 5.2.2], A is a direct product of homocyclic subgroups each admitting α. By assumption they must be cyclic, so α acts trivially on them. Thus $\alpha = 1$.

LEMMA 2.8. *Assume that* D *is not isomorphic to* $R_2(r, r - 1)$, $R_2(r, 2)$ *or* $R_1(2, 1)$. *If* Q *is an abelian subgroup of index* 2 *in* D, *then* $T(b_Q) = DC_G(Q)$.

Proof. By Lemma 2.3 $|T(b_Q) : DC_G(Q)|$ is odd. Thus, if $\mathrm{Aut}(Q)$ is a 2-group, the lemma is proved. In Lemma 2.2 we can see in which cases $\mathrm{Aut}(Q)$ is not a 2-group, using Lemma 2.7. It can be easily checked, that since we are excluding $R_2(r, r - 1)$, $R_2(r, 2)$, and $R_1(2, 1)$, $[D, D] = [D, Q] \subseteq \Phi(Q)$, the Frattini subgroup. Thus D acts trivially on $Q/\Phi(Q)$. Since any automorphism of odd order of Q acts nontrivially on $Q/\Phi(Q)$, we deduce that $DC_G(Q) \lhd T(b_Q)$. Let us now note that

$$Z(D) = \{u \in Q \mid |C_G(u) \cap T(b_Q) : C_G(Q)| \text{ is even}\}. \qquad (*)$$

Clearly, the inclusion $\subseteq$ holds. On the other hand, if u belongs to the right-hand side, then $|T(b_Q) : C_G(u) \cap T(b_Q)|$ is odd, and since $DC_G(Q)/C_G(Q)$ is the only Sylow 2-subgroup of $T(b_Q)/C_G(Q)$, we get $D \subseteq C_G(u)$, so $u \in Z(D)$.

Now $(*)$ shows that $Z(D)$ is $T(b_Q)$-invariant. However, by inspection we see that neither $Q/Z(D)$ nor $Z(D)$ are homocyclic, using Lemma 2.2. Consequently $T(b_Q)/C_G(Q)$ is a 2-group by [6, Theorem 5.3.2], and the lemma is proved.

LEMMA 2.9. *If $D \cong R_2(r, 2)$, $r \geqslant 4$ the conclusion of Lemma 2.8 still holds. It also holds if $D \cong R_2(3, 2)$, if Q is of type $(2^3, 2, 2)$.*

Proof. Assume that Q is of type $(2^r, 2, 2)$, $r \geqslant 3$, and that $|T(b_Q) : DC_G(Q)| = 3$. Since $[D, Q]$ is not contained in $\Phi(Q)$, we get that $T(b_Q)/C_G(Q)$ is dihedral of order 6. Let $\rho \in T(b_Q) - C_G(Q)$, such that $|\rho C_G(Q)| = 3$ (as element of $T(b_Q)/C_G(Q)$). By [6, Theorem 5.2.2] we may write $Q = Q_1 \times Q_2$, where Q_i is ρ-invariant, Q_1 is elementary abelian of order 4 and Q_2 is cyclic of order 2^r. We note that $C_Q(\rho) = Q_2$. Let $v \in D - Q$. Then $Q_2{}^v = C_Q(\rho^v) = C_Q(\rho) = Q_2$, since $\rho^v \equiv \rho^{-1} \pmod{C_G(Q)}$. Thus $Q_2 \lhd D$. This is a contradiction, since D has no normal cyclic subgroup of order 2^r.

LEMMA 2.10. *Let Q be an abelian subgroup of index 2 in D. The following conditions are equivalent: Let $u \in Q$.*

(i) $e_u := |T(b_Q) \cap C_G(u) : C_G(Q)|$ *is even.*

(ii) *u is conjugate in $T(b_Q)$ to an element of $Z(D)$.*

Proof. (ii) $\Rightarrow$ (i): Suppose $u^t \in Z(D)$, $t \in T(b_Q)$. Then

$$e_u = |T(b_Q) \cap C_G(u) : C_G(Q)| = |T(b_Q) \cap C_G(u^t) : C_G(Q)|.$$

However, $D \subseteq T(b_Q) \cap C_G(u^t)$, so e_u is even.

(i) $\Rightarrow$ (ii): Assume e_u even. Put $M = T(b_Q)$. Then $b^* = b_Q^{M \cap C_G(u)}$ has a larger defect group than Q by [3, (6A)]. Let D_1 be a defect group for b^*. Then D_1 is also a defect group for $(b^*)^M = b_Q{}^M$. Since D is also a defect group for $b_Q{}^M$, we get $D_1 \sim_M D$. Since $u \in Z(D_1)$, u is M-conjugate to an element in $Z(D)$.

PROPOSITION 2.11. *Suppose D is* not *one of the following groups:*

$$P_1(r, 1), P_2(r, 1), P_3(r, 2), R_1(2, 1), R_2(r, r-1), r \geqslant 2, R_2(r, r), r \geqslant 2, \text{ quaternion order } 8.$$

Then we can construct a set S of representatives for the conjugacy classes of subsections for B as follows: Let K be a complete set of representatives for the conjugacy classes of elements in D. Then

$$S = \bigcup_{x \in K} \{(x, b_{C_D(x)}^{C_G(x)})\}.$$

If $|D| = 2^n$ and $|Z(D)| = 2^a$, then $|K| = |S| = 2^{n-2} + 3 \cdot 2^{a-1}$.

Proof. Since we have made some exclusions in the statement of the proposition, we have the following:

(i) $\mathcal{M}_D = \mathcal{O}(D, b)$ (Lemma 2.4).

(ii) For all $Q \in \mathcal{M}_D : T(b_Q) = N_D(Q)\, C_G(Q)$ (Lemmas 2.9 and 2.10 and the proof of Lemma 2.4).

For $P_2(r, 1)$ and $P_3(r, 2)$, (i) may fail and for the rest (and $P_2(r, 1)$) (ii) may fail (cf. Lemma 2.2). If Q is any proper nonabelian subgroup of D, then Q has three maximal subgroups by Lemma 2.1, and two of these are conjugate in $N_D(Q)$. Moreover, Q is contained in a unique maximal subgroup of D. An inductive argument shows that two nonabelian subgroups of the same order lying in the same maximal subgroup of D are conjugate in D. Let A, A_1, and A_2 be representatives for the conjugacy classes of maximal abelian subgroups of D. $|D : A| = 2$. Write $|D : A_i| = 2^k$, and define subgroups M_{ij} of D, $i = 1, 2$, $j = 1,..., k$, as follows: $M_{ik} = A_i$, $M_{ij} = N_D(M_{i,j+1})$, $j = 1,..., k-1$, $i = 1, 2$. Then by Lemma 2.5

$$W = \{D, A, M_{ij}, i = 1, 2, j = 1,..., k\}$$

is a set of representatives for the strong conjugacy classes of elements in $\mathcal{O}_0(D, b)$.

We apply [5, (6C)]. For $Q \in W$ we determine a set I_Q for the $T(b_Q)$-conjugacy classes of the set

$$\{u \in Z(Q) \mid e_u = |T(b_Q)\, C_G(u) : QC_G(Q)| \text{ is odd}\}.$$

For any $Q \in W$ we have $T(b_Q) = N_D(Q)\, C_G(Q)$ by (ii) above. If $Q = D$ then obviously $I_Q = Z(D)$.

$Q = A$: By Lemma 2.10, e_u is odd if and only if $u \in A - Z(D)$. So I_A can be chosen as a set of representatives for the D-conjugacy classes in $A - Z(D)$.

$Q = M_{ij}$, $i = 1, 2$, $1 \leqslant j \leqslant k - 1$: Then M_{ij} is nonabelian and $Z(M_{ij}) = Z(D)$. So for any $u \in Z(M_{ij})$ e_u is even. I_Q is empty.

$Q = M_{ik}$: For elements in $Q - Z(D)$ e_u is odd. We can choose I_Q as a set of representatives for $M_{i,k-1}$-conjugacy classes of $M_{ik} - Z(D)$. $(M_{i0} = D)$. Put $I = \bigcup_{Q \in W} I_Q$. Let us show that any element of D is D-conjugate to exactly one element in I. This is clear for elements in $Z(D) = I_D$. Let $u \in D - Z(D)$. By Lemma 2.1, $C_D(u)$ is abelian and conjugate in D to A, M_{1k}, or M_{2k}. So u is D-conjugate to an element in one of these groups, i.e., to an element of I. So we need only show that $|I|$ is the number of conjugacy classes in D. Let $|D| = 2^n$, $|Z(D)| = 2^a$. Then

$$|I| = 2^a + \tfrac{1}{2}(2^{n-1} - 2^a) + 2 \cdot \tfrac{1}{2}(2^{a+1} - 2^a) = 2^{n-2} + 3 \cdot 2^{a-1}.$$

On the other hand, we may write D as a disjoint union of subsets

$$D = Z(D) \cup (A - Z(D)) \cup (M_{11} - \Phi(D)) \cup (M_{21} - \Phi(D)).$$

Each of these subsets is D-invariant. For $x \in M_{i1} - \Phi(D)$ $|C_D(x)| = 2^{a+1}$, so the set $M_{i1} - \Phi(D)$ contains 2^{a-1} conjugacy classes. For $x \in A - Z(D)$, $C_G(x) = A$. So D contains

$$2^a + \tfrac{1}{2}(2^{n-1} - 2^a) + 2^a = 2^{n-2} + 3 \cdot 2^{a-1} = |I|$$

conjugacy classes. This proves the proposition.

In the following results S is again a set of representatives for the conjugacy classes of subsections for B.

PROPOSITION 2.12. *Suppose $D \cong R_2(r, r)$ or D is quaternion of order* 8.

(i) *If $|T(b) : DC_G(D)| = 1$, then S can be chosen as in Proposition* 2.11.

(ii) *If $|T(b) : DC_G(D)| = 3$, then let A be a maximal subgroup in D. Let K be a set of D-conjugacy classes in A. Then*

$$S = \bigcup_{x \in K} \{(x, b_{C_D(x)}^{C_G(x)})\}.$$

If $|D| = 2^n$, then $|S| = 2^{n-1} - 2^{n-3}$.

Proof. (i) is proved like Proposition 2.11. (ii): By Lemma 2.6, $W = \{D, A\}$. Again $I_D = Z(D)$. By Lemma 2.10, I_A is a set of representatives for D-conjugacy classes in $A - Z(D)$. The result follows.

PROPOSITION 2.13. *Suppose $D \cong R_2(r, r-1)$, $r \geqslant 3$. Let A be the subgroup of type $(2^{r-1}, 2^{r-1}, 2)$.*

(i) *If $T(b_A) = DC_G(A)$, then S can be chosen as in Proposition* 2.11.

(ii) *If $|T(b_A) : DC_G(A)| = 3$, let K be a set of representatives for the D-conjugacy classes in $(D - A) \cup Z(D)$. Then S can be chosen similar to the preceding propositions.*

Proof. (i) is proved as Proposition 2.11. For $r = 3$ we apply Lemma 2.9.

(ii): If A_1 and A_2 are the maximal subgroups in D different from A, then $W = \{D, A, A_1, A_2\}$. From the subgroups different from A we get subsections as in Proposition 2.11. Let us show that there are no subsections from A. Let $\rho \in T(b_A) - C_G(A)$, $|\rho C_G(A)| = 3$ in $T(b_A)/C_G(A)$. We can write $A = BC$, where B is homocyclic of type $(2^{r-1}, 2^{r-1})$, $|C| = 2$ and B and C are ρ-invariant. Since $C \subseteq Z(D)$, $Z = Z(D) \cap B$ is of index 2 in B. Z is not ρ-invariant, because then ρ would stabilize the normal series $A \supseteq B \supseteq Z$ of A, contrary to [6, Theorem 5.3.2]. Consequently $B = Z \cup Z^\rho \cup Z^{\rho^2}$, so $A = Z(D) \cup Z(D)^\rho \cup Z(D)^{\rho^2}$. Thus no element in A satisfies the inertial condition by Lemma 2.10.

PROPOSITION 2.14. *Let $D \simeq R_2(2, 1)$ and let A be the elementary subgroup of order* 8. *Let $e_A = |T(b_A) : DC_G(A)|$. Then $e_A = 1$, 3 or 7. If $e_A = 1$, the conclusion of Proposition* 2.11 *holds. If $e_A = 3$ or 7 the conclusion of Proposition* 2.13(ii) *holds.*

The proof is similar to that of Proposition 2.15 and we omit it.

The only cases left are D dihedral ($P_1(r, 1)$, $R_2(2, 1)$) or generalized quaternion ($P_3(r, 2)$) or quasidihedral ($P_2(r, 1)$). These have been handled in [4, 8]. Thus the subsections have been determined in all cases.

One may try to continue the analysis to determine explicitly the number of characters in 2-blocks with a generalized Redei defect group. But apparently serious difficulties arise. In the analysis of the decomposition numbers too many cases have to be considered. Perhaps a different approach to the problem would be useful. The author has been able to prove results of the following type:

Suppose $D \simeq P_1(r, s)$, where $1 \leqslant s - 1 < r$. Let, in the presentation of Proposition 1.12, $A = \langle x, y^2 \rangle$ and $z = y^{2^{s-1}}$. Assume that x is not fused to $x^i z$ in $N_G(A)$ for any i. Then

$$k(B) = 2^{r+s-2} + 3 \cdot 2^{s-1}, \qquad l(B) = 1$$
$$k_0(B) = |D : [D, D]| = 2^{s-1}, \qquad k_1(B) = k(B) - k_0(B).$$

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