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Pontryagin Type Conditions for Differential Inclusions with Free Time

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Optimality conditions for differential inclusion problems, due to Kaskosz and Lojasiewicz, involve a costate equation and a pointwise maximizing property of the optimal velocity, expressed in terms of a *Carathéodory selection* of the differential inclusion. Such conditions have been extended in various directions, notably to permit unilateral state constraints. Here we add to earlier extensions, principally by allowing free endtimes. This is accomplished even though the data are required to be merely measurable in the time variable. The results are obtained by applying recent optimality conditions for free time problems, involving a Hamiltonian inclusion, to an auxiliary problem and a simple limiting argument. © 1992 Academic Press, Inc.

1. INTRODUCTION

Consider the following dynamic optimization problem:

$$\begin{aligned}&\text{Minimize } g(a, x(a), b, x(b))\\&\text{over arcs } x(\cdot)\in AC([a, b]; \mathbb{R}^n) \text{ satisfying}\\&\quad \dot{x}(t)\in F(t, x(t)) \qquad \text{a.e. } [a, b],\\&\quad (a, x(a), b, x(b))\in S,\\&\quad h(t, x(t))\leqslant 0, \qquad \forall t\in [a, b]\cap J.\end{aligned} \tag{P}$$

The data involved are a multifunction $F: \mathbb{R}\times\mathbb{R}^n \rightrightarrows \mathbb{R}^n$, a function $h: \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, and sets $S\subset\mathbb{R}^{1+n+1+n}$ and $J\subset\mathbb{R}$. The problem is a *free time problem*; i.e., the initial and terminal times, a and b, are included among the choice variables. The arcs are subject to endpoint and unilateral state constraints.

Let $z(\cdot)\in AC([\alpha, \beta]; \mathbb{R}^n)$, with $\beta>\alpha$, be a local minimizer for (P). This means that $z(\cdot)$ satisfies the constraints of (P) and that there exists a $w>0$ such that

$$g(\alpha, z(\alpha), \beta, z(\beta))\leqslant g(a, x(a), b, x(b))$$

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for any arc $x(\cdot) \in AC([a, b]; \mathbb{R}^n)$ also satisfying the constraints of (P) and for which

$$|a-\alpha| < w, \qquad |b-\beta| < w,$$

and

$$x(t) \in z(t) + wB, \qquad \text{for all} \quad t \in [\alpha - w, \beta + w]. \tag{1.1}$$

Here B is the open unit ball in $\mathbb{R}^n$. In interpreting conditions such as (1.1), we regard $z(\cdot)$ and $x(\cdot)$ as having been extended to all of $[\alpha - w, \beta + w]$ by constant extrapolation.

The arc $z(\cdot) \in AC([\alpha, \beta]; \mathbb{R}^n)$ and the parameter $w > 0$ will remain fixed in what follows. Our aim is to give conditions satisfied by the local minimizer $z(\cdot)$ under hypotheses (H1)–(H5) below. These make reference to the sets

$$\Omega = \{(t, x) \in \mathbb{R} \times \mathbb{R}^n : t \in (\alpha - w, \beta + w), |x - z(t)| < w\}.$$

$$\Omega_t = \{x \in \mathbb{R}^n : (t, x) \in \Omega\} = z(t) + wB.$$

$$D = \{(a, b, x\ y) \in \mathbb{R}^{2n+2} :$$

$$|a-\alpha| < w, |b-\beta| < w, |x - z(\alpha)| < w, |b - z(\beta)| < w\}.$$

(H1) *For each (t, x) in Ω, the set $F(t, x)$ is non-empty, compact, and convex.*

(H2) *For each $(t, x) \in \Omega$ the multifunction $s \to F(s, x)$ is measurable on some neighborhood of t; also, there is a non-negative $\psi_F \in L^1(\alpha - w, \beta + w)$ which is essentially bounded on $(\alpha - w, \alpha + w) \cup (\beta - w, \beta + w)$ and obeys*

$$F(t, x) \subset \psi_F(t)\, B, \qquad \forall (t, x) \in \Omega.$$

(H3) *There is a non-negative $K_F \in L^1(\alpha - w, \beta + w)$ which is essentially bounded on $(\alpha - w, \alpha + w) \cup (\beta - w, \beta + w)$ and obeys*

$$F(t, y) \subset F(t, x) + K_F(t)\, |y - x|\, B, \qquad \forall t \in (\alpha - w, \beta + w), \forall x, y \in \Omega_t.$$

(H4) *The function $h\colon \Omega \to \mathbb{R}$ is continuous and the set of times, J, where the state constraint applies is closed. There is a constant $K_h \geq 0$ such that*

$$|h(t, x) - h(t, y)| \leq K_h\, |x - y|, \qquad \forall t \in (\alpha - w, \beta + w), \forall x, y \in \Omega_t.$$

(H5) *The function $g\colon D \to \mathbb{R}$ is Lipschitz of rank K_g on D. The endpoint constraint set, $S \subset \mathbb{R}^{2n+2}$, is closed.*

The background to our investigations is now briefly reviewed. Consider first of all the special case of (P) where the endtimes are fixed, i.e., it is assumed that the endpoint constraint set S has the form

$$S = \{(\alpha, \zeta, \beta, \eta) : (\zeta, \eta) \in C\},$$

for some $[\alpha, \beta] \subset \mathbb{R}$, and closed $C \subset \mathbb{R}^{2n}$. Suppose also that the state constraint is absent. One of the first sets of optimality conditions given for such problems (*Hamiltonian inclusion conditions*) asserted the existence of a costate arc $p(\cdot)$ which satisfies certain transversality conditions and the *Hamiltonian inclusion*

$$(-\dot{p}(t), \dot{x}(t)) \in \partial H(t, x(t), p(t)), \qquad \text{a.e. } [\alpha, \beta]. \tag{1.2}$$

Here $H(\cdot, \cdot, \cdot)$ is the Hamiltonian

$$H(t, x, p) := \operatorname*{Sup}_{e \in F(t,x)} \langle p, e \rangle,$$

and ∂H denotes the (Clarke) generalized gradient in the (x, p) variables.

In [4], Kaskosz and Lojasiewicz proved a different kind of optimality condition for (P). Here the Hamiltonian inclusion is replaced by conditions involving a *Carathéodory selection*, $\phi(\cdot, \cdot)$, of $F(\cdot, \cdot)$ (defined below) for which $\dot{z}(t) = \phi(t, z(t))$. The conditions are

$$-\dot{p}(t) \in \langle p(t), \partial_x \phi(t, z(t)) \rangle, \qquad \text{a.e. } [\alpha, \beta],$$

$$\langle p(t), \phi(t, z)) \rangle = \operatorname*{Max}_{e \in F(t, z(t))} \langle p(t), e \rangle, \qquad \text{a.e. } [\alpha, \beta]. \tag{1.3}$$

Because condition (1.3) has something of the flavour of the maximum condition in the Pontryagin maximum principle of optimal control, the Kaskosz/Lojasiewicz conditions have been called conditions of *Pontryagin type*.

Pontryagin type conditions do not supercede Hamiltonian inclusion conditions but supplement them in a nontrivial way. Examples exist where arcs exist satisfying the constraints, for which the Pontryagin type conditions are satisfied but the Hamiltonian inclusion conditions are not (see [4]).

In [5] Loewen and Vinter showed that the Pontryagin type conditions could be derived by an application of the Hamiltonian inclusion conditions to an auxiliary problem, and a simple limiting argument. Subsequently, Frankowska and Kaskosz [3] showed how the techniques of [4] could be adapted to admit state constraints. Warga [7] independently derived Pontryagin type conditions in the presence of state constraints, for reparameterizations of optimal control problems.

All developments described thus far concern problems on a fixed time interval. By contrast, the Pontryagin type conditions we develop below relate to problems with *free endtimes*. What obstacles are there to such an extension? A well-known transformation technique [8] reduces a problem with free endtimes to a fixed endtime problem; application of standard fixed time optimality conditions to the transformed problem then provides a means of deriving optimality conditions for the original problem. This technique suffers from the shortcoming that the data must be "regular" (e.g., locally Lipschitz continuous) in the time variable as well as the state variable, due to the fact that the old time variable becomes a component of the new state variable. It is therefore of no avail in the present context, where the data are assumed to be merely *measurable* in the time variable. We must look elsewhere then for formulation and proof of the necessary conditions.

The simple idea underlying the proof of the Pontryagin type conditions in [5] is that, if $\phi(\cdot, \cdot)$ is a Carathéodory selection for which $\dot{z} = \phi(t, z)$, then $z(\cdot)$ remains a local minimizer when the dynamic constraint is replaced by

$$\dot{x}(t) \in \phi(t, x(t)) + \varepsilon\{F(t, x(t)) - \phi(t, x(t))\}, \qquad \text{a.e. } [a, b].$$

Here $\varepsilon \in (0, 1)$ is a parameter. We apply Hamiltonian inclusion necessary conditions, with reference to the new dynamics. Passage to the limit as $\varepsilon \downarrow 0$ gives the Pontryagin type conditions for the original problem.

Now this approach can be followed simply to derive Pontryagin type versions of known necessary conditions in a variety of settings. Here we provide a further illustration; we derive Pontryagin type conditions for problems with state constraints and free endtimes, where the data are assumed merely measurable in the time variable, by applying recent free time Hamiltonian inclusion necessary conditions [2] to the auxiliary problem.

The transversality condition in our optimality conditions involves the *essential value* of a function: given a measurable function $f: \mathbb{R} \times \mathbb{R}^n$ and a point $\tau \in \mathbb{R}$, we define the set of essential values of f at τ, denoted $\text{ess}_{t \to \tau} f(t)$, as

$$\underset{t \to \tau}{\text{ess}}\, f(t) = \{ y \in \mathbb{R}^n : \forall \varepsilon > 0, \mathscr{L}\text{-meas}\{ t \in (\tau - \varepsilon, \tau + \varepsilon) : |f(t) - y| < \varepsilon \} > 0 \}.$$

The symbol $\partial g(x)$ denotes the (Clarke) generalized gradient of the locally Lipschitz function g at x. (See [1] for the definitions and calculus.) We employ a rather specialized generalized gradient in relation to the state constraint function $h(\cdot, \cdot)$. It is

$$\partial_x^+ h(t, x) := \text{co}\{ \eta = \lim_{i \to \infty} \gamma_i : \gamma_i \in \partial_x h(t_i, x_i), t_i \to t, x_i \to x, h(t_i, x_i) > 0 \; \forall i \}.$$

The Euclidean distance function to the set $C \subset \mathbb{R}^n$ is denoted by $d_C(\cdot)$. The (Clarke) normal cone to C is denoted by $N_C(\cdot)$.

2. Pontryagin Type Conditions

We give conditions on the local minimizer $z(\cdot)$. These are stated in terms of *Carathéodory selections* of the multifunction F.

Definition 2.1. A mapping $\phi: \Omega \to \mathbb{R}^n$ is called a *Carathéodory selection* of the multifunction $F: \Omega \rightrightarrows \mathbb{R}^n$ if the single valued multifunction $\bar{F}(\cdot, \cdot) = \{\phi(\cdot, \cdot)\}$ obeys the hypotheses (H1)–(H3) (with a possibly different Lipschitz constant) and satisfies the selection condition

$$\phi(t, x) \subset F(t, x), \qquad \forall (t, x) \in \Omega.$$

It is shown in [4] that there always exists a Carathéodory selection, ϕ, of the multifunction, F, such that $\dot{z} = \phi(t, z)$.

Theorem 2.1. *Let* $z(\cdot) \in AC([\alpha, \beta]; \mathbb{R}^n)$ *be a local solution to* (P). *Assume for each endtime,* $t = \alpha, \beta$ *either* $h(t, z(t)) < 0$, *or else the t-component of the endpoint constraint set, S, is the single point* $\{t\}$. *Pick any Carathéodory selection* ϕ *of F with the property that* $\dot{z}(t) = \phi(t, z(t))$ a.e. $[\alpha, \beta]$. *Then there exist a constant* $\lambda \leqslant 0$, *an absolutely continuous function* $p: [\alpha, \beta] \to \mathbb{R}^n$, *a measurable function* $\gamma: [\alpha, \beta] \to \mathbb{R}^n$, *a non-negative measure* $\mu \in C^*([\alpha, \beta]; \mathbb{R})$, *such that* $\lambda + |p|_\infty + \mu([\alpha, \beta]) = 1$ *and for all sufficiently large one has*

$$-\dot{p}(t) \in \partial_x H\left(t, z(t), p(t) + \int_{[\alpha, t)} \gamma \, d\mu\right), \qquad \text{a.e. } [\alpha, \beta], \tag{2.1}$$

$$H\left(t, z(t), p(t) + \int_{[\alpha, t)} \gamma \, d\mu\right) = \left\langle p(t) + \int_{[\alpha, t)} \gamma \, d\mu, \phi(t, z(t))\right\rangle, \quad \text{a.e. } [\alpha, \beta], \tag{2.2}$$

$$h \in \operatorname{co} \operatorname*{ess}_{t \to \alpha} H(t, z(\alpha), p(\alpha)), \tag{2.3}$$

$$k \in \operatorname{co} \operatorname*{ess}_{t \to \beta} H\left(t, z(\beta), p(\beta) + \int_{[\alpha, \beta)} \gamma \, d\mu\right), \tag{2.4}$$

$$\left(-h, k, p(\alpha), -p(\beta) - \int_{[\alpha, \beta]} \gamma \, d\mu\right) \in r \, \partial d_S(\alpha, \beta, z(\alpha), z(\beta)) + \lambda \, \partial g(\alpha, \beta, z(\alpha), z(\beta)), \tag{2.5}$$

$$\gamma(t) \in \partial_x^+ h(t, z(t)), \qquad \mu\text{-a.e. } [\alpha, \beta], \tag{2.6}$$

and

$$\operatorname{Supp}(\mu) \subset \{t \in [\alpha, \beta] \cap J : \partial_x^+ h(t, z(t)) \neq \varnothing\}. \tag{2.7}$$

In the case where the state constraint is either inactive along the arc $z(\cdot)$ or absent altogether the conclusions of the theorem remain valid with $\mu \equiv 0$, (and $\lambda + |p|_\infty = 1$). The fact that, here, $\mu \equiv 0$ follows from the fact that, if $h(t, x(t)) < 0$, then $\partial_x^+ h(t, x(t)) = \varnothing$.

The form of the state constraint may seem somewhat restrictive, but many constraints can be reduced to this form. Consider, for example, the constraint, $f(t, x(t)) \in C$, adopted in [7], in which $f\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^m$ is essentially bounded, $f(, x)$ is measurable for all x and $f(t, \cdot)$ is C^1 for all t. An equivalent formulation of this constraint is $h(t, x(t)) \leqslant 0$, where $h(t, x(t)) = d_C(f(t, x(t)))$. The set $\partial_x^+ h(t, x(t))$ is contained in the set $\{d \cdot f_x(t, x(t)) : d \in N_C(f(t, x(t)))\}$, and the latter set can be used instead of $\partial_x^+ h(t, x(t))$ in the necessary conditions.

3. Proof of Theorem 2.2

The starting point of the proof is the following free time necessary conditions due to Clarke, Loewen, and Vinter [2].

THEOREM 3.1. *Let $z(\cdot) \in \mathrm{AC}([\alpha, \beta]; \mathbb{R}^n)$ be a local solution to* (P). *Assume for each endtime, $t = \alpha, \beta$ either $h(t, z(t)) < 0$, or else the t-component of the endpoint constraint set, S, is the single point $\{t\}$. Then there exist constants $\lambda \geqslant 0$, h, and k, an absolutely continuous function $p\colon [\alpha, \beta] \to \mathbb{R}^n$, a measurable function $\gamma\colon [\alpha, \beta] \to \mathbb{R}^n$, a non-negative measure $\mu \in C^*([\alpha, \beta]; \mathbb{R})$, such that $\lambda + |p|_\infty + \mu([\alpha, \beta]) = 1$ and for r sufficiently large one has*

$$(-\dot{p}(t), \dot{z}(t)) \in \partial H\left(t, z(t), p(t) + \int_{[\alpha, t)} \gamma \, d\mu\right), \qquad \text{a.e. } [\alpha, \beta],$$

$$h \in \operatorname{co} \operatorname*{ess}_{t \to \alpha} H(t, z(\alpha), p(\alpha)),$$

$$k \in \operatorname{co} \operatorname*{ess}_{t \to \beta} H\left(t, z(\beta), p(\beta) + \int_{[\alpha, \beta)} \gamma \, d\mu\right),$$

$$\left(-h, k, p(\alpha), -p(\beta) - \int_{[\alpha, \beta]} \gamma \, d\mu\right) \in r\, \partial d_S(\alpha, \beta, z(\alpha), z(\beta)) + \lambda\, \partial g(\alpha, \beta, z(\alpha), z(\beta)),$$

$$\gamma(t) \in \partial_x^+ h(t, z(t)), \qquad \mu\text{-a.e. } [\alpha, \beta],$$

and

$$\operatorname{Supp}(\mu) \subset \{t \in [\alpha, \beta] \cap J : \partial_x^+ h(t, z(t)) \neq \varnothing\}.$$

The following stability property of the operation of taking essential values is easily proved, (cf. [2, Lemma 1.2]).

LEMMA 3.2. *Let* $k(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ *and* $k_i(t, x) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots$, *be functions which satisfy the following conditions. For some neighbourhoods, T and X of* τ *and* ξ, *respectively,*

(i) *They are essentially bounded on* $T \times X$.

(ii) $k(\cdot, x)$ *and* $k_i(\cdot)$ *are measurable on T for each* $x \in X$.

(iii) $k(t, \cdot)$ *and* $k_i(t, \cdot)$ *are continuous on X, uniformly in* $t \in T$.

(v) $k_i \to k$ *uniformly on this neighborhood.*

Let v *and* v_i, $i = 1, 2, \ldots$, *be numbers, and* ξ *and* ξ_i, $i = 1, 2, \ldots$, *be points in* $\mathbb{R}^n$ *satisfying* $v_i \in \operatorname{co} \operatorname{ess}_{t \to \tau} k_i(t, \xi_i)$, $v_i \to v$, $\xi_i \to \xi$. *Then* $v \in \operatorname{co} \operatorname{ess}_{t \to \tau} k(t, \xi)$.

We turn to the proof of Theorem 3.1. For each $\varepsilon > 0$, consider the multifunction

$$F_\varepsilon(t, x) = \phi(t, x) + \varepsilon\{F(t, x) - \phi(t, x)\},$$

and the optimization problem (P_ε), obtained from (P) by replacing F with F_ε. Note that F_ε inherits from F and ϕ the hypotheses (H1)–(H3) except that the Lipschitz constant this time will be $k_\varepsilon(t) = (1 - \varepsilon) K_\phi(t) + \varepsilon K_F(t)$, where $K_\phi(\cdot)$ is the Lipschitz constant associated with the single valued multifunction $\{\phi(\cdot, \cdot)\}$. The convexity condition on F, together with the selection condition imposed on ϕ, implies that on Ω we have $F_\varepsilon(t, x) \subset F(t, x)$. Having noted this, consider the optimization problem (P_ε), obtained from (P) by replacing F with F_ε. Since $F_\varepsilon \subset F$, arcs admissible for (P_ε) are also admissible for (P), so that $\inf(\mathrm{P}_\varepsilon) \geqslant \inf(\mathrm{P})$. Note also that $z(\cdot)$ is admissible for (P_ε), and since inf(P) is obtained at $z(\cdot)$, $\inf(\mathrm{P}_\varepsilon)$ is obtained there also. We apply Theorem 3.1 to (P_ε) to deduce the following.

There exist constants $\lambda_\varepsilon \geqslant 0$, h_ε, and k_ε, an absolutely continuous function $p_\varepsilon : [\alpha, \beta] \to \mathbb{R}^n$, a measurable function $\gamma \in C^*([\alpha, \beta]; \mathbb{R})$, such that $\lambda_\varepsilon + |p_\varepsilon|_\infty + \mu_\varepsilon([\alpha, \beta]) = 1$ and for r sufficiently large one has

$$(-\dot{p}_\varepsilon(t), \dot{z}(t)) \in \partial H_\varepsilon\left(t, z(t), p_\varepsilon(t) + \int_{[\alpha, t)} \gamma_\varepsilon \, d\mu_\varepsilon\right), \qquad \text{a.e. } [\alpha, \beta], \tag{3.1}$$

$$h_\varepsilon \in \operatorname{co} \operatorname*{ess}_{t \to \alpha} H_\varepsilon(t, z(\alpha), p_\varepsilon(\alpha)), \tag{3.2}$$

$$k_\varepsilon \in \operatorname{co} \operatorname*{ess}_{t \to \beta} H_\varepsilon\left(t, z(\beta), p_\varepsilon(\beta) + \int_{[\alpha, \beta)} \gamma_\varepsilon \, d\mu_\varepsilon\right), \tag{3.3}$$

$$\left(-h_\varepsilon, k_\varepsilon, p_\varepsilon(\alpha), -p_\varepsilon(\beta)-\int_{[\alpha,\beta]}\gamma_\varepsilon\, d\mu_\varepsilon\right)\in r\,\partial d_S(\alpha, \beta, z(\alpha), z(\beta))$$

$$+\lambda\,\partial g(\alpha, \beta, z(\alpha), z(\beta)), \qquad (3.4)$$

$$\gamma_\varepsilon(t)\in\partial_x^+ h(t, z(t)), \qquad \mu_\varepsilon\text{-a.e. } [\alpha, \beta], \qquad (3.5)$$

and

$$\operatorname{Supp}(\mu_\varepsilon)\subset\{t\in[\alpha,\beta]\cap J : \partial_x^+ h(t, z(t))\neq\varnothing\}, \qquad (3.6)$$

where $H_\varepsilon(t, x, p) := (1-\varepsilon)\langle p, \phi(t, x)\rangle+\varepsilon H(t, x, p)$.

Expanding (3.1) we obtain

$$(-\dot{p}_\varepsilon(t), \dot{z}(t))\in(1-\varepsilon)\left[\left\langle p_\varepsilon(t)+\int_{[\alpha,t)}\gamma_\varepsilon\, d\mu_\varepsilon, \partial_x\phi(t, z(t))\right\rangle\times\{\phi(t, z(t))\}\right]$$

$$+\varepsilon\partial H\left(t, z(t), p_\varepsilon(t)+\int_{[\alpha,t)}\gamma_\varepsilon\, d\mu_\varepsilon\right), \qquad \text{a.e. } [\alpha, \beta]. \qquad (3.7)$$

Now apply [1, Proposition 3.2.4] to the second component of (3.7),

$$\varepsilon\phi(t, z(t))=\dot{z}(t)-(1-\varepsilon)\,\phi(t, z(t))\in\varepsilon\partial_p H\left(t, z(t), p_\varepsilon(t)+\int_{[\alpha,t)}\gamma_\varepsilon\, d\mu_\varepsilon\right). \qquad (3.8)$$

It follows that

$$H\left(t, z(t), p_\varepsilon(t)+\int_{[\alpha,t)}\gamma_\varepsilon\, d\mu_\varepsilon\right)$$

$$=\left\langle p_\varepsilon(t)+\int_{[\alpha,t)}\gamma_\varepsilon\, d\mu_\varepsilon, \phi(t, z(t))\right\rangle, \qquad \text{a.e. } [\alpha, \beta]. \qquad (3.9)$$

Applying the same proposition to the first component of (3.7) we deduce

$$-\dot{p}_\varepsilon(t)\in(1-\varepsilon)\left\langle p_\varepsilon(t)+\int_{[\alpha,t)}\gamma_\varepsilon\, d\mu_\varepsilon, \partial_x\phi(t, z(t))\right\rangle$$

$$+\varepsilon\left|p_\varepsilon(t)+\int_{[\alpha,t)}\gamma_\varepsilon\, d\mu_\varepsilon\right| k(t)\,\bar{B}. \qquad (3.10)$$

Let $\varepsilon=1/i$, and define

$$(\lambda_i, p_i(\cdot), \mu_i(\cdot), h_i, k_i) := \frac{(\lambda_\varepsilon, p_\varepsilon(\cdot), \mu_\varepsilon(\cdot), h_\varepsilon, k_\varepsilon)}{\lambda_\varepsilon+|p_\varepsilon(\alpha)|+\mu_\varepsilon([\alpha, \beta])}$$

and

$$(\gamma_i(\cdot) := \gamma_\varepsilon(\cdot).$$

We re-state (3.9) and (3.10) in terms of λ_i, $p_i(\cdot)$, $\mu_i(\cdot)$, h_i, k_i, and $\gamma_i(\cdot)$:

$$H\left(t, z(t), p_i(t) + \int_{[\alpha, t)} \gamma_i \, d\mu_i\right)$$

$$= \left\langle p_i(t) + \int_{[\alpha, t)} \gamma_i \, d\mu_i, \phi(t, z(t))\right\rangle, \qquad \text{a.e. } [\alpha, \beta], \tag{3.11}$$

$$-\dot{p}_i(t) \in \left\langle p_i(t) + \int_{[\alpha, t)} \gamma_i \, d\mu_i, \partial_x \phi(t, z(t))\right\rangle$$

$$+ \frac{1}{i} \left| p_i(t) + \int_{[\alpha, t)} \gamma_i \, d\mu_i \right| (K_F(t) + K_\phi(t)) \bar{B}. \tag{3.12}$$

Equations (3.4)–(3.6) can be re-stated in terms of λ_i, $p_i(\cdot)$, $\mu_i(\cdot)$, h_i, k_i, and $\gamma_i(\cdot)$ and look exactly as before except that "ε" is replaced by "i." (3.13)

We re-state (3.2) and (3.3).

$$k_i \in \underset{t \to \alpha}{\text{co ess}} \{\langle p_i(t), \phi(t, z(t))\rangle$$

$$+ \frac{1}{i} \{H(t, z(\alpha), p_i(\alpha)) - \langle p_i(t), \phi(t, z(t))\rangle\}\}, \tag{3.14}$$

$$h_i \in \underset{t \to \beta}{\text{co ess}} \left\{\left\langle p_i(t) + \int_{[\alpha, \beta)} \gamma_i \, d\mu_i, \phi(t, z(t))\right\rangle\right.$$

$$+ \frac{1}{i} \left\{H\left(t, z(\beta), p_i(\beta) + \int_{[\alpha, \beta)} \gamma_i \, d\mu_i\right)\right.$$

$$\left.\left. - \left\langle p_i(t) + \int_{[\alpha, \beta)} \gamma_i \, d\mu_i, \phi(t, z(t))\right\rangle\right\}\right\}. \tag{3.15}$$

Equations (3.11)–(3.15) are perturbed versions of the assertions of the theorem.

Convergence

Now since $\lambda_i + |p_i(\alpha)| + \mu_i([\alpha, \beta]) = 1$ for all i, the sequences $\{\lambda_i\}$ and $\{p_i(\alpha)\}$ converge (along subsequences which we shall not relabel) to $\lambda \in [0, 1]$ and $p(\alpha) \in \bar{B}$, respectively. Also, we have $\{\mu_i\}$ converges weak star (along another subsequence) to a measure μ and that $\mu_i([\alpha, \beta]) \to$

$\mu([\alpha, \beta])$, where μ is a non-negative measure in $C^*([\alpha, \beta]; \mathbb{R}^n)$. Since each μ_i has support contained in the set

$$\{t \in [\alpha, \beta] \cap J : \partial_x^+ h(t, z(t)) \neq \emptyset\},$$

it follows that the support of μ is contained in it.

The set valued map $t \to \partial_x^+ h(t, z(t))$ is uniformly bounded, convex, and of closed graph on Ω. Hence from [6, Lemma 4.5] we deduce that there exists a μ-integrable function $\gamma(\cdot)$: $[\alpha, \beta] \to \mathbb{R}^n$ satisfying

$$\gamma(t) \in \partial_x^+ h(t, z(t)), \qquad \mu\text{-a.e.},$$

and that for some subsequence $\gamma_i \, d\mu_i$

$$\gamma_i \, d\mu_i \xrightarrow{*} \gamma \, d\mu.$$

Lemma 4.3 of [6] allows us to assert further that

$$\int_{[\alpha, t)} \gamma_i \, d\mu_i \to \int_{[\alpha, t)} \gamma \, d\mu, \qquad \text{a.e. } [\alpha, \beta].$$

The bounded nature of $\partial_x^+ h$ and μ_i, μ implies that for some constant L,

$$\left| \int_{[\alpha, t)} \gamma_i \, d\mu_i \right|, \left| \int_{[\alpha, t)} \gamma \, d\mu \right| \leqslant L, \qquad \forall t \in [\alpha, \beta],$$

holds.

This together with (3.12) implies

$$|-\dot{p}_i(t)| \leqslant 2(k(t) + k_\phi(t)) \, |p_i(t)| + 2L(K_F(t) + K_\phi(t)) \, \bar{B}.$$

Recalling that $\{p_i(\alpha)\} \to p(\alpha) \in \bar{B}$, we deduce via Gronwall's lemma that the sequence $\{p_i(\cdot)\}$ is uniformly bounded and equicontinuous and thus converges (along a further subsequence) to an arc $p(\cdot)$. Theorem 3.1.7 of [1] applied to (3.12) allows us to deduce (2.1) for the same arc $p(\cdot)$.

Equation (2.4) follows directly from the limiting version of (3.4) and the remark (3.13). Equation (2.2) follows from the limiting version of (3.11). The essential value conditions (2.3) and (2.4) follow directly from (3.14), (3.15), and Lemma 3.2.

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