Nested Bounds for the Perron Root of a Nonnegative Matrix

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Dedicated to Alexander M. Ostrowski
on the occasion of his ninetieth birthday.

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ABSTRACT

It is well known that for a nonnegative matrix A, the smallest row sum $R'(A)$ and the largest row sum $R''(A)$ provide lower and upper bounds, respectively, for the Perron root of A. These bounds are generalized for a partitioned nonnegative matrix A. The new bounds are better than $R'(A)$ and $R''(A)$, and they can be further improved by a refinement of the partition. Known monotonicity and convergence properties of $R'(A)$ and $R''(A)$ are generalized for the new bounds.

1. INTRODUCTION AND NOTATIONS

It is very well known (see, for example, [9, p. 46]) that a nonnegative square matrix A has a nonnegative eigenvalue ρ, called the *Perron root* of A, which is greater than or equal to the absolute value of every other eigenvalue of A. To this eigenvalue ρ there corresponds a right (left) eigenvector with nonnegative components, called a *right (left) Perron vector* of A. Moreover, if A is irreducible, then the Perron root of A is positive, it has algebraic and geometric multiplicities equal to one, and every right (left) Perron vector of A (unique up to a positive multiplicative factor) has only positive components.

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We shall denote the Perron root of a nonnegative matrix X by $r(X)$, and the set of all right [left] Perron vectors of X by $\mathrm{RPV}(X)$ [$\mathrm{LPV}(X)$].

If X is a nonnegative $n\times n$ matrix, we shall denote the sum of the elements of the ith row of X (called the ith row sum of X) by $R_i(X)$ $(i=1,\dots,n)$, and we set

$$R'(X)=\min_i R_i(X),\qquad R''(X)=\max_i R_i(X).$$

If A is a nonnegative $n\times n$ matrix, it is known [9, p. 31] that

$$R'(A)\leqslant r(A)\leqslant R''(A), \tag{1}$$

and in the case of an irreducible A these inequalities are strict, unless all the $R_i(A)$'s are equal. It is also known [10] that

$$\begin{aligned} R'(A)\leqslant \left[R'(A^2)\right]^{1/2}\leqslant \left[R'(A^4)\right]^{1/4}\leqslant\cdots\leqslant r(A)\leqslant\cdots \\ \leqslant \left[R''(A^4)\right]^{1/4}\leqslant \left[R''(A^2)\right]^{1/2}\leqslant R''(A), \end{aligned} \tag{2}$$

$$\lim_{q\to+\infty}\left[R''(A^q)\right]^{1/q}=r(A), \tag{3}$$

and, if A is irreducible, then

$$\lim_{q\to+\infty}\left[R'(A^q)\right]^{1/q}=r(A). \tag{4}$$

In the present paper we shall generalize the results expressed by the relations (1)–(4). For example, the inequality (1) can be generalized in the following manner (for a more general version, see Theorem 2).

Let $\{t_0,t_1,\dots,t_m\}$ be a set of integers, satisfying

$$0=t_0<t_1<\cdots<t_m=n,$$

denote

$$\Gamma_i=\{t_{i-1}+1,t_{i-1}+2,\dots,t_i\}\qquad(i=1,\dots,m),$$

and consider the partition

$$\Pi=\{\Gamma_1,\Gamma_2,\dots,\Gamma_m\}$$

of the set $\{1,2,\dots,n\}$ into m nonempty subsets. Let

$$A=(A_{ij})_{i,\,j=1,\dots,m}$$

be the partitioning of the nonnegative $n\times n$ matrix $A=(a_{ij})$, induced by Π in the obvious manner. In each of the sets Γ_i $(i=1,\dots,m)$, we select a member $s_i\in\Gamma_i$, and we denote $\sigma=(s_1,\dots,s_m)$. To this m-tuple σ we associate an $m\times m$ matrix A_σ whose (i,j) entry is the sum of those entries from the s_ith row of A which belong to the block A_{ij}. Clearly, for a fixed partition Π (i.e., for a fixed partitioning of the matrix A), the number of these matrices A_σ is equal to the product of the cardinalities of Γ_i $(i=1,\dots,m)$, i.e., to $\prod_{i=1}^{m}(t_i-t_{i-1})$. We denote $r_\sigma(A)=r(A_\sigma)$. Then,

$$\min_\sigma r_\sigma(A)\leqslant r(A)\leqslant \max_\sigma r_\sigma(A), \tag{5}$$

where the minimum and the maximum are taken over all possible σ's (for a fixed partitioning of A).

If $\Pi=\{\{1,\dots,n\}\}$ (inducing the coarsest partitioning of A, i.e., "no partitioning"), then the inequalities (5) yield the classical inequalities (1). If $\Pi=\{\{1\},\dots,\{n\}\}$ (inducing the finest partitioning of A), then the inequalities (5) collapse into trivial equalities. If $\Pi=\{\{1\},\{2,3,\dots,n\}\}$, then we obtain from (5) a result of Hall and Porsching [6]. If $\Pi=\{\{1\},\{2\},\dots,\{k\},\{k+1,k+2,\dots,n\}\}$ for some $k\in\{1,2,\dots,n-1\}$, then we obtain a result of one of the authors, announced in [4].

We introduce some more notation:

X^{T} is the transpose of the matrix (vector) X;

$X<Y$, $X\leqslant Y$ are meant componentwise, X and Y being matrices (vectors) of the same dimension;

$\mathbf{R}^q$ is the vector space of all column q-tuples of real numbers;

$\mathbf{R}^{p\times q}$ is the vector space of all $p\times q$ matrices of real numbers;

$(x)_i$ is the ith component of the vector $x\in\mathbf{R}^q$.

We shall make use of the following propositions.

PROPOSITION 1 [3]. *If* $B\in\mathbf{R}^{n\times n}$, $B\geqslant 0$, $x\in\mathbf{R}^n$, $x>0$, *then*

$$\min_i\frac{(Bx)_i}{(x)_i}\leqslant r(B)\leqslant \max_i\frac{(Bx)_i}{(x)_i}. \tag{6}$$

PROPOSITION 2 [3]. *Let $B \in \mathbf{R}^{n\times n}$, $B \geqslant 0$, $x \in \mathbf{R}^n$, $x > 0$, and let ρ be a nonnegative number.*

(i) *if $Bx = \rho x$, then $\rho = r(B)$;*
(ii) *if $Bx < \rho x$, then $\rho > r(B)$;*
(iii) *if $Bx > \rho x$, then $\rho < r(B)$.*

(Proposition 2 is merely a reformulation of Proposition 1.)

PROPOSITION 3. *Let $B \in \mathbf{R}^{n\times n}$, $B \geqslant 0$, $x \in \mathbf{R}^n$, $0 \leqslant x \neq 0$, and let ρ be a nonnegative number. If $Bx \geqslant \rho x$, then $\rho \leqslant r(B)$.*

Proof. If $\rho = 0$, then the statement is trivially true. Assume $\rho > 0$. The inequality $Bx \geqslant \rho x$ implies $B^q x \geqslant \rho^q x$ for all positive integers q. Consequently, $(\rho^{-1}B)^q x \geqslant x$ for $q = 1,2,\dots$, and thus $(\rho^{-1}B)^q \nrightarrow 0$ as $q \to +\infty$. This, in turn, implies that $r(\rho^{-1}B) \geqslant 1$, since otherwise we would have $(\rho^{-1}B)^q \to 0$ [9, p. 13]. Consequently, $r(B) \geqslant \rho$. ∎

2. PRELIMINARIES

Let n be a positive integer, $n \geqslant 2$, and let $\Pi = \{\Gamma_1, \dots, \Gamma_m\}$ be a partition of the set $\{1,\dots,n\}$, where each Γ_i is nonempty. Clearly, m is a positive integer, satisfying $1 \leqslant m \leqslant n$. The sets $\Gamma_1, \dots, \Gamma_m$ will be called the *classes* of Π, and we shall write $i \overset{\Pi}{\sim} j$ if i and j belong to the same class.

To the partition Π we associate an $n \times m$ matrix

$$P_\Pi = (p_{ij}),$$

defined by

$$p_{ij} = \begin{cases} 1 & \text{if} \quad i \in \Gamma_j, \\ 0 & \text{if} \quad i \notin \Gamma_j. \end{cases}$$

(The 1's in the jth column of P_Π indicate the elements that belong to Γ_j.)

By $\mathbb{S}(\Pi)$ we shall denote the Cartesian product of the sets $\Gamma_1, \dots, \Gamma_m$, and the members of $\mathbb{S}(\Pi)$ will be called *selections*.

Let $\sigma=(s_1,\ldots,s_m)\in\mathbb{S}(\Pi)$. (Occasionally, σ will denote also the set $\{s_1,\ldots,s_m\}$.) To $\sigma\in\mathbb{S}(\Pi)$ we associate the mapping

$$\phi_\sigma:\{1,\ldots,n\}\to\{s_1,\ldots,s_m\},$$

$$\phi_\sigma(k)=s_j \qquad \forall k\in\Gamma_j \qquad (j=1,\ldots,m).$$

Clearly, $\phi_\sigma(k)=\phi_\sigma(l)$ if and only if $k\overset{\Pi}{\sim}l$.

To the selection $\sigma=(s_1,\ldots,s_m)\in\mathbb{S}(\Pi)$ we associate also the $m\times n$ matrix

$$L_\sigma=(l_{ij}),$$

defined by

$$l_{ij}=\begin{cases}1 & \text{if} \quad s_i=j,\\ 0 & \text{if} \quad s_i\neq j.\end{cases}$$

(The ith row of L_σ is the s_ith n-dimensional standard unit vector.) We also introduce the $n\times n$ matrix

$$J_\sigma=P_\Pi L_\sigma. \tag{7}$$

(The ith row of J_σ is the $\phi_\sigma(i)$th n-dimensional standard unit vector.)

It is easy to see that

$$L_\sigma P_\Pi=I_m, \tag{8}$$

where I_m is the $m\times m$ identity matrix. From (7) and (8) it follows at once that

$$J_\sigma J_\tau=J_\tau \qquad [\sigma,\tau\in\mathbb{S}(\Pi)]. \tag{9}$$

Let $x=(\xi_1,\ldots,\xi_n)^{\mathsf T}\in\mathbf{R}^n$ and let $\sigma\in\mathbb{S}(\Pi)$. We define

$$x_\sigma=\left(\xi_{\phi_\sigma(1)},\ldots,\xi_{\phi_\sigma(n)}\right)^{\mathsf T}\in\mathbf{R}^n.$$

We have

$$x_\sigma=J_\sigma x. \tag{10}$$

A vector $x=(\xi_1,\dots,\xi_n)^\mathsf{T}\in\mathbf{R}^n$ is said to be Π-*constant* if $\xi_i=\xi_j$ whenever $i\overset{\Pi}{\sim}j$. The set of all Π-constant vectors with positive components will be denoted by $\mathbf{S}_\Pi$.

LEMMA 1. *Let $x\in\mathbf{R}^n$. The following statements are equivalent:*

(i) *x is Π-constant;*
(ii) *$x=x_\sigma$ for some $\sigma\in\mathbb{S}(\Pi)$;*
(iii) *$x=x_\tau$ for all $\tau\in\mathbb{S}(\Pi)$.*

The proof is trivial.

Let $x\in\mathbf{R}^n$, $x\geqslant 0$. We denote

$$\langle x\rangle_\Pi=\{x_\sigma:\sigma\in\mathbb{S}(\Pi)\}.$$

If we consider the set $\langle x\rangle_\Pi$ partially ordered componentwise, then it is clear that $\langle x\rangle_\Pi$ has a first and a last element. A selection $\alpha\in\mathbb{S}(\Pi)$ is said to be *minimizing for x* if x_α is the first element of $\langle x\rangle_\Pi$; similarly, a selection $\beta\in\mathbb{S}(\Pi)$ is said to be *maximizing for x* if x_β is the last element of $\langle x\rangle_\Pi$. Obviously, a minimizing (maximizing) selection for x need not be unique.

We illustrate these very elementary but rather cumbersome concepts by an example. Let $n=5$ and let $\Pi=\{\{1,2\},\{3,4,5\}\}$. Then,

$$\mathbb{S}(\Pi)=\{(1,3),(1,4),(1,5),(2,3),(2,4),(2,5)\}.$$

If $x=(11,12,14,13,13)^\mathsf{T}$, then $\langle x\rangle_\Pi=\{y,z,u,v\}$, where $y=x_{(1,3)}=(11,11,14,14,14)^\mathsf{T}$, $z=x_{(1,4)}=x_{(1,5)}=(11,11,13,13,13)^\mathsf{T}$, $u=x_{(2,3)}=(12,12,14,14,14)^\mathsf{T}$, $v=x_{(2,4)}=x_{(2,5)}=(12,12,13,13,13)^\mathsf{T}$. The selections $(1,4)$ and $(1,5)$ are minimizing for x, while $(2,3)$ is maximizing for x.

LEMMA 2. *Let $x\in\mathbf{R}^n$, $x\geqslant 0$. Then*

(i) *the selection $\alpha\in\mathbb{S}(\Pi)$ is minimizing for x if and only if $x_\alpha\leqslant x$;*
(ii) *the selection $\beta\in\mathbb{S}(\Pi)$ is maximizing for x if and only if $x\leqslant x_\beta$.*

The proof is trivial.

Let $A=(a_{ij})\in\mathbf{R}^{n\times n}$, $A\geqslant 0$. We denote the columns of A by $a_1,\dots,a_n$ and its rows by $b_1^\mathsf{T},\dots,b_n^\mathsf{T}$. Let $\sigma=(s_1,\dots,s_m)\in\mathbb{S}(\Pi)$. We define an $m\times m$ matrix

$$A_\sigma=(a_{\sigma,ij}),$$

where

$$a_{\sigma,ij}=\sum_{k\in\Gamma_j}a_{s_ik}\qquad(i,j=1,\dots,m).$$

For each fixed partition Π of $\{1,\dots,n\}$ and for each fixed matrix $A\in\mathbf{R}^{n\times n}$, $A\geqslant 0$, we have introduced $\operatorname{card}\mathbb{S}(\Pi)$ matrices A_σ. If Π is the coarsest partition of $\{1,\dots,n\}$ (i.e., "no partition"), then $m=1$, $\operatorname{card}\mathbb{S}(\Pi)=n$, and the n matrices A_σ are the n 1×1 matrices $(R_i(A))$ $(i=1,\dots,n)$. If Π is the finest partition of $\{1,\dots,n\}$, then $m=n$, $\operatorname{card}\mathbb{S}(\Pi)=1$, and there is only one matrix A_σ, namely $A_\sigma=A$.

We consider an arbitrary but fixed partition Π of the set $\{1,\dots,n\}$. For each selection $\sigma\in\mathbb{S}(\Pi)$ we define

$$r_\sigma(A)=r(A_\sigma).$$

We also define

$$r'_\Pi(A)=\min_{\sigma\in\mathbb{S}(\Pi)}r_\sigma(A),\qquad r''_\Pi(A)=\max_{\sigma\in\mathbb{S}(\Pi)}r_\sigma(A).$$

In addition to A_σ, we associate to $A\in\mathbf{R}^{n\times n}$, $A\geqslant 0$, and to $\sigma=(s_1,\dots,s_m)\in\mathbb{S}(\Pi)$ two $n\times n$ matrices A'_σ and A''_σ, defined by

$A'_\sigma=$ the $n\times n$ matrix whose ith row is $b^{\mathsf{T}}_{\phi_\sigma(i)}$ $(i=1,\dots,n)$,

$A''_\sigma=$ the $n\times n$ matrix whose jth column is $\sum_{i\in\Gamma_k}a_i$ if $j=s_k$ $(k=1,\dots,m)$ and a column of zeros if $j\notin\sigma$.

One can easily see that we have

$$A_\sigma=L_\sigma AP_\Pi,\tag{11}$$

$$A'_\sigma=P_\Pi L_\sigma A=J_\sigma A,\tag{12}$$

$$A''_\sigma=AP_\Pi L_\sigma=AJ_\sigma.\tag{13}$$

Also, if $x\in\mathbf{R}^n$, then

$$A''_\sigma x=Ax_\sigma.\tag{14}$$

It is known [8, p. 200] that if $X \in \mathbf{R}^{m\times n}$, $Y \in \mathbf{R}^{n\times m}$ $(n \geqslant m)$, then the monic characteristic polynomials $\psi_{XY}(\lambda)$ of XY and $\psi_{YX}(\lambda)$ of YX satisfy the relation $\psi_{YX}(\lambda) = \lambda^{n-m}\psi_{XY}(\lambda)$. (In [8] the proof is given only for the case $m = n$; the case $n > m$ can be reduced to the case of square matrices by bordering X (Y) with $n - m$ rows (columns) of zeros.)

Consequently, taking into account the equalities (11)–(13), it follows that the nonzero eigenvalues of A_σ, A'_σ, A''_σ coincide. In particular,

$$r_\sigma(A) = r(A'_\sigma) = r(A''_\sigma) \qquad [\sigma \in \mathcal{S}(\Pi)]. \tag{15}$$

In the sequel the relations (9)–(15) will be used without any further reference to them.

Since both P_Π and L_σ have rank m, it follows from (11)–(13) that the matrices A_σ, A'_σ, A''_σ have ranks at most $\min\{\operatorname{rank} A, m\}$.

Since, by definition, the rows of A'_σ coincide with some of the rows of the given matrix A, the matrix A'_σ can be considered as an approximation for A, at least in the frequent case where the elements of A vary slowly in dependence on the row index. (This is true, for instance, when A has been obtained by the discretization of an integral equation with a continuous kernel.) On the other hand, as we have seen, the $m \times m$ matrix A_σ has the same nonzero eigenvalues as A'_σ. Therefore, one can even expect that the m largest eigenvalues of A will be approximated by the eigenvalues of A_σ.

To illustrate these heuristic considerations, we consider the following example. Let $k(s,t) = \min\{s,t\}$, $0 \leqslant s,t \leqslant 1$. Then, by discretization, the integral equation

$$\int_0^1 k(s,t)y(t)\,dt = \lambda y(s)$$

leads, for example, to the matrix eigenvalue problem $Kx = \lambda x$, where $x = (y(\frac{1}{4}), y(\frac{1}{2}), y(\frac{3}{4}), y(1))^{\mathrm{T}}$ and

$$K = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

Taking, for example, the partition $\Pi = \{\{1,2\},\{3,4\}\}$, we obtain for the matrices

A_σ

$$A_{(1,3)}=\frac{1}{16}\begin{pmatrix}2 & 2\\ 3 & 6\end{pmatrix},\qquad A_{(1,4)}=\frac{1}{16}\begin{pmatrix}2 & 2\\ 3 & 7\end{pmatrix},$$

$$A_{(2,3)}=\frac{1}{16}\begin{pmatrix}3 & 4\\ 3 & 6\end{pmatrix},\qquad A_{(2,4)}=\frac{1}{16}\begin{pmatrix}3 & 4\\ 3 & 7\end{pmatrix},$$

having eigenvalues $\{0.4476, 0.0523\}$, $\{0.5, 0.0625\}$, $\{0.5172, 0.0453\}$, $\{0.5625, 0.0625\}$, respectively. The eigenvalues of K are 0.5182, 0.0625, 0.0266, and 0.0177.

3. THE MAIN RESULTS

THEOREM 1. *Let* $A\in\mathbf{R}^{n\times n}$, $A\geqslant 0$, *let* $x\in\operatorname{RPV}(A)$, *and let* Π *be a partition of* $\{1,\dots,n\}$.

(i) *If* $\alpha\in\mathbb{S}(\Pi)$ *is minimizing for* x *and* $(x)_i>0$ *for all* $i\notin\alpha$, *then* $r_\alpha(A)\leqslant r(A)$;

(ii) *if* $\beta\in\mathbb{S}(\Pi)$ *is maximizing for* x, *then* $r(A)\leqslant r_\beta(A)$.

Proof. (i): We have $A'_\alpha x=J_\alpha Ax=r(A)J_\alpha x=r(A)x_\alpha\leqslant r(A)x$. Let $y\in\operatorname{LPV}(A'_\alpha)$. Then, $r_\alpha(A)y^{\mathsf T}x=y^{\mathsf T}A'_\alpha x\leqslant r(A)y^{\mathsf T}x$. If $y^{\mathsf T}x>0$, then we obtain at once that $r_\alpha(A)\leqslant r(A)$. Assume $y^{\mathsf T}x=0$. Then $(y)_j=0$ whenever $(x)_j>0$. In particular, $(y)_j=0$ for all $j\notin\alpha$. But the jth row $b^{\mathsf T}_{\phi_\alpha(j)}$ of A'_α may differ from the jth row $b_j^{\mathsf T}$ of A only when $j\notin\alpha$. Consequently, $y^{\mathsf T}A=y^{\mathsf T}A'_\alpha=r_\alpha(A)y^{\mathsf T}$. Hence, $r_\alpha(A)\leqslant r(A)$.

(ii): We have $A'_\beta x=J_\beta Ax=r(A)J_\beta x=r(A)x_\beta\geqslant r(A)x$ and then, by Proposition 3, $r(A)\leqslant r(A'_\beta)=r_\beta(A)$. ∎

REMARK 1. In Theorem 1(i) we cannot omit the assumption that $(x)_i>0$ for all $i\notin\alpha$. Indeed, let

$$A=\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix},\qquad x=\begin{pmatrix}1\\ 0\\ 0\end{pmatrix},$$

and consider the partition $\Pi=\{\{1\},\{2,3\}\}$. The selection $\alpha=(1,2)$ is minimizing for x, but $r_\alpha(A)=2>1=r(A)$.

THEOREM 2. *If A is a nonnegative $n \times n$ matrix and Π is a partition of* $\{1,\dots,n\}$, *then*

$$r'_{\Pi}(A) \leqslant r(A) \leqslant r''_{\Pi}(A). \tag{16}$$

Proof. The right-hand side inequality of (16) follows at once from Theorem 1(ii). To prove the left-hand inequality, first we assume that $A > 0$. Let $x \in \mathrm{RPV}(A)$. Then $x > 0$ and, by Theorem 1(i), we have $r_\alpha(A) \leqslant r(A)$, where α is a minimizing selection for x. Then, $r'_{\Pi}(A) \leqslant r_\alpha(A) \leqslant r(A)$. Since the functions r and r'_{Π} depend continuously on the elements of A, the inequality $r'_{\Pi}(A) \leqslant r(A)$ holds for any $A \geqslant 0$. ■

COROLLARY 1 [2, 5]. *Let A be a nonnegative $n \times n$ matrix, and let Π be a partition of* $\{1,\dots,n\}$ *into m classes. Define*

$$A' = (a'_{ij}) \in \mathbf{R}^{m \times m}, \qquad A'' = (a''_{ij}) \in \mathbf{R}^{m \times m},$$

where

$$a'_{ij} = \min_{\sigma \in \mathbb{S}(\Pi)} a_{\sigma,ij}, \qquad a''_{ij} = \max_{\sigma \in \mathbb{S}(\Pi)} a_{\sigma,ij}.$$

(The matrices A' and A'' are the greatest lower bound and the least upper bound, respectively, of the matrices A_σ $[\sigma \in \mathbb{S}(\Pi)]$ in the natural, i.e., componentwise, partial ordering of $\mathbf{R}^{m\times m}$.) Then,

$$r(A') \leqslant r(A) \leqslant r(A'').$$

REMARK 2. It is easy to see that if the underlying partition Π is the coarsest partition of $\{1,\dots,n\}$ (i.e., "no partitioning"), then Theorem 2 yields the well-known inequalities (1). For a fixed $j \in \{1,\dots,n\}$, taking the partition $\Pi = \{\{j\},\{1,\dots,n\}\setminus j\}$, we reobtain a result of Hall and Porsching [6].

If we assume that the matrix A is irreducible, then Theorem 2 can be strengthened.

THEOREM 3. *Let A be a nonnegative irreducible $n \times n$ matrix, let $x \in \mathrm{RPV}(A)$, and let Π be a partition of* $\{1,\dots,n\}$.

(i) *If x is Π-constant, then*

$$r(A) = r'_{\Pi}(A) = r''_{\Pi}(A);$$

(ii) *if x is not Π-constant, then*

$$r'_\Pi(A) < r(A) < r''_\Pi(A).$$

Proof. (i): Let $\sigma \in \mathcal{S}(\Pi)$. Then $A'_\sigma x = J_\sigma Ax = r(A)J_\sigma x = r(A)x_\sigma = r(A)x$. By Proposition 2, $r(A) = r(A'_\sigma) = r_\sigma(A)$.

(ii): Let $\alpha \in \mathcal{S}(\pi)$ be a minimizing selection for x, and denote $B = \frac{1}{2}(A''_\alpha + A)$. Then $(B - A)x = \frac{1}{2}(A''_\alpha - A)x = \frac{1}{2}A(x_\alpha - x)$. Let $y \in \mathrm{LPV}(B)$. Since A is irreducible and $B \geqslant \frac{1}{2}A$, it follows that B is irreducible and, consequently, $y > 0$. We have $[r(B) - r(A)]y^{\mathsf{T}}x = y^{\mathsf{T}}(B - A)x = \frac{1}{2}y^{\mathsf{T}}A(x_\alpha - x)$. But $y^{\mathsf{T}}x > 0$, $y^{\mathsf{T}}A > 0$, and since α is minimizing for x, we have $x_\alpha \leqslant x$ (Lemma 2). Since x is not Π-constant, we have $x_\alpha \neq x$ (Lemma 1). Then $y^{\mathsf{T}}A(x_\alpha - x) < 0$. Consequently,

$$r(B) < r(A). \tag{17}$$

Theorem 2 applied to B gives

$$r'_\Pi(B) \leqslant r(B). \tag{18}$$

For an arbitrary $\sigma \in \mathcal{S}(\Pi)$ we have $B''_\sigma = BJ_\sigma = \frac{1}{2}(A''_\alpha + A)J_\sigma = \frac{1}{2}(AJ_\alpha J_\sigma + AJ_\sigma) = \frac{1}{2}(AJ_\sigma + AJ_\sigma) = AJ_\sigma = A''_\sigma$, which implies $r'_\Pi(B) = r'_\Pi(A)$. This last equality, together with (17) and (18), yields $r'_\Pi(A) < r(A)$. The proof of $r(A) < r''_\Pi(A)$ is entirely similar.

LEMMA 3. *Let $A \in \mathbf{R}^{n\times n}$, $A > 0$, let Π be a partition of $\{1,\dots,n\}$, let $\alpha \in \mathcal{S}(\Pi)$ be such that $r_\alpha(A) = r'_\Pi(A)$, and let $u \in \mathrm{RPV}(A''_\alpha)$. Then*

(i) $0 < u_\alpha \leqslant u$ (*i.e., α is minimizing for u*);
(ii) $\min_i (u)_i/(u_\alpha)_i = 1$;
(iii) $A''_\sigma u \geqslant r_\alpha(A)u \quad \forall \sigma \in \mathcal{S}(\Pi)$;
(iv) $Au \geqslant r_\alpha(A)u$.

Proof. (i): Let $\Pi = \{\Gamma_1,\dots,\Gamma_m\}$, and let $\alpha = (s_1,\dots,s_m)$ be a selection such that $r_\alpha(A) = r'_\Pi(A)$. If $i \in \alpha$, then $(u_\alpha)_i = (u)_i$. Assume that $i \notin \alpha$. Without loss of generality we may assume that $i \in \Gamma_1$. Consider the selection $\sigma = (i, s_2,\dots,s_m)$, and let $y \in \mathrm{LPV}(A''_\sigma)$. From $A''_\alpha u = r_\alpha(A)u$ and $y^{\mathsf{T}}A''_\sigma = r_\sigma(A)y^{\mathsf{T}}$ it follows that

$$[r_\sigma(A) - r_\alpha(A)]y^{\mathsf{T}}u = y^{\mathsf{T}}(A''_\sigma - A''_\alpha)u. \tag{19}$$

On the other hand,

$$\begin{aligned}(A''_\sigma - A''_\alpha)u = A(u_\sigma - u_\alpha) &= \sum_{j=1}^{n} \left[(u)_{\phi_\sigma(j)} - (u)_{\phi_\alpha(j)}\right] a_j \\ &= \sum_{j\in\Gamma_1} \left[(u)_i - (u)_{s_1}\right] a_j = \left[(u)_i - (u)_{s_1}\right] \sum_{j\in\Gamma_1} a_j.\end{aligned}$$

Now from (19), taking into account that $r_\alpha(A) \leqslant r_\sigma(A)$, we obtain that $[(u)_i - (u)_{s_1}]\sum_{j\in\Gamma_1} y^{\mathsf{T}} a_j \geqslant 0$. Since the inequalities $0 \leqslant y \neq 0$ and $A > 0$ imply that $\sum_{j\in\Gamma_1} y^{\mathsf{T}} a_j > 0$, we have $(u)_{s_1} \leqslant (u)_i$, or $(u_\alpha)_i \leqslant (u)_i$. Consequently, the last inequality holds for all $i \in \{1,\dots,n\}$, i.e., $u_\alpha \leqslant u$. In order to prove that $u_\alpha > 0$, we premultiply both sides of $A''_\alpha u = r_\alpha(A)u$ by J_α. We obtain $A'_\alpha u_\alpha = r_\alpha(A)u_\alpha$. But $A > 0$ implies $A'_\alpha > 0$ and, consequently, $u_\alpha > 0$.

(ii): This statement follows from (i) and from the fact that $(u_\alpha)_i = (u)_i$ whenever $i \in \alpha$.

(iii): We have $A''_\sigma u \geqslant A''_\sigma u_\alpha = AJ_\sigma J_\alpha u = AJ_\alpha u = Au_\alpha = A''_\alpha u = r_\alpha(A)u$ for all $\sigma \in \mathbb{S}(\Pi)$.

(iv): We have $Au \geqslant Au_\alpha = A''_\alpha u = r_\alpha(A)u$. ■

LEMMA 4. *Let $A \in \mathbf{R}^{n\times n}$, $A > 0$, let Π be a partition of $\{1,\dots,n\}$, let $\beta \in \mathbb{S}(\Pi)$ be such that $r_\beta(A) = r''_\Pi(A)$, and let $v \in \mathrm{RPV}(A''_\beta)$. Then*

(i) $0 < v \leqslant v_\beta$ *(i.e., β is maximizing for v);*
(ii) $\max_i (v)_i/(v_\beta)_i = 1$;
(iii) $A''_\sigma v \leqslant r_\beta(A)v \quad \forall \sigma \in \mathbb{S}(\Pi)$;
(iv) $Av \leqslant r_\beta(A)v$.

Proof. Except for the left-hand side inequality of statement (i), the proof of Lemma 4 is entirely similar to that of Lemma 3. To prove that $v > 0$ we note that the relation $A''_\beta v = r_\beta(A)v$ implies $Av_\beta = r_\beta(A)v$. Since $A > 0$ and $0 \leqslant v_\beta \neq 0$, we obtain at once that $v > 0$. ■

REMARK 3. From Lemmas 3(iv) and 4(iv) we can obtain a new proof of Theorem 2. One need only apply Proposition 2 and use the standard continuity argument for $A \geqslant 0$.

We recall that S_Π denotes the set of all Π-constant vectors with positive components.

THEOREM 4. *Let A be a positive $n \times n$ matrix, and let Π be a partition of $\{1,\ldots,n\}$. Then*

(i) $r'_{\Pi}(A) = \max_{x \in \mathrm{S}_{\Pi}} \min_i \dfrac{(Ax)_i}{(x)_i}$;
(ii) $r''_{\Pi}(A) = \min_{x \in \mathrm{S}_{\Pi}} \max_i \dfrac{(Ax)_i}{(x)_i}$.

Proof. Let $\alpha \in \mathbb{S}(\Pi)$ be such that $r_\alpha(A) = r'_\Pi(A)$, and let $x \in \mathrm{S}_\Pi$. Then, applying Proposition 1 to A''_α, we have

$$r_\alpha(A) = r(A''_\alpha) \geqslant \min_i \frac{(A''_\alpha x)_i}{(x)_i} = \min_i \frac{(Ax_\alpha)_i}{(x)_i} = \min_i \frac{(Ax)_i}{(x)_i}.$$

On the other hand, if $u \in \mathrm{RPV}(A''_\alpha)$, then by Lemma 3(i) we have $u_\alpha > 0$ and

$$\min_i \frac{(Au_\alpha)_i}{(u_\alpha)_i} = \min_i \frac{(A''_\alpha u)_i}{(u_\alpha)_i} = r_\alpha(A) \min_i \frac{(u)_i}{(u_\alpha)_i} = r_\alpha(A),$$

where we have taken into account Lemma 3(ii). Since $u_\alpha \in \mathrm{S}_\Pi$, statement (i) is proved. The proof of statement (ii) is entirely similar. ■

In the next theorem we show that if the initial partition of $\{1,\ldots,n\}$ is replaced by one of its refinements, then the bounds given by Theorem 2 are, in general, improved.

THEOREM 5. *Let A be a nonnegative $n \times n$ matrix, let Π_1 be a partition of $\{1,\ldots,n\}$, and let Π_2 be a refinement of Π_1. Then,*

(i) $r'_{\Pi_2}(A) \geqslant r'_{\Pi_1}(A)$;
(ii) $r''_{\Pi_2}(A) \leqslant r''_{\Pi_1}(A)$.

Proof. If $A > 0$, then both statements follow at once from Theorem 4, since obviously $\mathrm{S}_{\Pi_1} \subseteq \mathrm{S}_{\Pi_2}$. If $A \geqslant 0$, then we apply the standard continuity argument.

COROLLARY 2. *If A is a nonnegative $n \times n$ matrix and Π is a partition of $\{1,\ldots,n\}$, then*

(i) $r'_\Pi(A) \geqslant R'(A)$;
(ii) $r''_\Pi(A) \leqslant R''(A)$.

THEOREM 6. *If A is a nonnegative $n \times n$ matrix and Π is a partition of $\{1,\dots,n\}$, then for all positive integers q we have*

(i) $r'_{\Pi}(A^q) \geqslant [r'_{\Pi}(A)]^q$,
(ii) $r''_{\Pi}(A^q) \leqslant [r''_{\Pi}(A)]^q$.

Proof. For $A \geqslant 0$ and $z \in \mathbf{R}^n$, $z > 0$, we denote, for convenience,

$$f_z(A) = \min_i \frac{(Az)_i}{(z)_i}, \qquad g_z(A) = \max_i \frac{(Az)_i}{(z)_i}.$$

If $B \in R^{n \times n}$, $B \geqslant 0$, a simple computation shows that $f_z(AB) \geqslant f_z(A) f_z(B)$, $g_z(AB) \leqslant g_z(A) g_z(B)$, whence $f_z(A^q) \geqslant [f_z(A)]^q$, $g_z(A^q) \leqslant [g_z(A)]^q$. Now, from Theorem 4 we obtain at once the desired result, first for $A > 0$ and then, by continuity, for any $A \geqslant 0$. ∎

THEOREM 7. *If A is a nonnegative $n \times n$ matrix and Π is a partition of $\{1,\dots,n\}$, then*

$$r(A) = \lim_{q \to +\infty} \left[r''_{\Pi}(A^q) \right]^{1/q}.$$

If, in addition, A is irreducible, then

$$r(A) = \lim_{q \to +\infty} \left[r'_{\Pi}(A^q) \right]^{1/q}.$$

Proof. Applying Corollary 2 and Theorem 2 to A^q, we obtain

$$\begin{aligned} \left[R'(A^q)\right]^{1/q} &\leqslant \left[r'_{\Pi}(A^q)\right]^{1/q} \leqslant r(A) \\ &\leqslant \left[r''_{\Pi}(A^q)\right]^{1/q} \leqslant \left[R''(A^q)\right]^{1/q}. \end{aligned} \tag{20}$$

The first assertion of the theorem follows from the last two inequalities of (20) if we let $q \to +\infty$ and if we take into account that R'' is an operator norm [1, pp. 67, 78]. The second assertion of the theorem follows in the same manner from the first two inequalities of (20), but now we have to take into account that under the assumption of the irreducibility of A we have $\lim_{q \to +\infty} [R'(A^q)]^{1/q} = r(A)$. ∎

COROLLARY 3. *Let A be a nonnegative irreducible $n \times n$ matrix, and let Π be a partition of $\{1,\dots,n\}$. If $\sigma \in \mathcal{S}(\Pi)$, then*

$$\lim_{q \to +\infty} \left[r_\sigma(A^q) \right]^{1/q} = r(A).$$

Proof. This statement is an immediate consequence of Theorem 7 and of the definitions of r'_Π and r''_Π. ■

COROLLARY 4. *If A is a nonnegative $n \times n$ matrix and Π is a partition of $\{1,\dots,n\}$, then the sequence $[r''_\Pi(A^{2^q})]^{2^{-q}}$ $(q = 1,2,\dots)$ is nonincreasing and converging to $r(A)$. If, in addition, A is irreducible, then the sequence $[r'_\Pi(A^{2^q})]^{2^{-q}}$ $(q = 1,2,\dots)$ is nondecreasing and converging to $r(A)$.*

Proof. These statements are immediate consequences of Theorems 6 and 7. ■

4. EXAMPLES

We conclude the paper with two numerical examples.

EXAMPLE 1. Let

$$A = \begin{pmatrix} 9 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 9 & 1 & 1 & 1 \\ 1 & 10 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

The inequalities (1) yield $3 < r(A) < 15$.

If we take the partition $\Pi_1 = \{\{1\},\{2,3,4,5\}\}$, then we have to consider the matrices

$$A_{(1,2)} = \begin{pmatrix} 9 & 3 \\ 0 & 3 \end{pmatrix}, \qquad A_{(1,3)} = \begin{pmatrix} 9 & 3 \\ 0 & 12 \end{pmatrix},$$

$$A_{(1,4)} = \begin{pmatrix} 9 & 3 \\ 1 & 14 \end{pmatrix}, \qquad A_{(1,5)} = \begin{pmatrix} 9 & 3 \\ 0 & 4 \end{pmatrix}.$$

Taking into account that for nonnegative $n \times n$ matrices X and Y we have $r(X) \leqslant r(Y)$ whenever $X \leqslant Y$ [9, p. 46], we do not have to compute the Perron roots of all these matrices. It can be seen at once that $r'_{\Pi_1}(A) = r(A_{(1,2)}) = 9$ and $r''_{\Pi_1}(A) = r(A_{(1,4)}) \cong 14.541$. Thus, $9 < r(A) < 14.542$.

If we take the partition $\Pi_2 = \{\{1\},\{2\},\{3,4,5\}\}$, which is a refinement of Π_1, then we have to consider the matrices

$$A_{(1,2,3)} = \begin{pmatrix} 9 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 9 & 3 \end{pmatrix}, \qquad A_{(1,2,4)} = \begin{pmatrix} 9 & 0 & 3 \\ 0 & 0 & 3 \\ 1 & 10 & 4 \end{pmatrix},$$

$$A_{(1,2,5)} = \begin{pmatrix} 9 & 0 & 3 \\ 0 & 0 & 3 \\ 0 & 1 & 3 \end{pmatrix}.$$

Once again, comparing the entries of these matrices, it is clear that $r'_{\Pi_2}(A) = r(A_{(1,2,5)}) = 9$ and $r''_{\Pi_2}(A) = r(A_{(1,2,4)}) = 10$. Thus, $9 < r(A) < 10$.

If we consider the partition $\Pi_3 = \{\{1\},\{2,5\},\{3,4\}\}$, which is a refinement of Π_1 and where we have a small variation between the rows belonging to the same class, then we have to consider the matrices

$$A_{(1,2,3)} = \begin{pmatrix} 9 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 10 & 2 \end{pmatrix}, \qquad A_{(1,2,4)} = \begin{pmatrix} 9 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 11 & 3 \end{pmatrix},$$

$$A_{(1,5,3)} = \begin{pmatrix} 9 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 10 & 2 \end{pmatrix}, \qquad A_{(1,5,4)} = \begin{pmatrix} 9 & 1 & 2 \\ 0 & 2 & 2 \\ 1 & 11 & 3 \end{pmatrix}.$$

(Note that the notation $A_{(1,2,3)}$, for example, does not reflect the dependence of this matrix on the considered partition; this explains the occurrence in this example of two distinct matrices denoted by $A_{(1,2,3)}$.)

Comparing the entries of these four matrices, it is clear that $r'_{\Pi_3}(A) = r(A_{(1,2,3)}) = 9$ and $r''_{\Pi_3}(A) = r(A_{(1,5,4)}) \cong 9.609$. Thus, $9 < r(A) < 9.610$. We actually have $r(A) \cong 9.288$.

EXAMPLE 2. Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 2 & 3 & 5 \end{pmatrix}.$$

We consider the partition $\Pi = \{\{1,2\},\{3\}\}$. Then

$$A_{(1,3)} = \begin{pmatrix} 2 & 2 \\ 5 & 5 \end{pmatrix}, \qquad A_{(2,3)} = \begin{pmatrix} 3 & 3 \\ 5 & 5 \end{pmatrix},$$

whose Perron roots are 7 and 8, respectively. Consequently $7 < r(A) < 8$. These bounds are better than those obtained by several other methods [7, p. 158]. If we consider the matrix A^2 rather than A, then, with the same partition as above, we obtain $53 < r(A^2) < 60$, whence $7.280 < r(A) < 7.746$. We actually have $r(A) = (7+\sqrt{65})/2 \cong 7.531$.

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