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ORIGINAL ARTICLE

Operator's Fuzzy Norm and Some Properties

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Abstract In this paper, a concept of operator's fuzzy norm is introduced for the first time in general t-norm setting. Ideas of fuzzy continuous operators, fuzzy bounded linear operators are given with some properties of such operators studied in this general setting.

Keywords Fuzzy norm · Fuzzy continuous operator · Fuzzy bounded linear operator

1. Introduction

The problem of defining fuzzy norm on a linear space was first initiated by Katsaras [1] and afterwards C. Felbin [2], Cheng & Mordeson [3], came up with their definitions of fuzzy norms approaching from different perspectives. Some authors worked on related topics in fuzzy setting [4-6]. In [7], we have also taken a definition of fuzzy norm slightly different from that of Cheng & Mordeson with a view to establish a complete decomposition of a fuzzy norm into crisp norms. Interestingly, this decomposition theorem played a crucial role in developing fuzzy functional analysis [8-11]. However, for doing so, we had to restrict the underlying $'t'$-norm in the triangle inequality of fuzzy norm to be the $'t'_{\min}$. This has become a bit of uncomfortable

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situation in the sense that uncertainty processing through fuzzy theory demands as much generality as possible in the underlying t-norm. Because of this we have tried to address this problem in two directions. In [12], this has been taken care of by using the concept of 'generating spaces of quasi-norm family' which plays the role of fuzzy norms in some sense of its decomposition under the general t-norm setting. On the other hand in [13, 14], we have tried to fuzzify the results of finite dimensional normed linear spaces with general t-norm but without using the decomposition technique.

With the latter approach, in this paper, we have been able to proceed further. The concept of fuzzy bounded linear operators, fuzzy continuous operators, fuzzy operator norm for fuzzy bounded linear operators and spaces of fuzzy bounded linear operators are introduced and their properties are studied.

The organization of the paper is as in the following:

Section 2 comprises some preliminary results. Definitions of fuzzy continuous operators, fuzzy bounded linear operators are introduced and relation between them are studied in Section 3. In Section 4, we introduce the idea of operator's fuzzy norm. Lastly in Section 5, completeness of $BF(X, Y)$ (set of all fuzzy bounded linear operators) is proved.

2. Preliminaries

Definition 2.1 [15] *A binary operation* $* : [0,\ 1] \times [0,\ 1] \rightarrow [0,\ 1]$ *is a t-norm if it satisfies the following conditions:*

(I) $*$ *is associative and commutative;*

(II) $a * 1 = a \ \ \forall a \in [0,\ 1]$;

(III) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$ *for each* $a,\ b,\ c,\ d \in [0,\ 1]$.

If $*$ *is continuous, then it is called continuous t-norm. Following are examples of some t-norms that are frequently used as fuzzy intersections defined for all* $a,\ b \in\ [0,\ 1]$.

(I) *Standard intersection:* $a * b = \min(a,\ b)$.

(II) *Algebraic product:* $a * b = ab$.

(III) *Bounded difference:* $a * b = \max(0,\ a + b - 1)$.

(IV) *Drastic intersection:*

$$a * b = \begin{cases} a, & for \quad b = 1, \\ b, & for \quad a = 1, \\ 0, & otherwise. \end{cases}$$

The relations among these t-norms are $a * b$ (Drastic)$\leq \max(0,\ a + b - 1) \leq\ ab \leq \min(a,\ b)$.

Definition 2.2 [13] *Let* U *be a linear space over the field* $\mathcal{F}$ (C *or* $\mathcal{R}$). *A fuzzy subset* N *of* $U \times \mathcal{R}$ ($\mathcal{R}$*- the set of all real numbers) is called a fuzzy norm on* U *if*

(NI) $\forall t \in \mathcal{R}$ *with* $t \leq 0,\ N(x,\ t) = 0$;

(NII) $(\forall t \in \mathcal{R}, t > 0, N(x,\ t) = 1)$ iff $x = \underline{0}$;

(NIII) $\forall t \in \mathcal{R},\ t > 0,\ N(cx,\ t) = N(x,\ \frac{t}{|c|})$ *if* $c \neq 0$;

(NIV) $\forall s, t \in \mathcal{R};\ x, u \in U$;
$N(x + u,\ s + t) \geq\ N(x,\ s) * N(u,\ t)$;

(NV) $N(x,\ \cdot)$ *is a non-decreasing function of* $\mathcal{R}$ and $\lim_{t\to\infty} N(x,\ t) = 1$. *The triplet* $(U,\ N,\ *)$ *will be referred to as a fuzzy normed linear space.*

Remark 2.1 From (NII) and (NIV), it follows that $N(x, \cdot)$ is a non-decreasing function of $\mathcal{R}$. So in the rest of the paper we take the modified form of (NV) by deleting the condition that $N(x,\ \cdot)$ is non-decreasing.

Assume that [7],
(NVI) $\forall t > 0,\ N(x,\ t) > 0$ implies $x = \underline{0}$.

Lemma 2.1 [14] *Let* $(U, N, *)$ *be a fuzzy normed linear space satisfying* (NVI) *and the underlying t-norm* $*$ *be continuous at* $(1,\ 1)$. *If* $\{x_1, x_2, \cdots, x_n\}$ *is a linearly independent set of vectors in X, then for each* $\alpha \in (0, 1)$, $\exists c_\alpha > 0$ *such that for any set of scalars* $\{\beta_1, \beta_2, \cdots, \beta_n\}$;

$$\bigwedge\{t > 0\ \mid\ N(x_1\beta_1 + x_2\beta_2 + \cdots + x_n\beta_n,\ t)\ \geq 1 - \alpha\}\ \geq c_\alpha \sum_{i=1}^{n} |\beta_i|.$$

Definition 2.3 [14] *Let* $(U, N, *)$ *be a fuzzy normed linear space and* $\alpha \in (0, 1)$.

(I) *A sequence* $\{x_n\}$ *in U is said to be* α*-fuzzy convergent if* $\exists x \in U$ *such that*
$$\lim_{n\to\infty} \bigwedge\{t > 0\ \mid\ N(x_n - x,\ t) > 1 - \alpha\} = 0.$$

(II) *A sequence* $\{x_n\}$ *in U is said to be* α*-fuzzy Cauchy if*
$$\lim_{m,n\to\infty} \bigwedge\{t > 0\ \mid N(x_n - x_m,\ t) > 1 - \alpha\} = 0.$$

3. Fuzzy Bounded Linear Operators

In this section, a concept of fuzzy bounded linear operator in general t-norm set is introduced.

Definition 3.1 *Let* $T \mid (X,\ N_1,\ *_1) \to (Y,\ N_2,\ *_2)$ *be a linear operator where* $(X,\ N_1,\ *_1)$ *and* $(Y,\ N_2,\ *_2)$ *are fuzzy normed linear spaces. T is said to be fuzzy bounded if for each* $\alpha \in (0,\ 1)$, $\exists M_\alpha > 0$ *such that*

$$N_1(x,\ \frac{t}{M_\alpha}) \geq 1 - \alpha\ \Rightarrow N_2(Tx,\ s) \geq \alpha\ \ \forall s > t,\ \ \forall t > 0. \tag{1}$$

Proposition 3.1 *Let* $T \mid (X, N_1, *_1) \to (Y, N_2, *_2)$ *be a linear operator where* $(X, N_1, *_1)$ *and* $(Y, N_2, *_2)$ *are fuzzy normed linear spaces. If T is fuzzy bounded, then the relation* (1) *is equivalent to the relation*

$$\bigwedge\{t>0 \mid N_2(Tx,\ t)\geq\alpha\}\ \leq M_\alpha \bigwedge\{t>0 \mid N_1(x,\ t)\geq 1-\alpha\}\ \forall x\in X. \qquad (2)$$

Proof First we show that (1) ⇒ (2).

Let $r > M_\alpha \bigwedge\{t>0 \mid N_1(x,\ t)\geq 1-\alpha\}$

$\Rightarrow \dfrac{r}{M_\alpha} > \bigwedge\{t>0 \mid N_1(x,\ t)\geq 1-\alpha\}$

$\Rightarrow \exists \dfrac{r'}{M_\alpha}$ such that $\dfrac{r}{M_\alpha} > \dfrac{r'}{M_\alpha} > \bigwedge\{t>0 \mid N_1(x,\ t)\geq 1-\alpha\}$

$\Rightarrow \exists \dfrac{r'}{M_\alpha}$ such that $N_1(x,\ \dfrac{r'}{M_\alpha})\geq 1-\alpha$

$\Rightarrow N_2(Tx,\ r)\geq\alpha$ by (1)

$\Rightarrow \bigwedge\{t>0 \mid\ N_2(Tx,\ \ t)\geq\alpha\}\ \leq M_\alpha \bigwedge\{t>0 \mid\ N_1(x,\ t)\geq 1-\alpha\}\ \ \forall x\in X.$

So (1) ⇒ (2).

Now we prove (2) ⇒ (1).

Assume that $N_1(x,\ \dfrac{t}{M_\alpha})\geq 1-\alpha$. So

$\bigwedge\{r>0 \mid N_1(x,\ r)\geq 1-\alpha\}\leq \dfrac{t}{M_\alpha}$

$\Rightarrow \bigwedge\{r>0 \mid N_2(Tx,\ r)\geq\alpha\}\leq t$

$\Rightarrow$ for any $s>t$, $\bigwedge\{r>0 \mid N_2(Tx,\ r)\geq\alpha\}\ < s$

$\Rightarrow N_2(Tx,\ s)\geq\alpha.$

Hence (2) ⇒ (1).

Note 3.1: *We denote the collection of all linear operators defined from a fuzzy normed linear space* $(X, N_1, *_1)$ *to another normed linear space* $(Y, N_2, *_2)$ *by* $L(X,\ Y)$ *and for fuzzy bounded linear operators we denote the collection by* $BF(X,\ Y)$.

Lemma 3.1 *Let* $(X, N, *)$ *be a fuzzy normed linear space and the underlying t-norm* $*$ *be continuous at* $(1,\ 1)$. *Then for each* $\alpha\in(0,\ 1)$, $\exists\beta\geq\alpha$ *such that*

$\bigwedge\{t>0 \mid N(x+y,\ t)\geq\alpha\}$

$\leq \bigwedge\{t>0 \mid N(x,\ t)\geq\beta\}+\bigwedge\{t>0 \mid N(y,\ t)\geq\beta\}\ \forall x,y\in X.$

Proof Since $*$ is continuous at $(1, 1)$, for each $\alpha\in(0, 1)$, we can find $\beta\in(0, 1)$ such that

$\beta*\beta\geq\alpha.$

Again $\beta\geq\beta*\beta\geq\alpha$. So $\beta\geq\alpha$. Now,

$\bigwedge\{t>0 \mid N(x,\ t)\geq\beta\}+\bigwedge\{t>0 \mid N(y,\ t)\geq\beta\}$

$=\bigwedge\{s+t>0 \mid N(x,\ s)\geq\beta, N(y,\ t)\geq\beta\}$

$\geq \bigwedge\{s+t>0 \mid N(x,\ s)*N(y,\ t)\geq\beta*\beta\}$

$\geq \bigwedge\{s+t>0 \mid N(x+y,\ s+t)\geq\alpha\}$ by (NIV).

Hence $\bigwedge\{t>0 \mid N(x+y,\ t)\geq\alpha\}\ \leq \bigwedge\{t>0 \mid N(x,\ s)\geq\beta\}+\bigwedge\{t>0 \mid N(y,\ t)\geq\beta\}\ \forall x,y\in X.$

Theorem 3.1 $BF(X, Y)$ *(set of all fuzzy bounded linear operators) is a subspace of* $L(X, Y)$ *(set of all linear operators) where* $(X, N_1, *_1)$ *and* $(Y, N_2, *_2)$ *are fuzzy normed linear spaces and* $*_2$ *is continuous at* $(1, 1)$.

Proof Take $T_1,\ T_2\in BF(X, Y)$.

Now by Lemma 3.1, we have for non-zero scalars k_1, k_2;

$\bigwedge\{t > 0 \mid N_2((k_1T_1 + k_2T_2)x, t) \geq \alpha\}$
$\leq \bigwedge\{t > 0 \mid N_2(k_1T_1x, t) \geq \beta\} + \bigwedge\{t > 0 \mid N_2(k_2T_2x, t) \geq \beta\}\ \forall x \in X,$
where β depends on α and $\beta \geq \alpha$.
i.e., $\bigwedge\{t > 0 \mid N_2((k_1T_1+k_2T_2)x, t) \geq \alpha\} \leq |k_1| \bigwedge\{t > 0 \mid N_2(T_1x, t) \geq \beta\} + |k_2| \bigwedge\{t > 0 \mid N_2(T_2x, t) \geq \beta\}$.

Since T_1 and T_2 are fuzzy bounded, $\exists M^1_{\beta(\alpha)},\ M^2_{\beta(\alpha)} > 0$ such that
$\bigwedge\{t > 0 \mid N_2(T_1x, t) \geq \beta\} \leq M^1_{\beta(\alpha)} \bigwedge\{t > 0 \mid N_1(x, t) \geq 1-\beta\}\ \forall x \in X$
and
$\bigwedge\{t > 0 \mid N_2(T_2x, t) \geq \beta\} \leq M^2_{\beta(\alpha)} \bigwedge\{t > 0 \mid N_1(x, t) \geq 1-\beta\}\ \forall x \in X.$
So, $\bigwedge\{t > 0 \mid N_2((k_1T_1 + k_2T_2)x, t) \geq \alpha\} \leq |k_1|M^1_{\beta(\alpha)} \bigwedge\{t > 0 \mid N_1(x, t) \geq 1-\beta\}$
$+|k_2|M^2_{\beta(\alpha)} \bigwedge\{t > 0 \mid N_1(x, t) \geq 1-\beta\}$.

Let $M_\alpha = |k_1|M^1_{\beta(\alpha)} + |k_2|M^2_{\beta(\alpha)}$. Then we have

$$\begin{aligned}&\bigwedge\{t > 0 \mid N_2((k_1T_1 + k_2T_2)x, t) \geq \alpha\}\\ &\quad\leq M_\alpha \bigwedge\{t > 0 \mid N_1(x, t) \geq 1-\beta\}\ \forall x \in X.\end{aligned} \tag{3}$$

Since $\beta \geq \alpha$, thus $1-\beta \leq 1-\alpha$, so,
$\{t > 0 \mid N_1(x, t) \geq 1-\alpha\} \subset \{t > 0 \mid N_1(x, t) \geq 1-\beta\}$
$\Rightarrow \bigwedge\{t > 0 \mid N_1(x, t) \geq 1-\alpha\} \geq \bigwedge\{t > 0 \mid N_1(x, t) \geq 1-\beta\}$.
So from (3) we get
$\bigwedge\{t > 0 \mid N_2((k_1T_1 + k_2T_2)x, t) \geq \alpha\} \leq M_\alpha \bigwedge\{t > 0 \mid N_1(x, t) \geq 1-\alpha\}\ \forall x \in X.$
Thus $k_1T_1 + k_2T_2 \in BF(X, Y)$.

Hence $BF(X, Y)$ is a subspace of $L(X, Y)$.

Definition 3.2 *An operator $T \mid (X_1, N_1) \to (X_2, N_2)$ is said to be fuzzy continuous at $x \in X$ if for every sequence $\{x_n\}$ in X_1 with $x_n \to x$ implies $Tx_n \to Tx$. i.e.,*
$\lim_{n\to\infty} N_1(x_n - x, t) = 1\ \forall t > 0$ *implies* $\lim_{n\to\infty} N_2(Tx_n - Tx, t) = 1\ \forall t > 0$.

Theorem 3.2 *Let $T \mid (X, N_1) \to (Y, N_2)$ be a linear operator where (X, N_1) and (Y, N_2) are fuzzy normed linear spaces. If T is fuzzy continuous at a point $x_0 \in X_1$, then T is fuzzy continuous everywhere in X_1.*

Proof Let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} N_1(x_n - x, t) = 1\ \forall t > 0$. i.e.,
$\lim_{n\to\infty} N_1(x_n - x + x_0 - x_0, t) = 1\ \forall t > 0 \Rightarrow x_n - x + x_0 \to x_0$.
Since T is continuous at x_0, it follows that $T(x_n - x + x_0) \to Tx_0$.
So,
$\lim_{n\to\infty} N_2(T(x_n - x + x_0) - T(x_0), t) = 1\ \forall t > 0,$
$\lim_{n\to\infty} N_2(T(x_n) - T(x), t) = 1\ \forall t > 0$ (since T is linear),
$T(x_n) \to Tx$.
Since x is arbitrary, it follows that T is continuous in X.

Theorem 3.3 *Let $T \mid (X_1, N_1) \to (X_2, N_2)$ be a linear operator where (X_1, N_1) and (X_2, N_2) are fuzzy normed linear spaces. If T is fuzzy bounded then it is fuzzy continuous but not conversely.*

Proof First suppose that T is fuzzy bounded. So for each $\alpha \in (0, 1)$, $\exists M_\alpha > 0$ such that

$\bigwedge\{t > 0 \mid N_2(Tx, t) \geq \alpha\} \leq M_\alpha \bigwedge\{t > 0 \mid N_1(x, t) \geq 1 - \alpha\}\ \forall x \in X_1$.

Let $\{x_n\}$ be a sequence in X_1 such that $x_n \to x$. Thus $\lim_{n\to\infty} N_1(x_n - x, t) = 1\ \forall t > 0$.

Let $\epsilon > 0$ be given. So for each $\alpha \in (0, 1)$, $\exists$ a positive integer $N(\alpha, \epsilon)$ such that

$N_1(x_n - x, \frac{\epsilon}{2M_\alpha}) > 1 - \alpha\ \forall n \geq N(\alpha, \epsilon)$

$\Rightarrow \bigwedge\{t > 0 \mid N_1(x_n - x, t) \geq 1 - \alpha\} \leq \frac{\epsilon}{2M_\alpha}\ \forall n \geq N(\alpha, \epsilon),\ \forall \alpha \in (0, 1)$

$\Rightarrow M_\alpha \bigwedge\{t > 0 \mid N_1(x_n - x, t) \geq 1 - \alpha\} \leq \frac{\epsilon}{2} < \epsilon\ \forall n \geq N(\alpha, \epsilon),\ \forall \alpha \in (0, 1)$

$\Rightarrow \bigwedge\{t > 0 \mid N_2(T(x_n - x), t) \geq \alpha\} < \epsilon\ \forall n \geq N(\alpha, \epsilon),\ \forall \alpha \in (0, 1)$

$\Rightarrow N_2(Tx_n - Tx, \epsilon) \geq \alpha\ \forall n \geq N(\alpha, \epsilon),\ \forall \alpha \in (0, 1)$

$\Rightarrow \lim_{n\to\infty} N_2(Tx_n - Tx, \epsilon) = 1$.

Since $\epsilon > 0$ is arbitrary, we have $\lim_{n\to\infty} N_2(Tx_n - Tx, t) = 1\ \forall t > 0$.
i.e., $\{Tx_n\} \to Tx$. Hence T is fuzzy continuous on X.

Converse result may not be true which is justified by the following example.

Example 3.1 Let us consider a normed linear space $(X, \|\cdot\|)$. Define two functions as in the following:

$$N_1(x, t) = \begin{cases} 1, & \text{if } t > \|x\|, \\ \frac{1}{2}, & \text{if } 0 < t \leq \|x\|, \\ 0, & \text{if } t \leq 0. \end{cases}$$

$$N_2(x, t) = \begin{cases} 1 - \frac{\|x\|}{t}, & \text{for } t \geq \|x\|, \\ 0, & \text{for } t < \|x\|. \end{cases}$$

It can be verified that N_1 and N_2 are fuzzy norm on X (please see Observation 1.2 [16]).

Define a linear operator $T \mid X \to X$ by $T(x) = 2x\ \forall x \in X$.

First we prove that T is fuzzy continuous.

For consider a sequence $\{x_n\}$ in X such that $x_n \to x$. Thus

$\lim_{n\to\infty} N_1(x_n - x, t) = 1\ \forall t > 0$

$\Rightarrow \lim_{n\to\infty} N_1(x_n - x, t) > \alpha\ \forall t > 0\ \forall \alpha \in (0, 1)$.

Choose $\alpha > \frac{1}{2}$. Then $\exists$ a positive integer $N(\alpha)$ such that

$\lim_{n\to\infty} N_1(x_n - x, t) > \alpha\ \forall t > 0\ \forall n \geq N(\alpha)$

$\Rightarrow \|x_n - x\| < t\ \forall t > 0\ \forall n \geq N(\alpha)$

$\Rightarrow \lim_{n\to\infty} \|x_n - x\| = 0$.

Now $\lim_{n\to\infty} N_2(Tx_n - Tx, t) = \lim_{n\to\infty} N_2(2x_n - 2x, t) = \lim_{n\to\infty} (1 - \frac{2\|x_n - x\|}{t}) = 1\ \forall t > 0$.
So $Tx_n \to Tx$. Thus T is fuzzy continuous.

Now we show that T is not fuzzy bounded.

Take $\alpha = \frac{1}{2}$. We have

$$\bigwedge\{t > 0 \mid N_2(Tx, t) \geq \frac{1}{2}\} = \bigwedge\{t > 0 \mid N_2(2x, t) = \frac{1}{2}\} = 2\|x\|. \tag{4}$$

Again,

$$\bigwedge\{t > 0 \mid N_1(x, t) \geq \frac{1}{2}\} = 0. \tag{5}$$

From (4) and (5), we observe that there does not exists $M_\alpha > 0$ for $\alpha = \frac{1}{2}$ for which $\bigwedge\{t > 0 \mid N_2(Tx, t) = \frac{1}{2}\} \leq \bigwedge\{t > 0 \mid N_1(x, t) \geq \frac{1}{2}\}$ holds $\forall x (\neq \underline{0}) \in X$.

Hence T is not fuzzy bounded.

Now to get some kind of converse result, we introduce below concepts of α-fuzzy continuity and α-fuzzy boundedness.

Definition 3.3 *Let $T \mid (X, N_1) \to (Y, N_2)$ be an operator where (X, N_1) and (Y, N_2) are fuzzy normed linear spaces and $\alpha \in (0, 1)$.*

(i) *T is said to be α-fuzzy continuous at $x \in X$ if for any sequence $\{x_n\}$ in X;*
$\lim_{n\to\infty} \bigwedge\{t > 0 \mid N_1(x_n - x, t) \geq 1 - \alpha\} = 0$ *implies*
$\lim_{n\to\infty} \bigwedge\{t > 0 \mid N_2(Tx_n - Tx, t) \geq \alpha\} = 0$.

(ii) *T is said to be α-fuzzy bounded if $\exists M_\alpha > 0$ such that*
$N_1(x, \frac{t}{M_\alpha}) \geq 1 - \alpha \Rightarrow N_2(Tx, s) \geq \alpha \;\; \forall s > t, \forall t > 0$.

Proposition 3.2 *Definition* 3.10 (ii) *is equivalent to the relation*

$$\bigwedge\{t > 0 \mid N_2(Tx, t) \geq \alpha\} \leq M_\alpha \bigwedge\{t > 0 \mid N_1(x, t) \geq 1 - \alpha\} \; \forall x \in X.$$

Proof Proof is similar to that of in Proposition 3.1.

Note 3.2: Denote
$d_\alpha = \bigwedge\{t > 0 \mid N_1(x,t) \geq \alpha\}, \; \alpha \in (0, 1)$.
By (NVI), it follows that for $x \neq \underline{0}$, $\bigwedge\{t > 0 \mid N_1(x, t) \geq \alpha\} > 0 \; \forall \alpha \in (0, 1)$.

Theorem 3.4 *Let $T \mid (X, N_1) \to (Y, N_2)$ be a linear operator where (X, N_1) and (Y, N_2) are fuzzy normed linear spaces and $\alpha \in (0, 1)$. Then T is α-fuzzy bounded iff it is α-fuzzy continuous.*

Proof First we assume that T is α-fuzzy bounded. Thus $\exists M_\alpha > 0$ such that
$\bigwedge\{t > 0 \mid N_2(Tx, t) \geq \alpha\} \leq M_\alpha \bigwedge\{t > 0 \mid N_1(x, t) \geq 1 - \alpha\}$.
Let $\{x_n\}$ be a sequence with $\lim_{n\to\infty} \bigwedge\{t > 0 \mid N_1(x_n - x, t) \geq 1 - \alpha\} = 0$.
Then
$\overline{\lim}_{n\to\infty} \bigwedge\{t > 0 \mid N_2(Tx_n - Tx, t) \geq \alpha\} \leq M_\alpha \lim_{n\to\infty} \bigwedge\{t > 0 \mid N_1(x_n - x, t) \geq 1 - \alpha\}$.
So, $\lim_{n\to\infty} \bigwedge\{t > 0 \mid N_2(Tx_n - Tx, t) \geq \alpha\} = 0$.

Hence T is α-fuzzy continuous.

Conversely, suppose that T is α-fuzzy continuous. If possible suppose that T is not α-fuzzy bounded. So $\exists$ a sequence $\{x_n\}$ in X such that
$\bigwedge\{t > 0 \mid N_2(Tx_n, t) \geq \alpha\} > n \bigwedge\{t > 0 \mid N_1(x_n, t) \geq 1 - \alpha\}, \; n = 1, 2, 3 \cdots$

Clearly, $x_n \neq \underline{0} \;\; \forall n$.

Let $x'_n = \dfrac{x_n}{n \bigwedge\{t > 0 \mid N_1(x_n, t) \geq 1 - \alpha\}}$ (by Note 3.2, denominator is positive for each n).

Then $\bigwedge\{t>0 \mid N_1(x'_n,\ t)\geq 1-\alpha\}=\frac{1}{n}$ for $n=1,2,3\cdots$

This implies that $\lim_{n\to\infty}\bigwedge\{t>0 \mid N_1(x'_n,\ t)\geq 1-\alpha\}=0$.

Since T is α-fuzzy continuous, it follows that

$$\lim_{n\to\infty}\bigwedge\{t>0 \mid N_2(Tx'_n-T\underline{0},\ t)\geq \alpha\}=0. \tag{6}$$

On the other hand we have

$\bigwedge\{t>0 \mid N_2(Tx'_n,\ t)\geq\alpha\}=\dfrac{\bigwedge\{t>0 \mid N_2(Tx_n,\ t)\geq\alpha\}}{n\bigwedge\{t>0 \mid N_1(x_n,\ t)\geq 1-\alpha\}}>1.$

i.e., $\bigwedge\{t>0 \mid N_2(Tx'_n,\ t)\geq\alpha\}>1\ \forall n$

$\Rightarrow \lim_{n\to\infty}\bigwedge\{t>0 \mid N_2(Tx'_n,\ t)\geq\alpha\}\neq 0$ which contradicts (6).

Hence T is α-fuzzy bounded.

4. Operator Fuzzy Norm in $BF(X,Y)$

In this section, we define fuzzy norm of fuzzy bounded linear operators.

Theorem 4.1 *Let* $(X,\ N_1,\ *_1)$ *and* $(Y,\ N_2,\ *_2)$ *be fuzzy normed linear spaces and* $*_2$ *be lower semicontinuous. Let* $BF(X,Y)$ *denote the set of all fuzzy bounded linear operators defined from X to Y. Then the mapping* $N \mid BF(X,Y)\times R\to[0,1]$ *defined by*

$$N(T,\ s)=\begin{cases}\vee\{\alpha\in(0,1)\} \mid \bigvee_{x\in X,x\neq 0}\bigwedge\{\dfrac{t}{d_{1-\alpha}}>0 \mid N_2(Tx,\ t)\geq\alpha\}\leq s, & \textit{for } (T,s)\neq(O,0)\\ 0, & \textit{for } (T,s)=(O,0)\end{cases}$$

is a fuzzy norm in $BF(X,Y)$ w.r.t. the underlying t-norm $*_2$.

Proof First we show that for $T\in BF(X,Y)$,

$$\bigvee_{x\in X,x\neq 0}\bigwedge\{\frac{t}{d_{1-\alpha}}>0 \mid N_2(Tx,\ t)\geq\alpha\}$$

exists for each $\alpha\in(0,1)$ and monotonically increasing w.r.t. α.

Since $T\in BF(X,Y)$ for each $\alpha\in(0,1),\exists M_\alpha>0$ such that

$\bigwedge\{t>0 \mid N_2(Tx,\ t)\geq\alpha\}\leq M_\alpha\bigwedge\{t>0 \mid N_1(x,\ t)\geq 1-\alpha\}\ \forall x\in X$

$\Rightarrow \bigwedge\{t>0 \mid N_2(Tx,\ t)\geq\alpha\}\leq M_\alpha d_{1-\alpha}\ \forall x\in X$

$\Rightarrow \bigwedge\{\frac{t}{d_{1-\alpha}}>0 \mid N_2(Tx,\ t)\geq\alpha\}\leq M_\alpha\ \forall x(\neq\underline{0})\in X$

$\Rightarrow \bigwedge\{\frac{t}{d_{1-\alpha}}>0 \mid N_2(Tx,\ t)\geq\alpha\}$ exists and $\leq M_\alpha\ \forall\alpha\in(0,1)$.

Take $\alpha>\beta$. So $1-\alpha<1-\beta$. Thus

$\bigwedge\{t>0 \mid N_2(Tx,\ t)\geq\alpha\}\geq\bigwedge\{t>0 \mid N_2(Tx,\ t)\geq\beta\}$

and

$\bigwedge\{t>0 \mid N_2(Tx,\ t)\geq 1-\beta\}\geq\bigwedge\{t>0 \mid N_2(Tx,\ t)\geq 1-\alpha\}$

$\Rightarrow \bigwedge\{\frac{t}{d_{1-\alpha}}>0 \mid N_2(Tx,\ t)\geq\alpha\}\geq\bigwedge\{\frac{t}{d_{1-\beta}}>0 \mid N_2(Tx,\ t)\geq\beta\}\ \forall x(\neq\underline{0})\in X$

$$\Rightarrow \bigvee_{x\in X,x\neq 0}\bigwedge\{\frac{t}{d_{1-\alpha}}>0 \mid N_2(Tx,\ t)\geq\alpha\}\geq\bigvee_{x\in X,x\neq 0}\bigwedge\{\frac{t}{d_{1-\beta}}>0 \mid N_2(Tx,\ t)\geq\beta\}.$$

Thus $\bigvee_{x\in X,x\neq 0}\bigwedge\{\frac{t}{d_{1-\alpha}}>0 \mid N_2(Tx,\ t)\geq\alpha\}$ is monotonically increasing w.r.t. α.

Now we verify the following:

(i) If $s<0$, then clearly $N(T,\ s)=0$.

For $s = 0$, we have

$N(T, s) = 0 \ \ \forall T \in BF(X, Y)$.

So (NI) holds.

(ii) $\forall s > 0, \ N(T, s) = 1$

$\Leftrightarrow \bigvee_{x\in X, x\neq \underline{0}} \bigwedge\{\frac{t}{d_{1-\alpha}} > 0 \mid \ N_2(Tx, \ t) \geq \alpha\} \leq s \ \forall \alpha \in (0, 1), \ \forall s > 0$

$\Leftrightarrow \bigvee_{x\in X, x\neq \underline{0}} \bigwedge\{\frac{t}{d_{1-\alpha}} > 0 \mid \ N_2(Tx, \ t) \geq \alpha\} = 0 \ \forall \alpha \in (0, 1)$

$\Leftrightarrow \bigwedge\{t \ > 0 \mid \ N_2(Tx, \ t) \geq \alpha\} = 0 \ \forall \alpha \in (0, 1), \ \forall x(\neq \underline{0}) \in X$

$\Leftrightarrow N_2(Tx, \ t) = 1 \ \forall x \in X, \ \ \forall t > 0$

$\Leftrightarrow Tx \ = \underline{0} \ \forall x \in X$

$\Leftrightarrow T \ = O$.

So (NII) holds.

(iii) For any scalar $\lambda \neq 0$, we have

$$
\begin{aligned}
N(\lambda T, \ s) &= \bigvee\{\alpha \in (0, 1) \mid \bigvee_{x\in X, x\neq \underline{0}} \bigwedge\{\frac{t}{d_{1-\alpha}} > 0 \mid \ N_2((\lambda T)(x), t) \geq \alpha\} \leq s\} \\
&= \bigvee\{\alpha \in (0, 1) \mid \bigvee_{x\in X, x\neq \underline{0}} \bigwedge\{\frac{t}{d_{1-\alpha}} > 0 \mid \ N_2(T(x), \ \frac{t}{|\lambda|}) \geq \alpha\} \leq s\} \\
&= \bigvee\{\alpha \in (0, 1) \mid |\lambda| \bigvee_{x\in X, x\neq \underline{0}} \bigwedge\{\frac{t}{d_{1-\alpha}} > 0 \mid \ N_2(T(x), \ t) \geq \alpha\} \leq s\} \\
&= N(T, \ \frac{s}{|\lambda|}).
\end{aligned}
$$

So (NIII) holds.

(iv) We have to show that

$$N(T_1 + T_2, \ s + t) \geq N(T_1, \ s) *_2 N(T_2, \ t) \ \ \forall s, t \in R.$$

If possible suppose that the above relation does not hold.

So $\exists s_0, \ t_0 \in R$ such that $N(T_1 + T_2, \ s_0 + t_0) \ < \ N(T_1, \ s_0) *_2 N(T_2, \ t_0)$.

Choose $\alpha_0 \in (0, 1)$ such that

$$N(T_1 + T_2, \ s_0 + t_0) \ < \ \alpha_0 < N(T_1, \ \ s_0) *_2 N(T_2, \ t_0). \tag{7}$$

Since $*_2$ is lower semicontinuous, $\exists \alpha_1, \ \alpha_2 \in (0, 1)$ where $N(T_1, \ s_0) > \alpha_1$ and $N(T_2, \ t_0) > \alpha_2$ such that $\alpha_1 *_2 \alpha_2 \ > \ \alpha_0$.

Now

$N(T_1, \ s_0) > \ \alpha_1$

$\Rightarrow \bigvee_{x\in X, x\neq 0} \bigwedge\{\frac{s}{d_{1-\alpha_1}} > 0 \mid N_2(T_1x, \ s) \geq \alpha_1\} \leq s_0$

$\Rightarrow \bigwedge\{\frac{s}{d_{1-\alpha_1}} > 0 \mid N_2(T_1x, \ s) \geq \alpha_1\} \leq s_0 \ \forall x(\neq 0) \in X$.

Similarly, $\bigwedge\{\frac{t}{d_{1-\alpha_2}} > 0 \mid N_2(T_2x, \ t) \geq \alpha_2\} \leq s_0 \ \forall x(\neq 0) \in X$. So,

$\bigwedge\{\frac{s+t}{d_{1-\alpha_0}} > 0 \mid \ N_2(T_1x, \ s) \geq \alpha_1, N_2(T_2x, \ t) \geq \alpha_2 \ \} \leq s_0 + t_0 \ \forall x(\neq 0) \in X$.

(Since $\alpha_1 \geq \alpha_1 *_2 \alpha_2 \ > \alpha_0$. So $1 - \alpha_0 \ > \ 1 - \alpha_1$. Similarly, $1 - \alpha_0 \ > \ 1 - \alpha_2$.)

Thus

$\bigwedge\{\frac{s+t}{d_{1-\alpha_0}} > 0 \mid N_2((T_1 + T_2), \ s + t) \geq \alpha_1 *_2 \alpha_2\} \leq s_0 + t_0 \ \forall x(\neq 0) \in X$

$\Rightarrow \bigwedge\{\frac{t'}{d_{1-\alpha_0}} > 0 \mid N_2((T_1 + T_2), \ t') > \alpha_0\} \leq s_0 + t_0 \ \forall x(\neq 0) \in X$

$\Rightarrow \bigwedge\{\frac{t'}{d_{1-\alpha_0}} > 0 \mid N_2((T_1 + T_2), \ t') \geq \alpha_0\} \leq s_0 + t_0 \ \forall x(\neq 0) \in X$

$$\Rightarrow \bigvee_{x\in X, x\neq 0} \bigwedge\{\frac{t'}{d_{1-\alpha_0}} > 0 \mid N_2((T_1+T_2),\ t') \geq \alpha_0\} \leq s_0 + t_0$$

$$\Rightarrow N(T_1+T_2,\ s_0+t_0) \geq \alpha_0$$

which contradicts the relation (7).

Hence $N(T_1+T_2,\ s_0+t_0) \geq N(T_1,\ s_0) *_2 N(T_2,\ t_0)$.

So (NIV) holds.

(v) From boundedness of T, for any $\alpha \in (0,1)$, $\exists M_\alpha$ such that

$$\bigvee_{x\in X, x\neq 0} \bigwedge\{\frac{t}{d_{1-\alpha}} > 0 \mid N_2(Tx,\ t) \geq \alpha\} \leq M_\alpha.$$

So for $s \geq M_\alpha$, $\bigvee_{x\in X, x\neq 0} \bigwedge\{\frac{t}{d_{1-\alpha}} > 0 \mid N_2(Tx,\ t) \geq \alpha\} \leq s$.

Thus, $\lim_{s\to\infty} N(T,\ s) = 1$. So, (NV) holds.

Hence N is a fuzzy norm and $BF(X,Y)$ is a fuzzy normed linear space.

5. Completeness of $BF(X,Y)$

In this section, we introduce the idea of l-fuzzy convergent sequence, l-fuzzy Cauchy sequence and l-fuzzy complete set and study the completeness of $BF(X,Y)$.

Definition 5.1 *Let $(X,N,*)$ be a fuzzy normed linear space.*

(I) *A sequence $\{x_n\}$ is said to be l-fuzzy convergent if $\exists x \in X$ such that*

$$\lim_{n\to\infty} \bigwedge\{t > 0 \mid N(x_n - x,\ t) > 1-\alpha\} = 0\ \forall\alpha \in (0,1).$$

(II) *$\{x_n\}$ is said to be l-fuzzy Cauchy sequence if*

$$\lim_{m,n\to\infty} \bigwedge\{t > 0 \mid N(x_n - x_m,\ t) > 1-\alpha\} = 0\ \forall\alpha \in (0,1).$$

(III) *$F \subset X$ is said to be l-fuzzy complete if every l-fuzzy Cauchy sequence is l-fuzzy convergent to some point in X.*

Proposition 5.1 *Let $(X,N,*)$ be a fuzzy normed linear space and $*$ be lower semi-continuous. Then limit of every l-fuzzy convergent sequence in X space is unique.*

Proof Let $\beta \in (0,1)$. By the lower semicontinuity of $*$, $\exists\alpha \in (0,1)$ such that $(1-\alpha) * (1-\alpha) > \beta$.

Let $\{x_n\}$ be an l-fuzzy convergent sequence in X which converges to two different limits x and y.

So $\lim_{n\to\infty} \bigwedge\{t > 0 \mid N(x_n - x,\ t) > 1-\alpha\} = 0\ \ \forall\alpha \in (0,1)$

and

$$\lim_{n\to\infty} \bigwedge\{t > 0 \mid N(x_n - y,\ t) > 1-\alpha\} = 0\ \ \forall\alpha \in (0,1).$$

Then for $\epsilon > 0$, there exists positive integers $N_1(\epsilon,\ \alpha)$ and $N_2(\epsilon,\ \alpha)$ such that

$\bigwedge\{t > 0 \mid\ N(x_n - x,\ \ t) > 1-\alpha\} < \frac{\epsilon}{2}\ \forall n \geq N_1(\epsilon,\alpha)$

and

$\bigwedge\{t > 0 \mid\ N(x_n - y, t) > 1-\alpha\} < \frac{\epsilon}{2}\ \ \forall n \geq N_2(\epsilon,\alpha)$.

So $N(x_n - x, \frac{\epsilon}{2}) > 1-\alpha\ \ \forall n \geq N_1(\epsilon,\alpha)$ and $N(x_n - y, \frac{\epsilon}{2}) > 1-\alpha\ \ \forall n \geq N_2(\epsilon,\alpha)$.

Let $N_0 = \max\{N_1,\ N_2\}$. Thus $N(x_n - x, \frac{\epsilon}{2}) > 1-\alpha\ \ \forall n \geq N_0(\epsilon,\alpha)$

and

$N(x_n - y, \frac{\epsilon}{2}) > 1-\alpha\ \ \forall n \geq N_0(\epsilon,\alpha)$.

Now, $N(x-y,\epsilon) = N(x_n - y - x_n + x, \frac{\epsilon}{2} + \frac{\epsilon}{2}) \geq N(x_n - x, \frac{\epsilon}{2}) * N(x_n - y, \frac{\epsilon}{2})$.
So,

$$N(x-y,\epsilon) \geq (1-\alpha) * (1-\alpha) > \beta. \tag{8}$$

Since $\epsilon > 0$ and $\beta \in (0,1)$ are arbitrary, from (8), it follows that
$\forall t > 0,\ N(x-y,\ t) > \alpha\ \forall \alpha \in (0,1)$
$\Rightarrow \forall t > 0,\ N(x-y,t) = 1$
$\Rightarrow x - y = \underline{0}$.
So $x = y$.

Theorem 5.1 *Let* $(X, N_1, *_1)$ *and* $(Y, N_2, *_2)$ *be fuzzy normed linear spaces and* $*_2$ *be lower semicontinuous. If* $(Y, N_2, *_2)$ *is l-fuzzy complete then BF(X, Y) is also l-fuzzy complete w.r.t. the underlying t-norm* $*_2$.

Proof Let $\{T_n\}$ be an l-fuzzy Cauchy sequence in $BF(X, Y)$. So
$\lim\limits_{m,n\to\infty} \bigwedge\{t > 0 \mid N(T_n - T_m,\ t)\ > 1-\alpha\} = 0\ \forall \alpha \in (0,1)$.
Thus for a given $\epsilon > 0$ and for $\alpha \in (0,1)$, $\exists N(\alpha,\epsilon)$ such that
$\bigwedge\{t > 0 \mid\ N(T_n - T_m,\ t)\ > 1-\alpha\}\ < \epsilon\ \forall m,\ n \geq N(\alpha,\epsilon)$
$\Rightarrow N(T_n - T_m\ ,\ \epsilon)\ > 1-\alpha\ \forall m, n \geq N(\alpha,\ \epsilon)$

$$\Rightarrow \bigvee_{x\in X,\ x\neq 0} \bigwedge\{\frac{s}{d_\alpha} > 0 \mid\ N_2(T_n x - T_m x,\ s) \geq 1-\alpha\} \leq \epsilon\ \forall m, n \geq N(\alpha,\ \epsilon) \tag{9}$$

$\Rightarrow \bigwedge\{\frac{s}{d_\alpha} > 0 \mid\ N_2(T_n x - T_m x,\ s) \geq 1-\alpha\} \leq \epsilon\ \forall m, n \geq N(\alpha,\ \epsilon)\ \forall x (\neq \underline{0}) \in X$
$\Rightarrow \lim\limits_{m,n\to\infty} \bigwedge\{s > 0 \mid\ N(T_n x - T_m x,\ s)\ \geq 1-\alpha\} = 0\ \forall x \in X,\ \forall \alpha \in (0,1)$
$\Rightarrow \lim\limits_{m,\ n\to\infty} \bigwedge\{s > 0 \mid\ N(T_n x - T_m x,\ s)\ > 1-\alpha\} = 0\ \forall x \in X,\ \forall \alpha \in (0,1)$.
Thus $\{T_n x\}$ is an α-fuzzy Cauchy sequence in Y. Since $\alpha \in (0,1)$ is arbitrary, it follows that $\{T_n x\}$ is l-fuzzy Cauchy sequence in Y for each $x \in X$.

So for each $x \in X$, there exists $y \in Y$ such that $\lim\limits_{n\to\infty} T_n(x) = y$.

Thus we can define a function T given by $\lim\limits_{n\to\infty} T_n(x) =\ T(x)$.

So,

$$\lim_{n\to\infty} \bigwedge\{t > 0 \mid\ N_2(T_n x - Tx,\ t)\ > 1-\alpha\} = 0\ \forall x \in X,\ \forall \alpha \in (0,1). \tag{10}$$

Now show that T is linear. We have
$\lim\limits_{n\to\infty} \bigwedge\{t > 0 \mid\ N_2(T_n x - Tx,\ t)\ > 1-\alpha\} = 0\ \forall x \in X,\ \forall \alpha \in (0,1)$.
For $\epsilon > 0$, $\exists n_1(\alpha,\epsilon)$ such that
$\bigwedge\{t > 0 \mid\ N_2(T_n x - Tx,\ \ t)\ > 1-\alpha\}\ < \frac{\epsilon}{2}\ \forall x \in X,\ \forall n \geq n_1(\alpha,\epsilon)$
$\Rightarrow N_2(T_n x - Tx,\ \frac{\epsilon}{2})\ > 1-\alpha\ \forall x \in X,\ \forall n \geq n_1(\alpha,\epsilon)$.
Similarly, $N_2(T_n y - Ty,\ \frac{\epsilon}{2})\ > 1-\alpha\ \forall y \in X,\ \forall n \geq n_2(\alpha,\epsilon)$.
Thus $N_2(T_n x - Tx,\ \ \frac{\epsilon}{2}) *_2 N_2(T_n y - Ty,\ \frac{\epsilon}{2}) \geq (1-\alpha) *_2 (1-\alpha)\ \forall n \geq n_0(\alpha)$, where $n_0(\alpha) = \max\ \{n_1(\alpha),\ n_2(\alpha)\}$.
So we get
$N_2(T_n(x+y) - (Tx + Ty),\ \epsilon) \geq (1-\alpha) *_2 (1-\alpha)\ \forall n \geq n_0(\alpha)$.

Let $\beta \in (0,1)$. Then $\exists \alpha = \alpha(\beta) \in (0,1)$ such that $(1-\alpha) *_2 (1-\alpha) \geq 1-\beta$. Thus we have

$N_2(T_n(x+y) - (Tx+Ty),\ \ \epsilon) \ > \ 1-\beta \ \forall n \geq n_0(\alpha, \epsilon)$.

$\Rightarrow \bigwedge\{t > 0 \mid N_2(T_n(x+y) - (Tx+Ty),\ t) > 1-\beta\} \leq \epsilon \ \forall n \geq n_0(\alpha, \epsilon)$

$\Rightarrow \lim_{n\to\infty} \bigwedge\{t > 0 \mid N_2(T_n(x+y) - (Tx+Ty),\ t) > 1-\beta\} = 0$.

Since $\beta \in (0,1)$ is arbitrary, it follows that $\{T_n(x+y)\}$ is l-fuzzy convergent and converges to $Tx+Ty$.

So $\lim_{n\to\infty} T_n(x+y) = \ Tx+Ty$. Hence $T(x+y) = \ Tx+Ty$.

Similarly, for any scalar λ, it can be shown that $T(\lambda x) = \lambda T(x)$. Hence T is linear.

Now we prove that T is fuzzy bounded.

Let $\gamma \in (0,1)$. By the lower semicontinuity of $*_2$, $\exists \alpha = \alpha(\gamma) \in (0,1)$ such that

$(1-\alpha) *_2 (1-\alpha) \geq \gamma$.

From (9), for $\alpha = \alpha(\gamma) \in (0,1)$, $\epsilon > 0$, $\exists$ a positive integer $N(\alpha(\gamma), \epsilon) \in \mathcal{N}$ such that

$\bigwedge\{\frac{s}{d_\alpha} > 0 \mid \ N_2(T_n x - T_m x,\ s) \geq 1-\alpha\} \leq \frac{\epsilon}{3} < \frac{\epsilon}{2} \ \forall m, n \geq N(\alpha(\gamma), \epsilon),\ \forall x(\neq \underline{0}) \in X$

$$\Rightarrow N_2(T_n x - T_m x, \frac{d_\alpha \epsilon}{2}) \geq 1-\alpha \ \forall m, n \geq N(\alpha(\gamma), \epsilon),\ \forall x(\neq \underline{0}) \in X. \quad (11)$$

From (10), it follows that, for $\alpha = \alpha(\gamma) \in (0,1)$, $\epsilon > 0$, $x(\neq \underline{0}) \in X$, $\exists$ a positive integer $N'(\alpha(\gamma), \epsilon, x) \in \mathcal{N}$ such that

$\bigwedge\{t > 0 \mid \ N_2(T_m x - Tx,\ t) > 1-\alpha\} < \frac{d_\alpha \epsilon}{2} \ \forall m \geq N'(\alpha(\gamma), \epsilon, x)$

$$\Rightarrow N_2(T_m x - Tx,\ \frac{d_\alpha \epsilon}{2}) > 1-\alpha \ \forall m \geq N'(\alpha(\gamma), \epsilon, x). \quad (12)$$

Now,

$$N_2(T_n x - Tx,\ \frac{d_\alpha \epsilon}{2} + \frac{d_\alpha \epsilon}{2})$$
$$= N_2(T_n x - T_m x + T_m x - Tx, \frac{d_\alpha \epsilon}{2} + \frac{d_\alpha \epsilon}{2})$$
$$\geq N_2(T_n x - T_m x, \frac{d_\alpha \epsilon}{2}) *_2 N_2(T_m x - Tx, \frac{d_\alpha \epsilon}{2})$$
$$\geq (1-\alpha) *_2 (1-\alpha) \geq \gamma \ \forall n \geq N(\alpha, \epsilon), \alpha \in (0,1), \forall x(\neq \underline{0}) \in X.$$

(By taking m suitably so as to satisfy (11) and (12)).

So,

$N_2(T_n x - Tx,\ d_\alpha \epsilon) \geq \gamma \ \forall n \geq N(\alpha, \epsilon), \alpha \in (0,1), \forall x(\neq \underline{0}) \in X$

$\Rightarrow \bigwedge\{t > 0 \mid N_2(T_n x - Tx,\ t) \geq \gamma\} \leq d_\alpha \epsilon \leq d_{1-\gamma}\epsilon$

(Since $1-\alpha > \gamma$, so $1-\gamma > \alpha$ and thus $d_{1-\gamma} \geq d_\alpha$)

$$\Rightarrow \bigwedge\{\frac{t}{d_{1-\gamma}} > 0 \mid N_2(T_n x - Tx,\ t) \geq \gamma\} < \epsilon,\ \forall n \geq N(\alpha(\gamma), \epsilon),\ \forall x(\neq \underline{0}) \in X. \quad (13)$$

Hence $T_n - T$ is bounded for all $n \geq N(\alpha(\gamma), \epsilon)$.

Since $BF(X, Y)$ is a linear space, so $T = (T - T_n) + T_n \in BF(X,\ Y)$.

Again from (13) we have

$N(T_n - T, \epsilon) \geq \gamma \ \forall n \geq N(\alpha(\gamma), \epsilon)$

$\Rightarrow \bigwedge\{s > 0 \mid \ N(T_n - T,\ s) \geq \gamma\} \leq \epsilon \ \forall n \geq N(\alpha(\gamma), \epsilon)$

$\Rightarrow \lim_{n\to\infty} \bigwedge\{s > 0 \mid N(T_n - T, s) \geq \gamma\} = 0.$

Since $\gamma \in (0, 1)$ is arbitrary it follows that

$\lim_{n\to\infty} \bigwedge\{s > 0 \mid N(T_n - T, s) \geq \alpha\} = 0 \ \forall \alpha \in (0, 1).$

So, $\{T_n\}$ is l-fuzzy convergent and converges to $T \in BF(X, Y)$. Thus $BF(X, Y)$ is l-fuzzy complete.

6. Conclusion

In an earlier paper, we introduced the concept of finite dimensional fuzzy normed linear spaces in general t-norm setting. Many authors published several papers on fuzzy normed linear spaces by using particular t-norm "min". The results of this paper are more general than the results related to the similar concept in particular t-norm "min".

It is interesting to note that, here boundedness of a linear operator implies its continuity but the converse is not true. It is an open problem to find an additional condition on continuity together which will be equivalent to boundedness.

We think that all the results of this paper are important for further development of operator theory (general t-norm setting) in fuzzy normed linear spaces.

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