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Lower bounds on several versions of signed domination number[☆]

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Abstract

Three numerical invariants of graphs concerning domination, which are named the signed domination number γ_s, the k-subdomination number γ_{ks} and the signed total domination number γ_{st}, are studied in this paper. For any graph, some lower bounds on γ_s, γ_{ks} and γ_{st} are presented, some of which generalize several known lower bounds on γ_s, γ_{ks} and γ_{st}, while others are considered as new. It is also shown that these bounds are sharp.

Keywords: Signed domination number; k-Subdomination number; Signed total domination number

1. Introduction

Let $G=(V,E)$ be a simple undirected graph with vertex set V and edge set E. The order of G is given by $n=|V|$ and its size by $m=|E|$. For $v \in V$, we denote by $d_G(v)$ the degree of v in G, by $N(v)$ the neighborhood of v and by $N[v]=N(v)\cup\{v\}$ its closed neighborhood. The maximum degree among the vertices of G is denoted by $\Delta(G)$ and the minimum degree by $\delta(G)$. For disjoint subsets A and B of vertices, we let $E(A,B)$ denote the set of edges between A and B. For $S\subseteq V$, $G[S]$ denotes the graph induced by S. Let $\bar{G}$ be the complement graph of G. For any real x, we denote $\lceil x\rceil$ for the minimum integer not less than x, and $\lfloor x\rfloor$ for the maximum integer not more than x. Finally, for a real-valued function $f: V\to R$ the weight of f is $w(f)=\sum_{v\in V} f(v)$, and, for $S\subseteq V$, we define $f(S)=\sum_{v\in S} f(v)$, so $w(f)=f(V)$. Other terminologies used in this paper will follow [1].

A *signed dominating function* of G is defined in [5] as a function f: $V\to\{-1,1\}$ such that $f(N[v])\geqslant 1$ for every $v\in V$. A *signed total dominating function* of G is defined in [16] as a function f: $V\to\{-1,1\}$ satisfying $f(N(v))\geqslant 1$ for all $v\in V$. The *signed domination number* for a graph G is $\gamma_s(G)=\text{minimum}\{w(f)|f$ is a signed dominating function of $G\}$. Similarly, the *signed total domination number* for a graph G is $\gamma_{st}(G)=\text{minimum}\{w(f)|f$ is a signed total dominating function of $G\}$.

For a positive integer k, a *k-subdominating function* of G is a function f: $V\to\{-1,1\}$ such that $f(N[v])\geqslant 1$ for at least k vertices v of G. The *k-subdomination number* for a graph G is defined as $\gamma_{ks}=\text{minimum}\{w(f)|f$ is a

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k-subdominating function of $G\}$. The concept of k-subdomination number was first introduced and studied by Cockayne and Mynhardt [4]. In the special cases where $k = |V|$ and $k = \lceil |V|/2 \rceil$, $\gamma_{ks}(G)$ is, respectively, the signed domination number $\gamma_s(G)$ [5] and the majority domination number $\gamma_{maj}(G)$ [2].

Because there are signed dominating functions (k-subdominating functions) in any graph G, e.g. $f(v) = 1$ for all $v \in V$, and because the number of these functions is finite, $\gamma_s(G)$ ($\gamma_{ks}(G)$) exists. Concerning a graph G without isolated vertices, there exists $\gamma_{st}(G)$ for the same reason.

Since the problems of determining the signed domination number, the signed total domination number and the k-subdomination number are NP-complete, many works on bounds on γ_s, γ_{st} and γ_{ks} were studied in [2–7,9,10,12–17]. Many results on γ_s, γ_{st} and γ_{ks} of certain special graphs have been reported. However, a few results on the lower bounds on γ_s, γ_{st} and γ_{ks} of general graphs are reported [3,12,15,17], but these results in the general cases do not coincide with those in the special ones. For example, from the existing results on the lower bounds on $\gamma_s(G)$ [15,17], the following result [10] cannot be wholly deduced: for any r-regular graph G of order n, $\gamma_s(G) \geqslant n/(r+1)$ for r even, and $\gamma_s(G) \geqslant 2n/(r+1)$ for r odd. On the other hand, the existing results on the lower bounds on $\gamma_s(G)$ [15,17] cannot be derived from the known results on the lower bounds on $\gamma_{ks}(G)$ [3,12] either.

Our main aim in this paper is to achieve some better results on the lower bounds on γ_s, γ_{st} and γ_{ks} for a general graph, so that there is a certain degree of coincidence among our results and those known results for the special graphs, that is, the results in some special cases can be deduced directly from the results in a general case. By using a simple and uniform approach, we derive some lower bounds on γ_s, γ_{ks} and γ_{st} in terms of several different graph parameters. Some of these bounds generalize several known lower bounds on γ_s, γ_{ks} and γ_{st}, while others are new. In addition, it is shown that these bounds are sharp.

2. Lower bounds on signed domination number

For any graph G, the following theorem was proved in [17].

Theorem A (*Zhang et al. [17]*). *For any graph G of order n and size m:*

(1) $\gamma_s(G) \geqslant (\delta + 2 - \Delta)n/(\delta + 2 + \Delta)$,
(2) $\gamma_s(G) \geqslant n - 2/3m$,
(3) $\gamma_s(G) \geqslant 2\lceil(-1 + \sqrt{1 + 8n})/2\rceil - n$,

and these bounds are sharp, where $\delta = \delta(G)$ and $\Delta = \Delta(G)$.

For a graph G, let $V_1 = \{v \in V | d_G(v) = 0\}$, $V_2 = \{v \in V | d_G(v) = 1\}$, $V_3 = \{v \in V | N(v) \cap V_2 \neq \emptyset\}$, $C(G) = V - (V_1 \cup V_2 \cup V_3)$ and $\delta^*(G) = \text{minimum}\{d_G(v) | v \in C(G)\}$. Obviously, if $C(G) = \emptyset$, then $\gamma_s(G) = |V(G)|$. Therefore, in the following discussion we assume, without loss of generality, that $C(G) \neq \emptyset$. Thus, $\delta^*(G) \geqslant \delta(G)$ and $\delta^*(G) \geqslant 2$.

In terms of $\delta^*(G)$, the following results were given in [15].

Theorem B (*Yin et al. [15]*). *For any graph G of order n and size m:*

(1) $\gamma_s(G) \geqslant (\delta^* + 2 - \Delta)n/(\delta^* + 2 + \Delta)$,
(2) $\gamma_s(G) \geqslant n - (4m/3)/(\delta^*/2 + 1)$,
(3) $\gamma_s(G) \geqslant 2\lceil(-\delta^* + \sqrt{\delta^{*2} + 8(\delta^* + 2)n})/4\rceil - n$, *where $\delta^* = \delta^*(G)$ and $\Delta = \Delta(G)$.*

It has been proved in [15] that these three results in Theorem B are, respectively, tighter than those three results in Theorem A because $\delta^*(G) \geqslant \delta(G)$ and $\delta^*(G) \geqslant 2$.

In this section, these known results will be extended and some new results will be given by introducing another graph parameter, namely, the number of vertices having odd degree in G. This graph parameter is henceforth denoted by $o(G)$ or o for short, i.e. $o(G)$ is the cardinality of the set $\{v \in V | d_G(v) \text{ is odd}\}$. We begin with the following lemma.

Lemma 1. *Let f denote any signed dominating function of the graph $G = (V, E)$. Let $P = \{v \in V | f(v) = 1\}$, $M = \{v \in V | f(v) = -1\}$, $p = |P|$, $n = |V|$ and $\mathrm{o} = \mathrm{o}(G)$. Therefore, we have*

(1) $\sum_{v \in P} d_G(v) \geqslant \sum_{v \in M} d_G(v) + 2(n - p) + \mathrm{o}$,
(2) $\sum_{v \in P} d_{G[P]}(v) \geqslant \sum_{v \in P} \lceil d_G(v)/2 \rceil$.

Proof. For $v \in V$, let $d_P(v)$ and $d_M(v)$ denote the numbers of vertices of P and of M, respectively, which are adjacent to v. Then, we have $d_G(v) = d_P(v) + d_M(v)$. Since $f(N[v]) \geqslant 1$, it is clear that $d_P(v) \geqslant d_M(v)$ for $v \in P$, and $d_P(v) \geqslant d_M(v) + 2$ for $v \in M$. Hence, if $v \in P$, then $d_P(v) \geqslant \lceil d_G(v)/2 \rceil$ and $d_M(v) \leqslant \lfloor d_G(v)/2 \rfloor$; if $v \in M$, then $d_P(v) \geqslant \lceil (d_G(v) + 2)/2 \rceil$ and $d_M(v) \leqslant \lfloor (d_G(v) - 2)/2 \rfloor$.

(1) Considering $E(P, M)$, we can deduce

$$\sum_{v \in P} \lfloor d_G(v)/2 \rfloor \geqslant |E(P, M)| \geqslant \sum_{v \in M} \lceil (d_G(v) + 2)/2 \rceil,$$

$$\sum_{v \in P} \lfloor d_G(v)/2 \rfloor + \sum_{v \in P} \lceil d_G(v)/2 \rceil \geqslant \sum_{v \in M} \lceil (d_G(v) + 2)/2 \rceil + \sum_{v \in P} \lceil d_G(v)/2 \rceil,$$

$$\sum_{v \in P} d_G(v) \geqslant \sum_{v \in V} \lceil d_G(v)/2 \rceil + |M|,$$

$$\sum_{v \in P} d_G(v) \geqslant \sum_{v \in V} d_G(v)/2 + (n - p) + \mathrm{o}/2,$$

$$\sum_{v \in P} d_G(v) \geqslant \sum_{v \in P} d_G(v)/2 + \sum_{v \in M} d_G(v)/2 + (n - p) + \mathrm{o}/2,$$

$$\sum_{v \in P} d_G(v) - \sum_{v \in P} d_G(v)/2 \geqslant \sum_{v \in M} d_G(v)/2 + (n - p) + \mathrm{o}/2.$$

Hence,

$$\sum_{v \in P} d_G(v) \geqslant \sum_{v \in M} d_G(v) + 2(n - p) + \mathrm{o}.$$

(2) Considering the graph $G[P]$ induced from P, we have $d_{G[P]}(v) = d_P(v)$ for $v \in P$. Since $d_P(v) \geqslant \lceil d_G(v)/2 \rceil$ for $v \in P$, then $\sum_{v \in P} d_{G[P]}(v) \geqslant \sum_{v \in P} \lceil d_G(v)/2 \rceil$. □

For any graph G, $\gamma_s(G)$ is related to all the following graph parameters: $|V(G)|$, $|E(G)|$, $\delta(G)$, $\Delta(G)$, $\delta^*(G)$, $\mathrm{o}(G)$, etc. From lemma 1, considering different combinations of these parameters, we obtain some new lower bounds for $\gamma_s(G)$, and these new results are independent from each other. Now, we are in such a position to prove the following results.

Theorem 1. *For any graph G of order n and size m, we have*

(1) $\gamma_s(G) \geqslant ((\delta^* + 2 - \Delta)n + 2\mathrm{o})/(\delta^* + 2 + \Delta)$,
(2) $\gamma_s(G) \geqslant (2m + 2n + \mathrm{o})/(\Delta + 1) - n$,
(3) $\gamma_s(G) \geqslant n - (2m - \mathrm{o})/(\delta^* + 1)$,
(4) $\gamma_s(G) \geqslant 2\lceil (-\delta^* + \sqrt{\delta^{*2} + 8(\delta^* + 2)n + 8\mathrm{o}})/4 \rceil - n$,
(5) $\gamma_s(G) \geqslant 2\lceil (1 + \sqrt{1 + 8(m + n) + 4\mathrm{o}})/4 \rceil - n$,

and these bounds are all sharp, where $\delta^* = \delta^*(G)$, $\Delta = \Delta(G)$ and $\mathrm{o} = \mathrm{o}(G)$.

Proof. Let f be a signed dominating function of G such that $w(f) = \gamma_s(G)$. Let $P = \{v \in V | f(v) = 1\}$, $M = \{v \in V | f(v) = -1\}$ and $p = |P|$. So, $\gamma_s(G) = p - (n - p) = 2p - n$.

According to Lemma 1, we have

$$\sum_{v\in P} d_G(v) \geqslant \sum_{v\in M} d_G(v) + 2(n-p) + \mathrm{o}. \tag{*}$$

Notice that $M \subseteq C(G)$, so $d_G(v) \geqslant \delta^*$ for $v \in M$. Thus,

(1) Considering $\Delta \geqslant d_G(v)$ and $\delta^* \leqslant d_G(v)$ for every $v \in V$, we get from (*)

$$\Delta p \geqslant \sum_{v\in M} d_G(v) + 2(n-p) + \mathrm{o} \geqslant (2+\delta^*)(n-p) + \mathrm{o}.$$

From this inequality, it is deduced that

$$p \geqslant ((\delta^*+2)n + \mathrm{o})/(\delta^*+2+\Delta).$$

Hence, $\gamma_s(G) = 2p - n \geqslant ((\delta^*+2-\Delta)n + 2\mathrm{o})/(\delta^*+2+\Delta)$.

(2) Considering $2m = \sum_{v\in V} d_G(v)$ and $\Delta \geqslant d_G(v)$ for every $v \in V$, we get from $(*)$

$$2\sum_{v\in P} d_G(v) \geqslant \sum_{v\in M} d_G(v) + \sum_{v\in P} d_G(v) + 2(n-p) + \mathrm{o},$$

$$2\sum_{v\in P} d_G(v) \geqslant \sum_{v\in V} d_G(v) + 2(n-p) + \mathrm{o},$$

$$2\sum_{v\in P} d_G(v) \geqslant 2m + 2(n-p) + \mathrm{o},$$

$$2\Delta p \geqslant 2m + 2(n-p) + \mathrm{o}.$$

From this inequality, it is deduced that

$$p \geqslant (2m + 2n + \mathrm{o})/(2\Delta + 2).$$

Hence, $\gamma_s(G) = 2p - n \geqslant (2m + 2n + \mathrm{o})/(\Delta + 1) - n$.

(3) Considering $2m = \sum_{v\in V} d_G(v)$ and $\delta^* \leqslant d_G(v)$ for every $v \in V$, we get from $(*)$

$$\sum_{v\in P} d_G(v) + \sum_{v\in M} d_G(v) \geqslant 2\sum_{v\in M} d_G(v) + 2(n-p) + \mathrm{o},$$

$$\sum_{v\in V} d_G(v) \geqslant 2\sum_{v\in M} d_G(v) + 2(n-p) + \mathrm{o},$$

$$2m \geqslant 2\sum_{v\in M} d_G(v) + 2(n-p) + \mathrm{o},$$

$$2m \geqslant 2(\delta^*+1)(n-p) + \mathrm{o}.$$

From the inequality, it is deduced that

$$p \geqslant n - (2m - \mathrm{o})/(2\delta^* + 2).$$

So, $\gamma_s(G) = 2p - n \geqslant n - (2m - \mathrm{o})/(\delta^* + 1)$.

(4) Consider $G[P]$. According to Lemma 1, we have $\sum_{v\in P} d_{G[P]}(v) \geqslant \sum_{v\in P} \lceil d_G(v)/2 \rceil$. On the other hand, since $G[P]$ is a simple graph, $p(p-1) \geqslant \sum_{v\in P} d_{G[P]}(v)$.Then,

$$p(p-1) \geqslant \sum_{v\in P} \lceil d_G(v)/2 \rceil \geqslant \sum_{v\in P} d_G(v)/2,$$

$$2p(p-1) \geqslant \sum_{v\in P} d_G(v) \geqslant \sum_{v\in M} d_G(v) + 2(n-p) + \mathrm{o} \geqslant (2+\delta^*)(n-p) + \mathrm{o}.$$

From the inequality, it is deduced that

$$p \geqslant \lceil (-\delta^* + \sqrt{\delta^{*2} + 8(\delta^* + 2)n + 8\text{o}})/4 \rceil.$$

So, $\gamma_s(G) = 2p - n \geqslant 2\lceil (-\delta^* + \sqrt{\delta^{*2} + 8(\delta^* + 2)n + 8\text{o}})/4 \rceil - n$.

(5) According to (4), we have

$$2p(p-1) \geqslant \sum_{v \in P} d_G(v).$$

From (2), we have

$$\sum_{v \in P} d_G(v) \geqslant m + (n - p) + \text{o}/2.$$

So,

$$2p(p-1) \geqslant m + (n - p) + \text{o}/2.$$

From the inequality, it is deduced that

$$p \geqslant (1 + \sqrt{1 + 8(m + n) + 4\text{o}})/4.$$

Hence, $\gamma_s(G) = 2p - n \geqslant 2\lceil (1 + \sqrt{1 + 8(m + n) + 4\text{o}})/4 \rceil - n$.

In order to show that these bounds in Theorem 1 are sharp, we first should notice the following observation, a proof of which is straightward and therefore omitted.

Observation. For a lower bound on γ_s, if there exists a graph whose γ_s reaches the lower bound, then the lower bound is sharp; for a graph G, if there exists a signed dominating function f on G such that $w(f)$ attains the lower bound, then $\gamma_s(G)$ attains the lower bound.

In the following, for every lower bound on γ_s, we will give a graph G and construct a signed dominating function f on G such that $w(f)$ attains the lower bound, and thus the lower bound is sharp. We also clarify that our bounds for some of these graphs are tight and the corresponding bounds given in Theorem B are not.

In fact, a trivial example such as $G = K_n$ suffices for this, where n is an even number not less than 4. It is easy to check that $\gamma_s(K_n) = 2$ attains all the five bounds in Theorem 1, while both those bounds in Theorem B(1) and (2) are not more than 1. Besides, we can construct a non-complete graph with an arbitrary large order whose γ_s reaches the lower bounds in Theorem 1(1)–(3) as follows. Letting t be a positive integer, we consider a cycle of length $2t$ and color the edges red and blue alternatively. For every red edge, we add an additional vertex being adjacent to both endpoints of this red edge. Let P contain the cycle vertices and M the "corona" vertices. The obtained graph is denoted by G. It is easy to see that the graph G is a graph with $n = 3t$, $m = 4t$, $\delta^* = 2$, $\Delta = 3$ and $\text{o} = 2t$. Define a function $f : V \to \{-1, 1\}$ as follows: $f(v) = 1$ for $v \in P$ and $f(v) = -1$ for $v \in M$. It is easy to check that f is a signed dominating function with $w(f) = t$ and that all of these bounds in Theorem 1(1)–(3) are also t, which implies that $\gamma_s(G)$ attains these bounds. However, $\gamma_s(G)$ does not attain the corresponding bounds given in Theorem B(1) and (2), which are $\lceil 3t/7 \rceil$ and $\lceil t/3 \rceil$, respectively. Next, we show that there is also a graph G different from K_n such that $\gamma_s(G)$ reaches the lower bounds in Theorem 1(4) and (5). Let G be a graph obtained from $K_6 \cup \bar{K}_6$ by joining each vertex of $\bar{K}_6$ to three vertices of K_6 in such a way thateach vertex of K_6 is joined to exactly three vertices in $\bar{K}_6$. The graph G is a graph with $n = 12$, $m = 33$, $\delta^* = 3$ and $\text{o} = 6$. Define a function $f : V \to \{-1, 1\}$ as follows: $f(v) = 1$ for $v \in K_6$ and $f(v) = -1$ for $v \in \bar{K}_6$. It is not difficult to prove that f is a signed dominating function of G, and $w(f) = 0$. We can verify that $\gamma_s(G)$ attains the bounds given in Theorem 1(4) and (5), both of which are 0, but $\gamma_s(G)$ does not attain the corresponding bound given in Theorem B(3), which is -2. □

Since $\delta^*(G) \geqslant 2$ and $\text{o}(G) \geqslant 0$, we easily deduce that the bound in Theorem 1(3) is tighter than the corresponding one in Theorem B(2) by directly comparing these two bounds. Besides, it is easy to see that Theorem B(1) and (3) are the special cases of Theorem 1(1) and (4) where $\text{o}(G) = 0$, respectively. Considering that each of both the results in Theorem 1(1) and (4) is an increasing function of $o(G)$ and the fact $o(G) \geqslant 0$, we deduce that they are tighter than those in Theorem B(1) and (3).

From Theorem 1(1) or (2) or (3), the following result can be immediately deduced.

Corollary 1 (*Henning [10]*). *For any r-regular graph G of order n, $\gamma_s(G) \geqslant n/(r+1)$ for r even, and $\gamma_s(G) \geqslant 2n/(r+1)$ for r odd. Furthermore, the bounds are sharp.*

3. Lower bounds on k-subdomination number

For $\gamma_{ks}(G)$ of any graph G, the results in Theorem C and D were presented in [3] and [12], respectively.

Theorem C (*Chang et al. [3]*). *If G be a graph of order n with degree sequence $d_1 \leqslant \ldots \leqslant d_n$, $\Delta = \Delta(G)$, then*

$$\gamma_{ks}(G) \geqslant \frac{2}{\Delta+1} \sum_{j=1}^{k} \left\lceil \frac{d_j+2}{2} \right\rceil - n.$$

Theorem D (*Kang et al. [12]*). *For any graph G of order n and size m, let $\Delta = \Delta(G)$ and $\delta = \delta(G)$. Then,*

$$\gamma_{ks}(G) \geqslant n - \frac{2m + (n-k)(\Delta+2)}{\delta+1}.$$

In the remaining part of this section, we will use a uniform approach to prove two results on γ_{ks}, one of which is the same as the result in Theorem C, and the other generalizes the result in Theorem D. We begin with the following lemma.

Lemma 2. *Given a graph G and a positive integer k. Let f be a k-subdominating function of G. Let $P = \{v \in V \mid f(v) = 1\}$, $M = \{v \in V \mid f(v) = -1\}$ and $p = |P|$. Let $P_1 = \{v \in V \mid f(v) = 1, f(N[v]) \geqslant 1\}$, $P_2 = P - P_1$, $M_1 = \{v \in V \mid f(v) = -1, f(N[v]) \geqslant 1\}$ and $M_2 = M - M_1$. Then,*

$$\sum_{v \in P} d_G(v) + |P_1| \geqslant \sum_{v \in P1 \cup M1} \lceil (d_G(v)+2)/2 \rceil.$$

Proof. For $v \in V$, let v be adjacent to exactly $d_P(v)$ vertices of P and exactly $d_M(v)$ vertices of M. Thus, $d_G(v) = d_P(v) + d_M(v)$. For $v \in P_1 \cup M_1$, since $f(N[v]) \geqslant 1$, it is clear that $d_P(v) \geqslant d_M(v)$ for $v \in P_1$, and $d_P(v) \geqslant d_M(v) + 2$ for $v \in M_1$. Hence, if $v \in P_1$, then $d_P(v) \geqslant \lceil d_G(v)/2 \rceil$ and $d_M(v) \leqslant \lfloor d_G(v)/2 \rfloor$; if $v \in M_1$, then $d_P(v) \geqslant \lceil (d_G(v)+2)/2 \rceil$ and $d_M(v) \leqslant \lfloor (d_G(v)-2)/2 \rfloor$.

Let us consider the set $E(P, M)$. So,

$$\sum_{v \in P1} \lfloor d_G(v)/2 \rfloor + \sum_{v \in P2} d_G(v) \geqslant |E(P,M)| \geqslant \sum_{v \in M1} \lceil (d_G(v)+2)/2 \rceil,$$

$$\sum_{v \in P1} \lfloor d_G(v)/2 \rfloor + \sum_{v \in P2} d_G(v) + \sum_{v \in P1} \lceil d_G(v)/2 \rceil \geqslant \sum_{v \in M1} \lceil (d_G(v)+2)/2 \rceil + \sum_{v \in P1} \lceil d_G(v)/2 \rceil,$$

$$\sum_{v \in P1} d_G(v) + \sum_{v \in P2} d_G(v) \geqslant \sum_{v \in M1} \lceil d_G(v)/2 \rceil + \sum_{v \in P1} \lceil d_G(v)/2 \rceil + |M_1|,$$

$$\sum_{v \in P} d_G(v) \geqslant \sum_{v \in P1 \cup M1} \lceil d_G(v)/2 \rceil + |M_1|.$$

Note that $M_1 \cap P_1 = \emptyset$. Hence,

$$\sum_{v \in P} d_G(v) + |P_1| \geqslant \sum_{v \in P1 \cup M1} \lceil (d_G(v)+2)/2 \rceil. \qquad \square$$

By Lemma 2, we can obtain the following results.

Theorem 2. *For any positive integer k and any graph G with degree sequence* $d_1 \leqslant \cdots \leqslant d_n$, *let* $m=|E(G)|, n=|V(G)|$ *and* $f_k = \sum_{j=1}^{k} \lceil (d_j+2)/2 \rceil$. *Then,*

(1) $\gamma_{ks}(G) \geqslant \frac{2f_k}{\Delta+1} - n$,
(2) $\gamma_{ks}(G) \geqslant n - \frac{4m+2n-2f_k}{\delta+1}$,

and these bounds are all sharp, where $\delta = \delta(G)$ *and* $\Delta = \Delta(G)$.

Proof. Let f be a k-subdominating function of G such that $w(f)=\gamma_{ks}(G)$, $P=\{v \in V | f(v)=1\}$, $M=\{v \in V | f(v)=-1\}$ and $p=|P|$. Let $P_1=\{v \in V | f(v)=1, f(N[v]) \geqslant 1\}$, $P_2=P-P_1$, $M_1=\{v \in V | f(v)=-1, f(N[v]) \geqslant 1\}$ and $M_2=M-M_1$. Then, $|M_1|+|P_1| \geqslant k$ and $\gamma_{ks}(G)=p-(n-p)=2p-n$.

By Lemma 2, we have

$$\sum_{v \in P} d_G(v) + |P_1| \geqslant \sum_{v \in P1 \cup M1} \lceil (d_G(v)+2)/2 \rceil. \tag{**}$$

(1) Considering $\Delta \geqslant d_G(v)$ for every $v \in V$, we get from (**)

$$\Delta p + p \geqslant \sum_{v \in P1 \cup M1} \lceil (d_G(v)+2)/2 \rceil.$$

Noting that $|M_1 \cup P_1| = |M_1| + |P_1| \geqslant k$, we have

$$p \geqslant \left(\sum_{v \in P1 \cup M1} \lceil (d_G(v)+2)/2 \rceil \right) / (\Delta+1) \geqslant f_k/(\Delta+1).$$

Thus, $\gamma_{ks}(G) = 2p-n \geqslant \frac{2f_k}{\Delta+1} - n$.

(2) Considering $2m = \sum_{v \in V} d_G(v)$ and $\delta \leqslant d_G(v)$ for every $v \in V$, we get from (**)

$$\sum_{v \in P} d_G(v) + \sum_{v \in M} d_G(v) + |P_1| \geqslant \sum_{v \in P1 \cup M1} \lceil (d_G(v)+2)/2 \rceil + \sum_{v \in M} d_G(v),$$
$$\sum_{v \in V} d_G(v) + |P_1| \geqslant \sum_{v \in P1 \cup M1} \lceil (d_G(v)+2)/2 \rceil + \sum_{v \in M} d_G(v),$$
$$2m + |P_1| \geqslant \sum_{v \in P1 \cup M1} \lceil (d_G(v)+2)/2 \rceil + \sum_{v \in M} d_G(v),$$
$$p \geqslant |P_1| \geqslant -2m + \sum_{v \in P1 \cup M1} \lceil (d_G(v)+2)/2 \rceil + \sum_{v \in M} d_G(v) \geqslant -2m + f_k + \delta(n-p),$$
$$p \geqslant \frac{-2m + \delta n + f_k}{\delta+1}.$$

Thus, $\gamma_{ks}(G) = 2p-n \geqslant \frac{-4m+(\delta-1)n+2f_k}{\delta+1} = n - \frac{4m+2n-2f_k}{\delta+1}$.

In the special case where $k=|V|$, considering that $2f_k = 2m+2n+o$ and that δ is actually δ^* in the process of the above proof, we can immediately get those two bounds in Theorem 1(2) and (3) from these two bounds in Theorem 2, respectively. Besides, those bounds in Theorem 1 are sharp, so there exist graphs whose γ_{ks}'s attain the bounds in Theorem 2. In this sense, the bounds in Theorem 2 are sharp. □

Notice that Theorem 2(1) is Theorem C. Considering that

$$f_k = \sum_{j=1}^{k} \left\lceil \frac{d_j+2}{2} \right\rceil = \sum_{j=1}^{k} \frac{d_j}{2} + k + o_k/2 = m - \sum_{j=k+1}^{n} \frac{d_j}{2} + k + o_k/2$$
$$\geqslant m - (n-k)\Delta/2 + k + o_k/2,$$

we get from Theorem 2(2)

$$\gamma_{ks}(G) \geqslant n - \frac{4m + 2n - 2f_k}{\delta + 1} \geqslant n - \frac{2m + (n-k)(\Delta + 2) - o_k}{\delta + 1} \geqslant n - \frac{2m + (n-k)(\Delta + 2)}{\delta + 1}.$$

This yields Theorem D, where o_k is the cardinality of the set $\{j | d_j \text{ is odd}, 1 \leqslant j \leqslant k\}$.

By setting $d_1 = d_2 = \ldots = d_n = r$ in Theorem 2(1) or (2), we have:

Corollary 2 (*Hattingh et al. [9]*). *For every r-regular* ($r \geqslant 2$) *graph G of order n*, $\gamma_{ks}(G) \geqslant -n + k(r+3)/(r+1)$ *for r odd, and* $\gamma_{ks}(G) \geqslant -n + k(r+2)/(r+1)$ *for r even.*

In the special case where $k = \lceil |V|/2 \rceil$, from Theorem 2(1) or (2), we easily obtain the following result.

Corollary 3 (*Henning [10]*). *For every r-regular* ($r \geqslant 2$) *graph G of order n*, $\gamma_{\text{maj}}(G) \geqslant (1-r)n/(2(r+1))$ *for r odd, and* $\gamma_{\text{maj}}(G) \geqslant -rn/(2(r+1))$ *for r even.*

4. Lower bounds on signed total domination number

In this section, we consider any graph G without isolated vertices. We will establish several lower bounds on γ_{st}in a way similar to the case of γ_{s}.

Lemma 3. *Given a graph G without isolated vertices. Let f be a signed total dominating function of G. Let* $P = \{v \in V | f(v) = 1\}$, $M = \{v \in V | f(v) = -1\}$, $p = |P|$, $n = |V|$ *and* $\text{o} = \text{o}(G)$. *Then,*

(1) $\sum_{v \in P} d_G(v) \geqslant \sum_{v \in M} d_G(v) + 2n - o$,
(2) $\sum_{v \in P} d_{G[P]}(v) \geqslant \sum_{v \in P} \lceil (d_G(v) + 1)/2 \rceil$.

Proof. For $v \in V$, let v be adjacent to $d_P(v)$ vertices of P and adjacent to $d_M(v)$ vertices of M. Thus, $d_G(v) = d_P(v) + d_M(v)$. For $v \in V$, since $f(N(v)) \geqslant 1$, we have $d_P(v) \geqslant d_M(v) + 1$. Therefore, $d_P(v) \geqslant \lceil (d_G(v) + 1)/2 \rceil$ and $d_M(v) \leqslant \lfloor (d_G(v) - 1)/2 \rfloor$ for $v \in V$.

(1) Consider the set $E(P,M)$. Thus,

$$\sum_{v \in P} \lfloor (d_G(v) - 1)/2 \rfloor \geqslant |E(P, M)| \geqslant \sum_{v \in M} \lceil (d_G(v) + 1)/2 \rceil,$$
$$\sum_{v \in P} \lfloor (d_G(v) - 1)/2 \rfloor + \sum_{v \in P} \lceil (d_G(v) + 1)/2 \rceil \geqslant \sum_{v \in M} \lceil (d_G(v) + 1)/2 \rceil + \sum_{v \in P} \lceil (d_G(v) + 1)/2 \rceil,$$
$$\sum_{v \in P} d_G(v) \geqslant \sum_{v \in V} \lceil (d_G(v) + 1)/2 \rceil,$$
$$\sum_{v \in P} d_G(v) \geqslant \sum_{v \in V} d_G(v)/2 + n/2 + (n - \text{o})/2,$$
$$\sum_{v \in P} d_G(v) \geqslant \sum_{v \in P} d_G(v)/2 + \sum_{v \in M} d_G(v)/2 + n - \text{o}/2.$$

Hence,

$$\sum_{v \in P} d_G(v) \geqslant \sum_{v \in M} d_G(v) + 2n - \text{o}.$$

(2) Considering graph $G[P]$ induced from P, we have $d_{G[P]}(v) = d_P(v)$ for $v \in P$. Since $d_P(v) \geqslant \lceil (d_G(v) + 1)/2 \rceil$ for $v \in P$, $\sum_{v \in P} d_{G[P]}(v) \geqslant \sum_{v \in P} \lceil (d_G(v) + 1)/2 \rceil$. $\square$

By Lemma 3, we can obtain the following theorem concerning the lower bounds on γ_{st}.

Theorem 3. *For any graph G without isolated vertices, let* $m=|E(G)|$ *and* $n=|V(G)|$. *Then,*

(1) $\gamma_{st}(G) \geqslant \frac{(\delta-\Delta+4)n-2o}{\delta+\Delta}$,
(2) $\gamma_{st}(G) \geqslant \frac{2m+2n-o}{\Delta}-n$,
(3) $\gamma_{st}(G) \geqslant n-\frac{2m-2n+o}{\delta}$,
(4) $\gamma_{st}(G) \geqslant 2\lceil (3-\delta+\sqrt{(\delta-3)^2+8(\delta+2)n-8o})/4\rceil - n$,
(5) $\gamma_{st}(G) \geqslant 2\lceil (3+\sqrt{9+8(m+n)-4o})/4\rceil - n$,

and these bounds are all sharp, where $\delta=\delta(G)$, $\Delta=\Delta(G)$ and $o=o(G)$.

Proof. A proof of these bounds in Theorem 3 is similar to the proof of those bounds in Theorem 1, and therefore is omitted. In the following, we will give a graph G such that $\gamma_{st}(G)$ attains all the five bounds, and thus these bounds are all sharp.

Let $t \geqslant 4$ be an integer. Take a graph H that is isomorphic to the complete graph K_t. To each $v \in V(H)$ assign a star $S(v)$ with $t-2$ edges. Identify the central vertex of $S(v)$ with v for each $v \in V(H)$. Denote the resulting graph by G, cf. [16]. Let $f : V(G) \to \{-1, 1\}$ be such that $f(v)=1$ for $v \in V(H)$ and $f(v)=-1$ otherwise. Obviously, f is a signed total dominating function of G and $w(f)=f(V(G))=t(3-t)$. It is easy to verify that all the five bounds in Theorem 3 are also $t(3-t)$, which implies that $\gamma_{st}(G)$ attains these bounds. □

From Theorem 3(1), (2) or (3), we can directly obtain the following result.

Corollary 4 (*Zelinka [16]*). *For any r-regular* ($r \geqslant 2$) *graph G of order n,* $\gamma_{st}(G) \geqslant n/r$ *for r being odd, and* $\gamma_{st}(G) \geqslant 2n/r$ *for r being even.*

Note. We have been recently informed from one of the referees that several other lower bounds on the signed total domination number of a general graph have been established in a different way by Henning [11].

Remark. We may view the concept γ_{ks} as a generalization of γ_s. Similarly, we may also generalize the concept γ_{st} as follows.

Definition 1. For a positive integer k, a *total k-subdomination function* G is a function $f : V \to \{-1, 1\}$ such that $f(N(v)) \geqslant 1$ for at least k vertices v of G. *The total k-subdomination number* for a graph G is defined as $\gamma_{kst}(G) =$ minimum $\{w(f)|f$ is a total k-subdominating function of $G\}$.

Quite analogous to the case of γ_{ks}, we have the following lemma, from which we can obtain Theorem 4 stating two different lower bounds on $\gamma_{kst}(G)$ of any graph G without isolated vertices. Their proofs are very similar to those in the case of γ_{ks}, and therefore are omitted.

Lemma 4. *Given a graph G without isolated vertices and a positive integer k. Let f be any total k-subdominating function of G. Let* $P=\{v \in V|f(v)=1\}$, $M=\{v \in V|f(v)=-1\}$ *and* $p=|P|$. *Let* $P_1=\{v \in V|f(v)=1, f(N(v)) \geqslant 1\}$, $P_2=P-P_1$, $M_1=\{v \in V|f(v)=-1, f(N(v)) \geqslant 1\}$ *and* $M_2=M-M_1$. *Then,*

$$\sum_{v\in P} d_G(v) \geqslant \sum_{v \in P1\cup M1} \lceil (d_G(v)+1)/2\rceil.$$

Theorem 4. *For any positive integer k and any graph G containing no isolated vertices with degree sequence* $d_1 \leqslant \cdots \leqslant d_n$, *let* $m=|E(G)|$, $n=|V(G)|$ *and* $g_k=\sum_{j=1}^{k}\lceil (d_j+1)/2\rceil$. *Then,*

(1) $\gamma_{kst}(G) \geqslant \frac{2g_k}{\Delta}-n$,
(2) $\gamma_{kst}(G) \geqslant n-\frac{4m-2g_k}{\delta}$,

and these bounds are all sharp, where $\delta=\delta(G)$ *and* $\Delta=\Delta(G)$.

It is easy to see that the results in Theorem 4(1) and (2) generalize the results in Theorem 3(2) and (3), respectively. Moreover, we can directly obtain the following result from Theorem 4(1) or (2).

Corollary 5. *For any positive integer k and every r-regular* $(r \geqslant 2)$ *graph G of order n*, $\gamma_{kst}(G) \geqslant (r+1)k/r - n$ *for r being odd*, *and* $\gamma_{kst}(G) \geqslant (r+2)k/r - n$ *for r being even*.

Note. We have been recently informed from one of the referees that an analogous theory for total k-subdominating functions has been studied in the Ph.D. dissertation by Harris [8].

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