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Binding Numbers and f-Factors of Graphs

MIKIO KANO

Akashi College of Technology, Akashi 674, Japan

AND

NORIHIDE TOKUSHIGE

Department of Computer Science, Meiji University, Higashimita, Tama-ku, Kawasaki 214, Japan

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Let G be a connected graph of order n, a and b be integers such that $1 \leqslant a \leqslant b$ and $2 \leqslant b$, and $f\colon V(G) \to \{a, a+1, ..., b\}$ be a function such that $\sum(f(x); x \in V(G)) \equiv 0 \pmod 2$. We prove the following two results: (i) If the binding number of G is greater than $(a+b-1)(n-1)/(an-(a+b)+3)$ and $n \geqslant (a+b)^2/a$, then G has an f-factor; (ii) If the minimum degree of G is greater than $(bn-2)/(a+b)$, and $n \geqslant (a+b)^2/a$, then G has an f-factor.

1. INTRODUCTION

We consider a finite graph G with vertex set $V(G)$ and edge set $E(G)$, which has neither loops nor multiple edges. For a vertex x of G, the neighborhood $N_G(x)$ of x in G is the set of vertices of G adjacent to x, and the degree $\deg_G(x)$ of x is $|N_G(x)|$. We denote by $\delta(G)$ the minimum degree of G. For a subset X of $V(G)$, let

$$N_G(X) := \bigcup_{x \in X} N_G(x).$$

We say that X is independent if $N_G(X) \cap X = \varnothing$. The *binding number* bind(G) of G is defined by

$$\mathrm{bind}(G) := \min \left\{ \frac{|N_G(X)|}{|X|} \mid \varnothing \neq X \subset V(G),\, N_G(X) \neq V(G) \right\}$$

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(cf. [12]). It is trivial by the definition that $\text{bind}(G) > c$ implies that for every subset X of $V(G)$, we have $N_G(X) = V(G)$ or $|N_G(X)| > c\,|X|$. It is also obvious that if $\text{bind}(G) > 1$, then G is connected. Let k be a positive integer and f be an integer-valued function defined on $V(G)$ (i.e., $f: V(G) \to \{..., 0, 1, 2, ...\}$). Then a spanning k-regular subgraph of G is called a *k-factor* of G, and a spanning subgraph F of G is called an f-factor if $\deg_F(x) = f(x)$ for all $x \in V(G)$.

In this paper, we study conditions on the binding number and on the minimum degree of a graph G which guarantee the existence of an f-factor in G. We begin with some known results.

THEOREM A (Anderson [1]). *If a graph G has even order and $\text{bind}(G) \geqslant 4/3$, then G has a 1-factor.*

THEOREM B (Woodall [12]). *If $\text{bind}(G) \geqslant 3/2$, then G has a Hamilton cycle, in particular, G has a 2-factor.*

Recently, Katerinis and Woodall [8] and Katerinis [6] found the following sufficient conditions for a graph to have a k-factor. These conditions were also obtained by Egawa and Enomoto [3] independently.

THEOREM C. *Let $k \geqslant 2$ be an integer and G be a graph of order n. Assume $n \geqslant 4k-6$ and kn is even. Then the following two statements hold:*

(i) *If $\text{bind}(G) > (2k-1)(n-1)/(kn-2k+3)$, then G has a k-factor* [8].

(ii) *If $\delta(G) \geqslant n/2$, then G has a k-factor* [6].

It is shown that the conditions in (i) and (ii) are best possible. Let us note that if $k \geqslant 3$ and $n \geqslant 4k-5$, then

$$2 - \frac{1}{k} \leqslant \frac{(2k-1)(n-1)}{kn-2k+3} < 2.$$

We now give our theorem, which is an extension of the above Theorem C. Moreover, the theorem gives a result concerning the following question: If $\text{bind}(G) > c \geqslant 2$, what factor does a graph G have?

THEOREM 1. *Let G be a connected graph of order n, a and b be integers such that $1 \leqslant a \leqslant b$ and $2 \leqslant b$, and $f: V(G) \to \{a, a+1, ..., b\}$. Suppose that $n \geqslant (a+b)^2/a$ and $\sum_{x \in V(G)} f(x) \equiv 0 \pmod 2$. If one of the following three conditions is satisfied, then G has an f-factor.*

$$\text{(i)}\quad \text{bind}(G) > (a+b-1)(n-1)/(an-(a+b)+3); \qquad (1)$$

(ii) $\delta(G) > (bn-2)/(a+b)$; (2)

(iii) $\delta(G) \geqslant ((b-1)n+a+b-2)/(a+b-1)$ (3)

and for every non-empty independent subset X of $V(G)$,

$$|N_G(X)| \geqslant \frac{(b-1)n+|X|-1}{a+b-1}. \tag{4}$$

We now show that the conditions (1) and (2) are best possible. If a graph G consists of n ($n \geqslant 2$) disjoint copies of a graph H, then we write $G = nH$. The *join* $G = A + B$ has $V(G) = V(A) \cup V(B)$ and $E(G) = E(A) \cup E(B) \cup \{xy \mid x \in V(A) \text{ and } y \in V(B)\}$. Let $c = \lceil b/a \rceil$, m be a positive integer, and $G = K_{2mb-2m-2c} + (ma-1)K_2$, where K_l denotes the complete graph of order l. Define a function f: $V(G) \to \{a, a+1, ..., b\}$ by

$$f(x) = \begin{cases} a & \text{if } x \in V(K_{2mb-2m-2c}) \\ b & \text{otherwise.} \end{cases}$$

Then G has no f-factor since for $S = V(K_{2mb-2m-2c})$ and $T = V(G) \backslash S$, we have

$$\gamma_G(S, T) = 2b - 2ac - 2 < 0 \qquad \text{(see Lemma 1).}$$

Moreover, we have

$$\text{bind}(G) = \frac{(a+b-1)(n-1)}{na-(a+b)+3+2(ac-b)}.$$

Note that for $X = V(G) \backslash (V(K_{2mb-2m-2c}) \cup \{u\})$, where $V(K_2) = \{u, v\}$, we obtain

$$\frac{|N_G(X)|}{|X|} = \frac{n-1}{2(ma-1)-1} = \frac{(a+b-1)(n-1)}{na-(a+b)+3+2(ac-b)} = \text{bind}(G).$$

Therefore, if b is divisible by a, then condition (1) is best possible.

Next, suppose that $a+b$ is even and there exist positive integers s and t such that $bs = at+2$ and $s+t$ is even. Let $G = (am+s)K_1 + K_{bm+t}$, where m is a positive integer, and let f be a function on $V(G)$ defined by

$$f(x) = \begin{cases} b & \text{if } \; x \in V((am+s)K_1), \\ a & \text{if } \; x \in V(K_{bm+t}). \end{cases}$$

Then G has no f-factor and

$$\delta(G) = bm + t = \frac{bn-2}{a+b}.$$

Hence condition (2) is also best possible in this sense.

Note that (iii) of Theorem 1 is an extension of results in [9, 13], which are obtained from (iii) by setting $a=b$. Similar results on 1-factor can be found in [2]. Moreover, a similar sufficient condition for a graph to have an $[a, b]$-factor, which is a spanning subgraph F such that $a \leqslant \deg_F(x) \leqslant b$ for all vertices x, can be found in [5], and similar sufficient conditions for a bipartite graph to have k-factors are given in [7, 4].

2. Proofs

Let G be a graph and S and T be disjoint subsets of $V(G)$. Then $G-S$ denotes the subgraph of G induced by $V(G)\backslash S$, and $e_G(S, T)$ denotes the number of edges of G joining a vertex in S to a vertex in T. Our proof of Theorem 1 is analogous to those of [3, 8, 9, 13] and depends on the following lemma, which is called the f-factor theorem.

Lemma 1 (Tutte [10, 11]). *Let G be a graph and $f\colon V(G) \to \{0, 1, 2, ...\}$ such that $\sum_{x \in V(G)} f(x) \equiv 0$ (mod 2). Then G has an f-factor if and only if*

$$\gamma_G(S, T) := \sum_{x \in S} f(x) + \sum_{x \in T} (\deg_{G-S}(x) - f(x)) - h_G(S, T) \geqslant 0$$

for all $S, T \subset V(G)$, $S \cap T = \varnothing$, where $h_G(S, T)$ denotes the number of components C of $G-(S \cup T)$ such that $\sum_{x \in V(C)} f(x) + e_G(V(C), T) \equiv 1$ (mod 2).

Moreover, the following useful congruence expression holds:

$$\gamma_G(S, T) \equiv \sum_{x \in V(G)} f(x) \equiv 0 \qquad (\mathrm{mod}\ 2). \tag{5}$$

Lemma 2 [12]. *Let G be a graph of order n. If $\mathrm{bind}(G) > c$, then $\delta(G) > ((c-1)\,n+1)/c$, and $|N_G(X)| > ((c-1)\,n + |X|)/c$ for all non-empty subsets X of $V(G)$ with $N_G(X) \neq V(G)$.*

Proof. Let $Y := V(G)\backslash N_G(X)$. Since $N_G(Y) \subseteq V(G)\backslash X$, we have $n - |X| \geqslant |N_G(Y)| > c\,|Y| = c(n - |N_G(X)|)$. Hence $|N_G(X)| > ((c-1)\,n + |X|)/c$, and so $\delta(G) > ((c-1)\,n+1)/c$.

Suppose that (1) in Theorem 1 holds. Then, by Lemma 2, we have

$$\delta(G) > \frac{(b-1)\,n+a+b-3}{a+b-1} \geqslant \frac{(b-1)\,n}{a+b-1}, \tag{6}$$

and

$$\begin{aligned} |N_G(X)| &> \frac{(b-1)\,n + a\,|X| + (b-3)(n-|X|)/(n-1)}{a+b-1} \\ &\geqslant \frac{(b-1)\,n + |X| - 2}{a+b-1} \end{aligned}$$

for every independent subset X of $V(G)$. Hence G satisfies (3) and (4). Therefore (i) of Theorem 1 is an immediate consequence of (iii) of the theorem, and so we shall prove (ii) and (iii) of the theorem.

Proof of (iii) *of Theorem* 1. Suppose that G satisfies the conditions (3) and (4), but has no f-factor. By Lemma 1 and (5), there exist disjoint subsets S and T of $V(G)$ such that

$$\sum_{x\in S} f(x) + \sum_{x\in T} (\deg_{G-S}(x) - f(x)) - w \leqslant -2,$$

where w denotes the number of components of $G-(S\cup T)$. Note that $S\cup T\neq\emptyset$ since $\gamma(\emptyset,\emptyset) = -h(\emptyset,\emptyset) = 0$, which follows from the assumption that G is connected and $\sum_{x\in V(G)} f(x) \equiv 0 \pmod 2$. In particular, we have

$$a\,|S| + \sum_{x\in T} (\deg_{G-S}(x) - b) - w \leqslant -2. \tag{7}$$

We choose S and T so that $|S|+|T|$ is as large as possible subject to $\gamma(S,T)<0$. Let $s := |S|$ and $t := |T|$. It is clear that

$$w \leqslant n - s - t. \tag{8}$$

If $w>0$ then let m denote the minimum order of components of $G-(S\cup T)$. Then

$$m \leqslant \frac{n-s-t}{w} \tag{9}$$

and

$$\delta(G) \leqslant m - 1 + s + t. \tag{10}$$

Moreover, it follows from the choice of S and T that

$$\text{if } a=b \text{ then } m \geqslant 3 \tag{11}$$

(cf. [8]). If $T\neq\emptyset$, let

$$h := \min\{\deg_{G-S}(x) \mid x\in T\}.$$

Then obviously

$$\delta(G) \leqslant h + s. \tag{12}$$

We consider five cases and derive a contradiction in each case.

Case 1. $T=\emptyset$. By (7) and (8), we have

$$as + 2 \leqslant w \leqslant n - s. \tag{13}$$

Hence we have by (6), (10), (9), and (12) that

$$\frac{(b-1)n}{a+b-1} < \delta(G) \leqslant m-1+s \leqslant \frac{n-s}{w} - 1 + s$$
$$\leqslant \frac{n-s}{as+2} - 1 + s$$
$$= \frac{n-2}{a+1} - \frac{(n-2-as-s)(as-a+1)}{(a+1)(as+2)}.$$

Since $n-2-sa-s \geqslant 0$ by (13), it follows that

$$\frac{(b-1)n}{a+b-1} < \frac{n-2}{a+1},$$

which implies $a(b-2)n < -2(a+b-1)$. This is clearly impossible since $b \geqslant 2$.

Case 2. $T \neq \emptyset$ and $h=0$. Let $Z := \{x \in T \mid \deg_{G-S}(x) = 0\} \neq \emptyset$ and $z = |Z|$. Since Z is independent, we have by (4)

$$\frac{(b-1)n+z-1}{a+b-1} \leqslant |N_G(Z)| \leqslant s. \tag{14}$$

On the other hand, we have by (7), (8), and the fact that $b-1 \geqslant 1$

$$as - bz + (1-b)(t-z) - (b-1)(n-s-t) \leqslant -2.$$

Hence

$$s \leqslant \frac{(b-1)n+z-2}{a+b-1}.$$

This contradicts (14).

Case 3. $T \neq \emptyset$ and $1 \leqslant h \leqslant b-1$, By (7), (8), and the fact that $b-h \geqslant 1$, we have

$$as + (h-b)t - (b-h)(n-s-t) \leqslant -2.$$

Thus

$$s \leqslant \frac{(b-h)n-2}{a+b-h}. \tag{15}$$

On the other hand, we obtain by (3) and (12) that

$$\frac{(b-1)n-1}{a+b-1}+1 \leqslant \delta(G) \leqslant s+h.$$

This inequality together with (15) gives us

$$\frac{(b-1)n-1}{a+b-1}+1-h \leqslant \frac{(b-h)n-2}{a+b-h}.$$

Hence

$$(h-1)\,an \leqslant (h-1)(a+b-1)(a+b-h)-(a+b+h-2).$$

This implies $h \geqslant 2$ and

$$an \leqslant (a+b-1)(a+b-h)-\frac{(a+b+h-2)}{h-1}.$$

This contradicts our assumption that $n \geqslant (a+b)^2/a$.

Case 4. $T \neq \varnothing$ and $h=b$. We have $w \geqslant as+2$ by (7), and so we obtain by (9) that

$$m \leqslant \frac{n-s-t}{w} \leqslant \frac{n-s-1}{as+2}. \tag{16}$$

If $b \geqslant 3$ then we get the following inequality from $an \geqslant (a+b)^2 > (a+b+1)(a+b-1)$:

$$an(b-2) > (a+b-1)(ab+b^2-2a-b-2). \tag{17}$$

By (3) and (12), we have

$$\frac{(b-1)n-1}{a+b-1}+1 \leqslant \delta(G) \leqslant h+s=b+s$$

and so

$$\begin{aligned} s &\geqslant \frac{(b-1)n-1}{a+b-1}-(b-1) && (18)\\ &= \frac{n-3}{a+1}+\frac{an(b-2)+(a+b-1)(3-(a+1)(b-1))-(a+1)}{(a+b-1)(a+1)} \\ &> \frac{n-3}{a+1}+\frac{(a+b-1)(b^2-a-2b)+2b+a-3}{(a+b-1)(a+1)}. && \text{(by (17))} \end{aligned}$$

Hence, if $b \geqslant 3$ then $s > (n-3)/(a+1)$, and so $m < 1$ by (16), a contradiction. If $a = b = 2$ then $s \geqslant (n-4)/3$ by (18), and so $m < 3$ by (16). This contradicts (11). Therefore we may assume that $a = 1$ and $b = 2$. By (16) and (18), we have $m = 1$. Thus it follows from (10) and (12) that

$$\delta(G) \leqslant s + t \qquad \text{and} \qquad \delta(G) \leqslant b + s = s + 2.$$

Hence, by (7) and (8), we obtain

$$\delta(G) \leqslant s + 2 = as + 2 \leqslant w \leqslant n - s - t \leqslant n - \delta(G).$$

Hence $\delta(G) \leqslant n/2$. This contradicts (3).

Case 5. $T \neq \emptyset$ and $h > b$. By (7), we have $as + (h-b)\,t - w \leqslant -2$, and so

$$w \geqslant as + t + 2 \geqslant s + t + 2. \tag{19}$$

Suppose that $m \geqslant 3$. Then, by (10) and (9), we have

$$\begin{aligned}\delta(G) &\leqslant m - 1 + s + t \leqslant m + w - 3\\ &\leqslant m + w - 3 + \frac{1}{3}(m-3)(w-3) = \frac{mw}{3} \leqslant \frac{n}{3}.\end{aligned}$$

This contradicts (4). Thus we may assume that $m \leqslant 2$. It follows from (8) and (19) that $s + t + 1 \leqslant n/2$. Then by (3) and (10), we have

$$\frac{(b-1)\,n}{a+b-1} < \delta(G) \leqslant s + t + 1 \leqslant \frac{n}{2}.$$

Thus $n(2b - a - 1) < 0$. This is impossible. Consequently, (iii) is proven.

Proof of (ii) *of Theorem* 1. This is almost identical to the proof of (iii). Since $n \geqslant (a+b)^2/a$, we have

$$\frac{bn-2}{a+b} \geqslant \frac{(b-1)\,n + a + b - 2}{a+b-1},$$

and so (4) still holds by (3). Thus Cases 1, 3, 4, and 5 carry over without modification from (iii) to (ii) because we don't use (4) in these cases. The only case that needs to change is the following:

Case 2. $T \neq \emptyset$ and $h = 0$. By (7) and (8), we have

$$-2 \geqslant as - bt - (n - s - t) \geqslant as - bt - b(n - s - t),$$

and so $s \leqslant (bn-2)/(a+b)$. Then (3) and (11) give

$$\frac{bn-2}{a+b} < \delta(G) \leqslant h+s = s \leqslant \frac{bn-2}{a+b},$$

a contradiction. Consequently the proof is complete.

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