Discrete Mathematics 85 (1990) 97–98
North-Holland

COMMUNICATION

ON THE EXISTENCE OF MINIMAL END-SEPARATORS

Norbert POLAT
Département de Mathématiques, Bâtiment Doyen Jean Braconnier, Université Claude Bernard (Lyon I), 43, Boulevard du 11 Novembre 1918, 69622 Villeurbanne Cedex, France
I.A.E. Université Jean Moulin (Lyon III), 15, quai Claude Bernard, 69239 Lyon Cedex 2, France

Communicated by G. Sabidussi
Received 10 April 1990

The purpose of this communication is to prove the existence of a minimal endseparator in any infinite graph, a problem which was set and almost solved by Sabidussi in [2]. The notation, terminology, and the basic tools are those of [2]. Throughout this paper, G stands for an infinite connected graph.

1. The set of circuit-connected trees of G, ordered by inclusion, is obviously inductive. Hence G has a spanning tree T whose core T_* is a maximal circuit-connected tree of G (with respect to inclusion). Such a tree will be said to be *quasi circuit-connected* (q.c.c.). Note that if the core T_* of a q.c.c. tree T of G is non-empty, then $G-T_*$ is rayless.

2. We recall the two following concepts:
2.1. An infinite subset S of $V(G)$ is *concentrated* if there is a ray R such that, for any finite set F of vertices, only finitely many elements of S do not belong to the component of $G-F$ containing a subray of R.
2.2. A set of vertices of G is *dispersed* in G if it has no concentrated subset.

Note that any subset of vertices of a tree T which is dispersed in some subtree of T, is also dispersed in T.

3. Lemma. *Let T be a tree, and let T' and T'' be subtrees of T. If $V(T' \cap T'')$ contains an infinite subset which is dispersed in T'' (thus in T), then $T' \cap T''$ has a vertex of infinite degree.*

Proof. Let A be an infinite set of vertices of $T' \cap T''$ which is dispersed in T''. Then A has no concentrated subset, thus, by [1, 3.11], it contains an infinite subset B such that, for some finite subset F of $V(T'')$, the intersection of B with any component of $T''-F$ has at most one vertex. But, since T'' is acyclic and F is finite, there are $x \in F$ and an infinite subset C of B whose intersection with any component of $T''-x$ has at most one element. Therefore, since $C \subseteq V(T')$, and since T' is a tree (thus connected and acyclic) $x \in V(T')$ and $\{[x, y_c]: c \in C\} \subseteq E(T')$

where, for $c \in C$, y_c denotes the only neighbor of x in the component of $T'' - x$ containing c. $\square$

4. Theorem. *Let G be a connected graph, and let T be a* q.c.c. *spanning tree of G. If D is a dendroid of G based on T such that D_T is locally cofinite in T_*, then $D^{(2)}$ is a minimal end-separator of G.*

Proof. (a) This first part of the proof is exactly that of Theorem (5.2) of [2], the last paragraph excepted. We recall the main points.

One supposed by way of contradiction that $D^{(2)}$ misses some 2-ended double ray Z. Let $X := Z \setminus D \cup \bigcup \{C(e, D) \setminus e : e \in Z \cap D\}$. One proved that if $Z \cap D_\infty \neq \emptyset$, then X is disconnected, and each of its components, as well as X itself, is 1-ended in G. On the other hand, if $Z \cap D_\infty = \emptyset$, then X is connected. In either case every component of X is infinite.

The following is now different from the sequel of the proof of (5.2). Since $X \subseteq T \setminus D = T \setminus D_T = T \setminus D_{T_*}$ and D_{T_*} is locally cofinite in T_*, we have that $X \cap T_*$ is locally finite. Hence, by Lemma 3, no component of X can contain an infinite dispersed subset of $V(Z \cap T_*)$.

(b) Let Z_0 be a ray of Z, and let $Z_1 = Z \setminus Z_0$. These two rays are inequivalent in G since Z is 2-ended. We distinguish two cases. In each we shall show that there is a component of X which contains an infinite dispersed subset of $V(Z \cap T_*)$, thus giving rise to a contradiction with the conclusion of (a).

Case 1: $Z \cap D^{(1)} = \emptyset$.

Then X is a tree which is rayless or 1-ended in G. Thus there is an i such that Z_i is equivalent with no ray of X. Hence $V(Z_i \cap T_*)$, which is an infinite subset of $V(X)$, is dispersed.

Case 2: $Z \cap D^{(1)} \neq \emptyset$.

Since every component of X contains a ray, and since X is 1-ended in G, there is an i such that Z_i is equivalent with no ray of X. Thus there are only finitely many components of X meeting Z_i. Hence, since $V(Z_i) \subseteq V(X)$, and since $Z_i \cap T_*$ is infinite, there must be one of these components, say Y, which contains an infinite subset of $V(Z_i \cap T_*)$; and this subset is then dispersed in Y. $\square$

The main result of this communication is an immediate consequence of 2 and 4.

5. Theorem. *Any graph has a minimal end-separator.*

References

[1] N. Polat, Aspects topologiques de la séparation dans les graphes infinis, I. Math. Z. 165 (1979) 73–100.

[2] G. Sabidussi, Dendroids, end-separators, and almost circuit-connected trees, in: G. Hahn, G. Sabidussi and R. Woodrow, eds., "Cycles and Rays" NATO ASI Ser. C 301 (Kluwer, Dordrecht, 1990) 221–236.