New upper solution bounds of the discrete algebraic Riccati matrix equation

Richard Davies*, Peng Shi, Ron Wiltshire

Faculty of Advanced Technology, University of Glamorgan, Pontypridd CF37 1DL, UK

Received 15 February 2006; received in revised form 8 January 2007

Abstract

In this note, we present upper matrix bounds for the solution of the discrete algebraic Riccati equation (DARE). Using the matrix bound of Theorem 2.2, we then give several eigenvalue upper bounds for the solution of the DARE and make comparisons with existing results. The advantage of our results over existing upper bounds is that the new upper bounds of Theorem 2.2 and Corollary 2.1 are always calculated if the stabilizing solution of the DARE exists, whilst all existing upper matrix bounds might not be calculated because they have been derived under stronger conditions. Finally, we give numerical examples to demonstrate the effectiveness of the derived results.

© 2007 Elsevier B.V. All rights reserved.

MSC: 15A24; 49N05; 49N10

Keywords: Matrix bound; Discrete algebraic Riccati equation

1. Introduction and preliminaries

The discrete Riccati equation is encountered in many control analysis and design problems, particularly in the field of optimal control. In practice, analytical solution of this equation is complicated, particularly when the dimensions of the system matrices are high. As such, a number of works have been presented over the past three decades for deriving solution bounds of this equation [2–5,7–20,22,24–26], to reduce the computational burdens required to solve it analytically. Not only do solution bounds provide estimates for the solution of this equation, but they can also be applied to deal with practical situations involving the solution of this equation. Several types of bounds have been obtained [2–5,7–20,22,24–26], including bounds for the minimal and maximal eigenvalues, summation, trace, product, determinant, and the solution matrix itself. Of these bounds, the matrix bounds are the most general, because they can directly offer all other types of bounds mentioned. However, viewing the literature [9,2,3,5,14–17,19,20], it appears that all proposed upper matrix bounds for the DARE have been developed under assumptions additional to the fundamental existence conditions for the DARE solution. Therefore, this note develops two upper matrix bounds, of which the bounds of Theorem 2.2 and Corollary 2.1 are always calculated if the stabilizing solution of the DARE exists. The derivation of these bounds use the idea of the controllability of the matrix pair \((A, B)\), in which it is well-known that

* Corresponding author.

E-mail addresses: rkdavies@glam.ac.uk (R. Davies), pshi@glam.ac.uk (P. Shi), rjwiltsh@glam.ac.uk (R. Wiltshire).
if \((A, B)\) is a stabilizable pair then there will always exist a matrix \(K\) such that \(A + BK\) is stable. In the literature, there
a number of papers have used this idea to aid in the solution of control and estimation problems. An example of such a
paper can be found from [6]. Using the upper matrix bound of Theorem 2.2, we then provide several eigenvalue bounds
for the DARE, including bounds for the minimal and maximal eigenvalues, trace and determinant of the solution.
Finally, we demonstrate the effectiveness of the derived bounds through numerical examples.

The following symbol conventions are used in this note. \(\mathbb{R}\) denotes the real number field. The inequality \(A \succ (\succeq) B\)
means \(A - B\) is a positive (semi-) definite matrix; \(\lambda_i(A)\) and \(\sigma_i(A)\) denote, respectively, the \(i\)th eigenvalue and \(i\)th
singular value of a matrix \(A\) for \(i = 1, 2, \ldots, n\) whereas \(\hat{\lambda}_i(A)\) and \(\hat{\sigma}_i(A)\) are arranged in the nonincreasing order (i.e.,
\(\hat{\lambda}_1(A) \geq \cdots \geq \hat{\lambda}_n(A)\) and \(\hat{\sigma}_1(A) \geq \cdots \geq \hat{\sigma}_n(A)\)), \(\text{tr}(X)\) and \(\det(X)\) denote, respectively, the trace
determinant of the \(n \times n\) matrix \(X\). The identity matrix with appropriate dimensions is represented by \(I\). \(X^T\) denotes
the transpose of matrix \(X\).

2. Main results

Consider the discrete algebraic Riccati equation (DARE):

\[
P = A^T PA - A^T PB(I + B^T PB)^{-1} B^T PA + Q, \tag{2.1}
\]

where \(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, Q \in \mathbb{R}^{n \times n}\) is a symmetric semi-positive definite matrix, and the matrix \(P\) is the unique
positive semi-definite solution of the DARE (2.1). To guarantee the existence of the stabilizing solution of the DARE,
it shall be assumed that \((A, B)\) is a stabilizable pair and \((A, Q)\) is a detectable pair.

Before developing the main results, we shall recall the following useful Lemmas.

**Lemma 2.1 (Bernstein [1])**. For any symmetric matrices \(X, Y \in \mathbb{R}^{n \times n}\) and \(1 \leq i, j \leq n\), the following inequality holds:

\[
\hat{\lambda}_{i+j-1}(X + Y) \leq \hat{\lambda}_j(X) + \hat{\lambda}_i(Y), \quad i + j \leq n + 1.
\]

**Lemma 2.2 (Bernstein [1])**. For any symmetric matrix \(X\), the following inequality holds:

\[
X \preceq \hat{\lambda}_1(X) I.
\]

**Lemma 2.3 (Kim and Park [5])**. Let matrices \(A, X, R, Y \in \mathbb{R}^{n \times n}\) with \(X, Y, R \succeq 0\) and \(X \succeq Y\). Then

\[
A^T (I + XR)^{-1}XA \succeq A^T (I + YR)^{-1}YA
\]

with strict inequality if \(A\) is nonsingular and \(X \succ Y\).

**Lemma 2.4 (Bernstein [1])**. For any matrices \(X, Y \in \mathbb{R}^{n \times n}\) and any matrix \(A \in \mathbb{R}^{n \times m}\), if \(X \succeq Y\) then

\[
A^T X A \succeq A^T Y A.
\]

Using these lemmas, we can now develop the main results of this note.

**Theorem 2.1.** Let \(P\) be the positive semi-definite solution of the DARE (2.1). If \(\sigma_1^2(A + BK) < 1\) then \(P\) has the upper bound

\[
P \preceq \kappa (A + BK)^T (A + BK) + Q + K^T K \equiv P_{du1} \tag{2.2}
\]

where the positive constant \(\kappa\) is defined by

\[
\kappa = \frac{\hat{\lambda}_1(Q + K^T K)}{1 - \sigma_1^2(A + BK)} \tag{2.3}
\]

and the matrix \(K \in \mathbb{R}^{m \times n}\) is chosen to stabilize \(A + BK\).
Applying Lemmas 2.2 and 2.3 to (2.12) gives

\[ A \equiv [K + (I + B^T PB)^{-1}B^TPA]^T(I + B^T PB)[K + (I + B^T PB)^{-1}B^TPA] \]
\[ = K^T K + K^T B^T PBK + A^T PBK + K^T B^T PA + A^T PB(I + B^T PB)^{-1}B^TPA \geq 0, \]

(2.4)

where \( K \in \mathbb{R}^{m \times n} \). Using the DARE (2.1), (2.4) becomes

\[ P \leq A^T PA + K^T K + K^T B^T PBK + A^T PBK + K^T B^T PA + Q. \]

(2.5)

By use of the matrix identity

\[ (A + BK)^T P (A + BK) = A^T PA + A^T PBK + K^T B^T PA + K^T B^T PBK \]

(2.5) becomes

\[ P \leq (A + BK)^T P (A + BK) + Q + K^T K. \]

(2.6)

By making use of Lemma 2.2, (2.6) becomes

\[ P \leq \lambda_1 (P) (A + BK)^T (A + BK) + Q + K^T K. \]

(2.7)

Introducing Lemma 2.1 to (2.7) gives

\[ \lambda_1 (P) \leq \lambda_1 (\lambda_1 (P) (A + BK)^T (A + BK) + Q + K^T K) \]
\[ \leq \sigma_1^2 (A + BK) \lambda_1 (P) + \lambda_1 (Q + K^T K). \]

(2.8)

If \( \sigma_1^2 (A + BK) < 1 \) then (2.8) infers \( \lambda_1 (P) \leq \kappa \), where \( \kappa \) is defined by (2.3).

Substituting (2.3) into (2.7) results in bound (2.2). This completes the proof of the theorem. \( \square \)

Having developed the upper bound \( P_{du1} \) of Theorem 2.1, we can suggest the following iterative algorithm to derive tighter upper matrix bounds for the solution of the DARE (2.1). Before doing so, we shall first modify the DARE (2.1) using the matrix identity [3]

\[ (I + ST)^{-1} = I - S(I + TS)^{-1}T. \]

(2.9)

Using this identity, the DARE becomes

\[ P = A^T (I + PBB^T)^{-1} PA + Q. \]

(2.10)

Using the transformed DARE (2.10), together with Lemma 2.3, we now have the following iterative algorithm to obtain sharper upper matrix solution bounds of the DARE (2.1). The derivation of this algorithm will follow partly along the lines of [3, Theorem 1].

**Algorithm 2.1.** Step 1: Set \( M_0 = P_{du1} \), where \( P_{du1} \) is defined by (2.2).

Step 2: Calculate

\[ M_k = A^T (I + M_{k-1} BB^T)^{-1} M_{k-1} A + Q, \quad k = 1, 2, \ldots. \]

(2.11)

Then \( M_k \) are upper solution bounds of the DARE (2.1). In fact, as \( k \to \infty \), \( M_{k+1} = M_k \) and \( M_\infty = \lim_{k \to \infty} M_k = P \), where \( P \) is the positive semi-definite solution of the DARE (2.1).

**Proof.** First, it will be shown that \( M_1 \leq M_0 \). Setting \( k = 1 \) in (2.11) gives

\[ M_1 = A^T (I + M_0 BB^T)^{-1} M_0 A + Q. \]

(2.12)

Applying Lemmas 2.2 and 2.3 to (2.12) gives

\[ M_1 \leq \lambda_1 (M_0) A^T [I + \lambda_1 (M_0) BB^T]^{-1} A + Q. \]

(2.13)
Now, let $N \equiv \lambda_{1/2}^{1/2}(M_0)I$. Applying the identity (2.9) to (2.13), and following along the lines of the proof of Theorem 2.1, we get

$$
M_1 \leq A^T(\lambda_1(M_0))A - \lambda_1(M_0)A^TNB[I + B^TN^2B]^{-1}B^TNA + Q
= A^TN^2A - [K + (I + B^TN^2B)^{-1}B^TN^2B][K + (I + B^TN^2B)^{-1}B^TN^2A]
+ K^T(K + K^TB^TN^2BK + A^TN^2BK + K^TB^TN^2A)
\leq (A + BK)^T N^2(A + BK) + (Q + K^T K)
= \lambda_1(M_0)(A + BK)^T(A + BK) + (Q + K^T K).
$$

From (2.2), we have, using Lemma 2.1, that

$$
\lambda_1(M_0) = \lambda_1(K(A + BK)^T(A + BK) + (Q + K^T K))
\leq \kappa \sigma_1^2(A + BK) + \lambda_1(Q + K^T K)
= \kappa \left\{ 1 - \frac{\lambda_1(Q + K^T K)}{\kappa} \right\} + \lambda_1(Q + K^T K) = \kappa,
$$

(2.15)

where the condition $\sigma_1^2(A + BK) < 1$ and (2.3) have been employed. Substituting (2.15) into (2.14) gives

$$
M_1 \leq \kappa(A + BK)^T(A + BK) + (Q + K^T K) \equiv M_0.
$$

Therefore, we have completely proven that $M_1 \leq M_0$. Assume now that $M_{k-1} \leq M_{k-2}$. By (2.11) and use of Lemma 2.3, we have that

$$
M_k = A^T(I + M_{k-1}BB^T)^{-1}M_{k-1}A + Q \leq A^T(I + M_{k-2}BB^T)^{-1}M_{k-2}A + Q = M_{k-1}.
$$

One can conclude, by means of induction, that $M_k \leq M_{k-1} \leq \cdots \leq M_1 \leq M_0$. Clearly, we have $M_k \geq 0$ for any $k$. Along the lines of [3, Theorem 1], it can be seen that $M_k$ is monotone decreasing and bounded, so there exists $M_\infty \geq 0$, with $M_\infty = \lim_{k \to \infty} M_k$, such that

$$
M_\infty = A^T(I + M_\infty BB^T)^{-1}M_\infty A + Q.
$$

Here, $M_\infty$ is merely the DARE solution with $P$ replaced by $M_\infty$. Hence, we arrive at the conclusion that this algorithm can obtain the exact solution of the DARE. □

**Remark 2.1.** Let $P_K$ be the solution of the inequality (2.6):

$$
P_K = (A + BK)^T P_K (A + BK) + Q + K^T K,
$$

(2.16)

where $K$ is any chosen matrix stabilizing $A + BK$. Then, the non-negative definite solution $P_K$ of the Lyapunov-type matrix equation (2.16) is an upper bound for the solution of the DARE (2.1). Along the lines of Algorithm 2.1, $P$ then has the upper bound

$$
P \leq A^T(I + P_K BB^T)^{-1}P_K A + Q \equiv P_1.
$$

(2.17)

Furthermore, we can successively improve the upper bounds by back-substituting them into (2.10).

Even though $K$ is chosen to stabilize $A + BK$, it is not always possible to fulfill the condition $\sigma_1^2(A + BK) < 1$. In the following theorem and corollary, we shall utilize a free matrix $D$ to get around this problem.

First, we shall modify the DARE (2.1), using the similarity transformation, to obtain the following modified DARE:

$$
\overline{P} = \overline{A}^T \overline{P} \overline{A} - \overline{A}^T \overline{P} \overline{B} (I + \overline{B}^T \overline{P} \overline{B})^{-1} \overline{B}^T \overline{P} \overline{A} + \overline{Q},
$$

(2.18)

where $\overline{P} = D^{-T} P D^{-1}$, $\overline{A} = D A D^{-1}$, $\overline{B} = D B$ and $\overline{Q} = D^{-T} Q D^{-1}$, and $D$ is a nonsingular matrix.
Theorem 2.2. The solution $P$ of the DARE (2.1) has the following upper matrix bound on its solution:

$$P \leq \mu (A + BK)^T D^T D(A + BK) + Q + K^T K \equiv P_{du2},$$ (2.19)

where $K$ is chosen to stabilize $A + BK$, the nonsingular matrix $D$ is chosen so that $\sigma_1^2 [D(A + BK)D^{-1}] < 1$, and the positive constant $\mu$ is defined by

$$\mu \equiv \frac{\lambda_1 [D^{-T}(Q + K^T K)D^{-1}]}{1 - \sigma_1^2[D(A + BK)D^{-1}]}.$$ (2.20)

Proof. By applying the method of Theorem 2.1 to the modified DARE (2.18), we get the following upper bound for $\overline{P}$:

$$\overline{P} \leq \mu (\overline{A} + \overline{Bk})^T (\overline{A} + \overline{Bk}) + (\overline{Q} + \overline{K^T K}),$$ (2.21)

where $\mu$ is defined by (2.20), and $\overline{K} = KD^{-1}$. By reverting to the original matrices in (2.21), and then applying Lemma 2.4, we are lead to the upper bound (2.19). This finishes the proof of the theorem. □

Now that we have finished the proof of Theorem 2.2, we can propose the following iterative algorithm to obtain more precise upper matrix solution bounds of the DARE, and hence its exact solution.

Algorithm 2.2. Step 1: Set $M_0 \equiv P_{du2}$, where $P_{du2}$ is defined by (2.19).

Step 2: Calculate

$$M_k = A^T (I + M_{k-1}BB^T)^{-1} M_{k-1}A + Q,$$ as $k \rightarrow \infty$.

Then $M_k$ are upper solution bounds of the DARE (2.1). In fact, as $k \rightarrow \infty$, $M_{k+1} = M_k$ and $M_{\infty} = \lim_{k \rightarrow \infty} M_k = P$, where $P$ is the positive semi-definite solution of the DARE (2.1).

Proof. The proof of this algorithm is similar to that of Algorithm 2.1. Therefore, we omit its proof. □

Corollary 2.1. Based on the analysis of Theorem 2.2, we have the following eigenvalue upper bounds for the solution of the DARE (2.1):

$$\lambda_i(P) \leq \lambda_i \left\{ \mu (A + BK)^T D^T D(A + BK) + Q + K^T K \right\}, \quad i = 1, 2, \ldots, n,$$

$$\text{tr}(P) \leq \text{tr} \left\{ \mu (A + BK)^T D^T D(A + BK) + Q + K^T K \right\},$$

$$\det(P) \leq \det \left\{ \mu (A + BK)^T D^T D(A + BK) + Q + K^T K \right\}.$$

Remark 2.2. When $A$ is stable, $K = 0$ and $D = I$, the results of this note decompose into the upper bound for the DARE reported in [5]. Therefore, this work can be considered to be a generalization of the upper bound presented in [5].

The following upper bound was reported in [15]:

$$P \leq \frac{\lambda_1(Q)}{1 + \eta \sigma_2^2(B)} A^T A + Q,$$ (2.22)

where $\eta \equiv \lambda_1([A^T Q^{-1} + BB^T]^{-1} A + Q)$. The calculation of this bound has to assume that $Q$ is nonsingular and (i) $BB^T$ is nonsingular and $1 + \eta \sigma_2^2(B) > \sigma_1^2(A)$ or (ii) $BB^T$ is singular and $\sigma_1^2(A) < 1$. It can be seen that when $K = 0$ and $D = I$, bound (2.19) is identical to bound (2.22) when $BB^T$ is singular. In these cases, the resulting bounds only work when $\sigma_1^2(A) < 1$. Furthermore, when $K = 0$ and $D = I$ in (2.19), and when $BB^T$ is nonsingular, the bounds (2.19)
and (2.22) become, respectively, the bounds $P_{U1}$ and $P_{U2}$, where

$$P_{U1} = \frac{\hat{\lambda}_1(Q)}{1 - \sigma^2_1(A)} A^T A + Q,$$

$$P_{U2} = \frac{\hat{\lambda}_1(Q)}{1 + \eta \sigma^2_n(B) - \sigma^2_1(A)} A^T A + Q.$$  

In this case, we have $P_{U2} \leq P_{U1}$, so bound (2.22) gives the tighter solution estimate than bound (2.19) for this case. Furthermore, we have, for this case, that the bound $P_{U1}$ only works when $\sigma^2_1(A) < 1$, whilst the bound $P_{U2}$ only works when $1 + \eta \sigma^2_n(B) > \sigma^2_1(A)$ and $Q$ is nonsingular.

For the remaining upper matrix bounds existing in the literature [9,14,16,19,20,2,3,17], one can see that the presented bounds cannot be compared with the existing ones by any mathematical method. However, comparison via a numerical example is always possible.

**Remark 2.3.** The condition $\sigma^2_1[D(A + BK)D^{-1}] < 1$ is equivalent to $\hat{\lambda}_1[D^{-T}(A + BK)^T D^T D(A + BK)D^{-1}] < 1$, which is equivalent to

$$D^{-T}(A + BK)^T D^T D(A + BK)D^{-1} < I,$$

(2.23)

Using Lemma 2.4, (2.23) is equivalent to the condition

$$(A + BK)^T P_D (A + BK) < P_D,$$

(2.24)

where $P_D = D^T D$. Since the pair $(A, B)$ is assumed to be stabilizable, there will always exist a matrix $K$ stabilizing $A + BK$. Then, since $A + BK$ is stable, there will always exist a symmetric matrix $P_D$ yielding (2.24) by the Stein theorem [21]. Therefore, the upper bounds of Theorem 2.2 and Corollary 2.1 are always calculated if the solution of the DARE exists. In fact, such a free matrix $D$ may be constructed via the following procedure:

**Step 1:** Design a matrix $K$ such that $A + BK$ is stable.

**Step 2:** Solve the following Lyapunov-type matrix equation for $P_D$:

$$P_D = (A + BK)^T P_D (A + BK) + c I,$$

(2.25)

where $c$ is any positive constant. The Lyapunov-type equation (2.25) is much more easily solved than the DARE (2.1).

**Step 3:** Having solved (2.25) for $P_D$, a possibility for the free matrix $D$ is $D = \sqrt{P_D}$. The square root of $P_D$ may be found by a number of methods, see for example [1].

Having followed Step 1 of the above procedure, an alternative to solving (2.25) is to use trial-and-error, i.e., try out various values of $P_D = P_D^T$ such that $P_D - (A + BK)^T P_D (A + BK) > 0$. Then, having found $P_D$, $D$ may then be found by following Step 3 of the above process. On the other hand, we may choose $D$ to be symmetric, and then follow this trial-and-error method. One can use the determinant criterion [21] for a positive definite matrix to aid us in finding such a possibility of $D$.

**Remark 2.4.** The tightness of the upper bounds developed in this note depend on the choice of the matrices $K$ and $D$. It is difficult to say which choice of $K$ and $D$ give the best upper bound for the DARE (2.1). Therefore, the choice of $K$ and $D$ which give the optimal upper bound remains an open question. However, the choice of $K$ and $D$ which give the optimal bounds could be considered as an optimization problem. Besides, for any $K$ and $D$ determined to meet the required conditions for satisfaction, one can easily get tighter upper matrix bounds for the DARE solution by using Algorithms 2.1 or 2.2, and hence its exact solution. It should also be noted that if $\sigma^2_1(A + BK) < 1$ is satisfied for a matrix $K$ chosen to stabilize $A + BK$, then a simple choice of $D = I$ will suffice in the calculation of the bounds.

**Remark 2.5.** In the literature, there are a number of ways that one can design a matrix $K$ so as to make the matrix $A + BK$ stable. For example, one may utilize pole assignment techniques to determine such a matrix $K$, as is presented in [23].
3. Numerical examples

In this section, we give numerical examples to demonstrate the effectiveness of our upper matrix bounds, and make comparisons, when possible, with existing results.

3.1. Example 1 [2, Example 1]

Consider the DARE (2.1) with
\[
A = \begin{bmatrix} 1.45 & -0.45 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.25 & 0.19 \\ 0.19 & 0.1444 \end{bmatrix}.
\]

Then, the unique positive definite solution of the DARE (2.1) is
\[
P_{\text{exact}} = \begin{bmatrix} 1.5989 & -0.1802 \\ -0.1802 & 0.2690 \end{bmatrix}
\]
with \(\lambda_n(P_{\text{exact}}) = 0.2450\), \(\lambda_1(P_{\text{exact}}) = 1.6229\), \(\text{tr}(P_{\text{exact}}) = 1.8679\) and \(\det(P_{\text{exact}}) = 0.3976\).

If we design \(K\) so that \(A + BK\) has eigenvalues situated at 0.25 and 0.2, then use of a pole placement technique [23] reveals that \(K = \begin{bmatrix} -10.4 \end{bmatrix}\), the upper bounds \(P_K\) and \(P_1\) for the solution of the DARE (2.1) are found by (2.16) and (2.17), respectively, to be
\[
P_K = \begin{bmatrix} 1.6879 & -0.0559 \\ -0.0559 & 0.3086 \end{bmatrix},
\]
\[
P_1 = \begin{bmatrix} 1.6035 & -0.1803 \\ -0.1803 & 0.2716 \end{bmatrix}.
\]

For \(P_1\), we have \(\lambda_n(P_1) = 0.2476\), \(\lambda_1(P_1) = 1.6275\), \(\text{tr}(P_1) = 1.8751\) and \(\det(P_1) = 0.4030\). The resulting bounds are very close to the real values.

Since the matrix \(BB^T\) is singular for this case, the upper matrix bounds of [9,14] cannot work for this case. Since \(\sigma_n^2(B) = 0\) and \(\sigma_1^2(A) > 1\), the upper matrix bound of [15] cannot work either. Since \(\lambda_n(Q) = 0\) and \(\sigma_n^2(A) < 1\), the upper matrix bound of [16] cannot work here. Because \(\eta \equiv \lambda_1(A^T[I - B(\lambda_1^{-1}(Q)I + B^T B)^{-1} B^T]A) > 1\), the upper matrix bound of [19] also cannot work for this case. For this example, the matrix \(A\) is not stable, so the upper matrix bound of [5] cannot be applied here. Furthermore, \(b_1 \equiv 2\sigma_1^2(B) - 2\lambda_1(\sigma_1^2(B)A^T A - A^T B B A) = 0\), so the upper matrix bound of [20] also cannot be used. Finally, the upper matrix bounds of [17] also cannot work for this case because \(BB^T\) is singular. However, we can try the upper matrix bounds of [2,3]. Since the results of [2] are merely a special case of those in [3] with \(M = zI\), where \(z\) is a positive constant, we shall only consider the results of [3].

With \(M = \begin{bmatrix} 4 & 0.1 \\ 0.1 & 0.5 \end{bmatrix}\), the upper matrix bound proposed in [3] gives
\[
P \leq \begin{bmatrix} 2.49 & -0.341 \\ -0.341 & 0.3064 \end{bmatrix}
\]
if there exists a positive semi-definite matrix \(M\) such that \(A^T(I + MBB^T)^{-1} MA + Q \leq M\).

3.2. Example 2 [18, Example]

Consider the DARE (2.1) with
\[
A = \begin{bmatrix} 1.1 & 0 \\ 0.1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix}.
\]
Then, the unique positive definite solution of the DARE (2.1) is
\[
P_{\text{exact}} = \begin{bmatrix} 3.3340 & 1 \\ 1 & 4 \end{bmatrix}
\]
with \(\lambda_* (P_{\text{exact}}) = 2.6130\), \(\lambda_1 (P_{\text{exact}}) = 4.7210\), \(\text{tr}(P_{\text{exact}}) = 7.3340\) and \(\text{det}(P_{\text{exact}}) = 12.336\).

If we assign \(A + BK\) to have eigenvalues of 0.2 and \(-0.1\), then application of a pole placement method [23] gives \(K = [-0.5 \ 0.1]\), the upper bounds \(P_K\) and \(P_1\) for the solution of the DARE (2.1) are found by (2.16) and (2.17), respectively, to be
\[
P_K = \begin{bmatrix} 3.3456 & 1.0377 \\ 1.0377 & 4.1438 \end{bmatrix},
\]
\[
P_1 = \begin{bmatrix} 3.3358 & 1 \\ 1 & 4 \end{bmatrix}.
\]

For \(P_1\), we have \(\lambda_* (P_1) = 2.6142\), \(\lambda_1 (P_1) = 4.7216\), \(\text{tr}(P_1) = 7.3358\) and \(\text{det}(P_1) = 12.3432\). The resulting bounds are very close to the real values.

By back-substituting \(P_1\) into (2.10), we get the following tighter bound for the solution of the DARE:
\[
P \leq \begin{bmatrix} 3.3340 & 1 \\ 1 & 4 \end{bmatrix}
\]
which is the same as the exact solution of the DARE (2.1). Further back-substitution into (2.10) will only give us the exact solution of the DARE.

For this example, the matrix \(BB^T\) is singular, so the upper matrix bounds of [9,14,17] cannot work. Also, \(\sigma_1^2 (A) > 1\), so the upper matrix bound of [15] also cannot work. The matrix \(A\) is not stable for this example, so the upper matrix bound of [5] cannot be applied here. Furthermore, since the matrix \(A\) is singular, the upper matrix bounds of [2] cannot work either. However, we find that the upper matrix bounds of [16,19,20,3] can work for this case.

For this case, the upper matrix bound derived in [16] gives
\[
P \leq \begin{bmatrix} 3.3341 & 1 \\ 1 & 4 \end{bmatrix}.
\]

The upper matrix bound proposed in [19] gives the estimate
\[
P \leq \begin{bmatrix} 3.3377 & 1 \\ 1 & 4 \end{bmatrix}.
\]

For this example, the upper matrix bound presented in [20] gives
\[
P \leq \begin{bmatrix} 3.3349 & 1 \\ 1 & 4 \end{bmatrix}.
\]

With \(M = \begin{bmatrix} 3.5 & 1 \\ 1 & 4 \end{bmatrix}\), the upper matrix bound obtained in [3] gives
\[
P \leq \begin{bmatrix} 3.4133 & 1 \\ 1 & 4 \end{bmatrix}
\]
if there exists a positive semi-definite matrix \(M\) such that \(A^T (I + MBB^T)^{-1} MA + Q \leq M\).

4. Conclusions

In this note, we have derived upper matrix bounds for the solution of the DARE. Using these matrix bounds, we then gave several eigenvalue bounds, including bounds for the individual eigenvalues, trace and determinant of the solution of the DARE. The ultimate achievement of this work is that the upper bounds developed in Theorem 2.2 and Corollary 2.1 are always calculated if the stabilizing solution of the DARE exists, whereas existing upper solution bounds for
the DARE might not be calculated, because they require restrictions of validity in addition to the existence conditions for the solution of the DARE. Finally, we gave numerical examples to demonstrate the effectiveness of the proposed results. The first numerical example suggests that our bounds can be tighter than existing ones for some cases, whilst for the second example, it appears that our bounds are not the tightest. As mentioned, the tightness of our bounds depend on the choice of $K$ and $D$ in which, unfortunately, we do not find a method such that the derived bounds are optimal. However, it is hoped that future research will propose a method(s) to determine the matrices $K$ and $D$ that can give sharper upper solution bounds.

References