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Quasialgebra Structure of the Octonions

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We show that the octonions are a twisting of the group algebra of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in the quasitensor category of representations of a quasi-Hopf algebra associated to a group 3-cocycle. In particular, we show that they are *quasialgebras* associative up to a 3-cocycle isomorphism. We show that one may make general constructions for quasialgebras exactly along the lines of the associative theory, including quasilinear algebra, representation theory, and an automorphism quasi-Hopf algebra. We study the algebraic properties of quasialgebras of the type which includes the octonions. Further examples include the higher 2^n-onion Cayley algebras and examples associated to Hadamard matrices. © 1999 Academic Press

1. INTRODUCTION

In this paper we provide a natural setting for the octonion algebra, namely as a *quasialgebra* or algebra in a symmetric monoidal or quasitensor category. Such categories have a tensor product and associativity isomorphisms $\Phi_{V,W,Z}\colon V \otimes (W \otimes Z) \cong (V \otimes W) \otimes Z$ for any three ob-

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jects, but these need not, however, be the trivial vector space isomorphisms. These categories also arise naturally as the representation categories of quasi-Hopf algebras [1]. Our first result is to identify the correct "octonion generating quasi-Hopf algebra" in the category of representations of which the octonions live as a quasialgebra. The categorical point of view then provides further general constructions on the octonions. Moreover, the framework has many other interesting quasiassociative algebras beyond these.

We note that our formulation of the octonions as quasialgebras is quite different from the usual treatment of such nonassociative algebras as "alternative algebras." The alternative axiom is much weaker than associativity and as a result most constructions for associative algebras do not go through for alternative ones. By contrast, in our new approach, the associativity axiom is retained up to isomorphism in the form

$$\cdot \circ (\cdot \otimes \mathrm{id}) = \cdot \circ (\mathrm{id} \otimes \cdot) \circ \Phi$$

where $\cdot$ is the product. We can then make use of Mac Lane's theorem [2], which asserts that any monoidal category is equivalent to a strictly associative one—i.e., we may make all categorical constructions for quasialgebras exactly as if they were strictly associative and simply put in the associators Φ as needed for compositions to make sense (Mac Lane's theorem ensures that all the ways to do this will give the same answer). Thus our result allows us to work with octonions exactly as if they were effectively associative. We demonstrate this in Sections 6 and 7 with notions of automorphisms and quasimatrix representations.

The second aspect of our result is the explicit form of the octonions as twistings of group algebras. We obtain them as examples of a particular class of quasialgebras $k_F G$ which we study in detail, where G is a group and F a 2-cochain (and where we work over a field k). We show how various algebraic properties of the octonions, including composition and alternativity properties, may be expressed in general in terms of F and a quasibicharacter $\mathscr{R}$ and 3-cocycle ϕ built from it. These are the main results of the paper and occupy Sections 3 and 4. Higher Cayley algebras and also (more trivially) the quaternions and complex numbers are of this $k_F G$ type, the latter two being associative as corresponding to F closed (a 2-cocycle) in these two cases. There are also many $k_F G$ quasialgebras included in our theory which are quite different from the octonions; some interesting classes of these are collected in Section 5.

Moreover, many of the properties of these $k_F G$ quasialgebras are related to the fact that they are in some sense "gauge equivalent" to the group algebras kG by a certain twisting operation. The general principle of "gauge equivalence" or twisting has been introduced for quasi-Hopf algebras by Drinfeld [1] and has also been used for ordinary associative Hopf algebras [3]. Our $k_F G$ are *not* themselves Hopf algebras or quasi-Hopf algebras but rather they relate to Drinfeld's theory as follows. Thus, behind our construction is a quasi-Hopf algebra $(k(G), \phi)$, where $k(G)$ is the usual Hopf algebra of functions on a (finite) group G but regarded trivially as a quasi-Hopf algebra via a 3-cocycle ϕ on G. In this sense our construction is very much the analogue, for discrete groups, of Drinfeld's construction of the quantum groups $U_q(g)$ as equivalent to quasi-Hopf algebras $(U(g), \phi)$, where ϕ is a Hopf-algebra 3-cocycle obtained by solving the Knizhnik-Zamalodchikov equations [1]. In our case the $(k(G), \phi)$ are obtained by twisting from the Hopf algebras $k(G)$ by cochains F. On the other hand, the group algebra kG lives in an obvious way in the category of $k(G)$-modules (or G-graded spaces) as an algebra; when we twist or "gauge transform" $k(G)$ into $(k(G), \phi)$ we must likewise gauge transform all the algebras on which it acts, which in our case sends kG to $k_F G$ as an algebra in the category of $(k(G), \phi)$ representations. This is the point of view by which our quasialgebras $k_F G$ are introduced, in Section 2. For technical reasons (to avoid G finite) we actually work with dual quasi-Hopf algebras and comodules rather than quasi-Hopf algebras and modules.

In the quantum groups literature this kind of extension of Drinfeld's twisting to algebraic structures living *in* representation categories has been introduced in [4], where it was used to express the transformation from q-deformed Euclidean space (which lives in the category of $U_q(so_4)$-modules) to q-deformed Minkowski space (which lives in the category of $U_q(so_{1,3})$-modules). In a similar way, a certain 2-cochain F (which we provide) gauge transforms the group algebra of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ ($\mathbb{Z}_2$ the cyclic group of order 2) into the octonions.

2. PRELIMINARIES: GENERAL CONSTRUCTIONS

An introduction to quantum groups, including quasitensor categories and quasi-Hopf algebras is in [5], the main notations of which we use here. In fact, the natural setting for us is the dual of Drinfeld's axioms [1], namely the notion of a dual quasi-Hopf algebra [6, 5].

Thus, a *dual quasibialgebra* is $(H, \Delta, \epsilon, \phi)$ where the coproduct Δ: $H \to H \otimes H$ and counit ϵ: $H \to k$ form a coalgebra (the axioms are those of a unital associative algebra with arrows reversed) and are multiplicative with respect to a "product" $H \otimes H \to H$. We write $\Delta h = \sum h_{(1)} \otimes h_{(2)}$ and $(\mathrm{id} \otimes \Delta) \circ \Delta h = (\Delta \otimes \mathrm{id}) \circ \Delta h = \sum h_{(1)} \otimes h_{(2)} \otimes h_{(3)}$, etc., as an explicit notation [7] for elements resulting from repeated applications of Δ. The product is required to be associative up to "conjugation" by ϕ in the sense

$$\sum a_{(1)} \cdot (b_{(1)} \cdot c_{(1)}) \phi(a_{(2)}, b_{(2)}, c_{(2)}) = \sum \phi(a_{(1)}, b_{(1)}, c_{(1)})(a_{(2)} \cdot b_{(2)}) \cdot c_{(2)} \tag{1}$$

for all $a, b, c \in H$, where ϕ is a unital 3-cocycle in the sense

$$\begin{aligned}\sum \phi(b_{(1)}, c_{(1)}, d_{(1)}) \phi(a_{(1)}, b_{(2)} \cdot c_{(2)}, d_{(2)}) \phi(a_{(2)}, b_{(3)}, c_{(3)}) \\ = \sum \phi(a_{(1)}, b_{(1)}, c_{(1)} \cdot d_{(1)}) \phi(a_{(2)} \cdot b_{(2)}, c_{(2)}, d_{(2)})\end{aligned} \tag{2}$$

for all $a, b, c, d \in H$, and $\phi(a, 1, b) = \epsilon(a)\epsilon(b)$ for all $a, b \in H$. We also require that ϕ is convolution-invertible in the algebra of maps $H^{\otimes 3} \to k$, i.e., that there exists ϕ^{-1}: $H^{\otimes 3} \to k$ such that

$$\begin{aligned}\sum \phi(a_{(1)}, b_{(1)}, c_{(1)}) \phi^{-1}(a_{(2)}, b_{(2)}, c_{(2)}) \\ = \epsilon(a)\epsilon(b)\epsilon(c) = \sum \phi(a_{(1)}, b_{(1)}, c_{(1)}) \phi^{-1}(a_{(2)}, b_{(2)}, c_{(2)})\end{aligned}$$

for all $a, b, c \in H$.

A dual quasibialgebra is a dual quasi-Hopf algebra if there is a linear map S: $H \to H$ and linear functionals α, β: $H \to k$ such that

$$\sum (Sa_{(1)}) a_{(3)} \alpha(a_{(2)}) = 1\alpha(a), \qquad \sum a_{(1)} Sa_{(3)} \beta(a_{(2)}) = 1\beta(a), \tag{3}$$

$$\begin{aligned}\sum \phi(a_{(1)}, Sa_{(3)}, a_{(5)}) \beta(a_{(2)}) \alpha(a_{(4)}) = \epsilon(a), \\ \sum \phi^{-1}(Sa_{(1)}, a_{(3)}, Sa_{(5)}) \alpha(a_{(2)}) \beta(a_{(4)}) = \epsilon(a)\end{aligned} \tag{4}$$

for all $a \in H$.

Finally, H is called *dual quasitriangular* if there is a convolution-invertible map $\mathscr{R}$: $H \otimes H \to k$ such that

$$\begin{aligned}\mathscr{R}(a \cdot b, c) = \sum \phi(c_{(1)}, a_{(1)}, b_{(1)}) \mathscr{R}(a_{(2)}, c_{(2)}) \phi^{-1}(a_{(3)}, c_{(3)}, b_{(2)}) \\ \times \mathscr{R}(b_{(3)}, c_{(4)}) \phi(a_{(4)}, b_{(4)}, c_{(5)}),\end{aligned} \tag{5}$$

$$\begin{aligned}\mathscr{R}(a, b \cdot c) = \sum \phi^{-1}(b_{(1)}, c_{(1)}, a_{(1)}) \mathscr{R}(a_{(2)}, c_{(2)}) \phi(b_{(2)}, a_{(3)}, c_{(3)}) \\ \times \mathscr{R}(a_{(4)}, b_{(3)}) \phi^{-1}(a_{(5)}, b_{(4)}, c_{(4)}),\end{aligned} \tag{6}$$

$$\sum b_{(1)} \cdot a_{(1)} \mathscr{R}(a_{(2)}, b_{(2)}) = \sum \mathscr{R}(a_{(1)}, b_{(1)}) a_{(2)} \cdot b_{(2)} \tag{7}$$

for all $a, b, c \in H$.

We recall also that a *corepresentation* or comodule under a coalgebra means a vector space V and a map β: $V \to V \otimes H$ obeying $(\mathrm{id} \otimes \Delta) \circ \beta =$

$(\beta \otimes \mathrm{id}) \circ \beta$ and $(\mathrm{id} \otimes \epsilon) \circ \beta = \mathrm{id}$. This is the notion of an action with arrows reversed. In the finite-dimensional case a coaction of H means an action of the associative algebra H^*.

A monoidal category is a category $\mathscr{C}$ of objects V, W, Z, etc., a functor $\otimes$: $\mathscr{C} \otimes \mathscr{C} \to \mathscr{C}$, and a natural transformation Φ: $((\otimes) \otimes\ ) \to (\otimes(\otimes))$ between the two functors $\mathscr{C} \times \mathscr{C} \times \mathscr{C} \to \mathscr{C}$, where Φ obeys Mac Lane's "pentagon identity" for equality of the two obvious isomorphisms [2]

$$(((V \otimes W) \otimes Z) \otimes U \cong V \otimes (W \otimes (Z \otimes U)))$$

built from Φ for any four objects V, W, Z, and U. A braided or "quasitensor" category is a monoidal one which has, in addition, a natural transformation Ψ: $\otimes \to \otimes^{\mathrm{op}}$ obeying two "hexagon" coherence conditions (see [8]).

The comodules $\mathscr{M}^H$ over a dual quasi-Hopf algebra form such a category with

$$\Phi_{V,W,Z}((v \otimes w) \otimes z) = \sum v^{(\bar{1})} \otimes (w^{(\bar{1})} \otimes z^{(\bar{1})}) \phi(v^{(\bar{2})}, w^{(\bar{2})}, z^{(\bar{2})})$$

for $v \in V$, $w \in W$, and $z \in Z$. Here $\beta(v) = \sum v^{(\bar{1})} \otimes v^{(\bar{2})}$ is a notation and the tensor product of two comodules is a comodule by composition with the product of H. In the quasitriangular case the category is braided [1], with $\Psi_{V,W}(v \otimes w) = \sum w^{(\bar{1})} \otimes v^{(\bar{1})} \mathscr{R}(v^{(\bar{2})}, w^{(\bar{2})})$. There is also a conjugate or dual coaction on V^* made possible by the antipode S. The converse is also true: namely any (braided) monoidal category with duals and with a multiplicative functor to the category of vector spaces (and some finiteness properties) comes from the comodules over a dual (quasitriangular) quasi-Hopf algebra [6].

If H is a dual quasi-Hopf algebra, then so is H_F with the new product, Φ, $\mathscr{R}$, α, β given by

$$\begin{aligned} a \cdot_F b &= \sum F^{-1}(a_{(1)}, b_{(1)}) a_{(2)} \cdot b_{(2)} F(a_{(3)}, b_{(3)}) \\ \phi_F(a, b, c) &= \sum F^{-1}(b_{(1)}, c_{(1)}) F^{-1}(a_{(1)}, b_{(2)} \cdot c_{(2)}) \phi(a_{(2)}, b_{(3)}, c_{(3)}) \\ &\quad \times F(a_{(3)} \cdot b_{(4)}, c_{(4)}) F(a_{(4)}, b_{(5)}) \qquad (8) \\ \alpha_F(a) &= \sum F(Sa_{(1)}, a_{(3)}) \alpha(a_{(2)}), \qquad \beta_F(a) = \sum F^{-1}(a_{(1)}, Sa_{(3)}) \beta(a_{(2)}) \\ \mathscr{R}_F(a, b) &= \sum F^{-1}(b_{(1)}, a_{(1)}) \mathscr{R}(a_{(2)}, b_{(2)}) F(a_{(3)}, b_{(3)}) \end{aligned}$$

for all $a, b, c \in H$. Here F is any convolution-invertible map F: $H \otimes H \to k$ obeying $F(a, 1) = F(1, a) = \epsilon(a)$ for all $a \in H$ (a 2-cochain). This is the dual version of the twisting operation or "gauge equivalence" of Drinfeld, so called because it does not change the category of comodules up to monoidal equivalence.

DEFINITION 2.1. Let H be a dual quasi-Hopf algebra. An H-comodule quasialgebra A is an algebra in the category of H-comodules. This means an H-comodule, a product map $\cdot$ associative in the category and equivariant under the coaction of H. Explicitly,

$$(a \cdot b) \cdot c = \sum a^{(\bar{1})} \cdot (b^{(\bar{1})} \cdot c^{(\bar{1})}) \phi(a^{(\bar{2})}, b^{(\bar{2})}, c^{(\bar{2})}),$$
$$\beta(a \cdot b) = \beta(a)\beta(b), \quad \forall a, b \in A$$

where the last expression uses the tensor product algebra in $A \otimes H$.

PROPOSITION 2.2. *If A is an H-comodule quasialgebra and $F\colon H \otimes H \to k$ a 2-cochain, then A_F with the new product*

$$a \cdot_F b = \sum a^{(\bar{1})} b^{(\bar{1})} F(a^{(\bar{2})}, b^{(\bar{2})})$$

and unchanged unit is an H_F-comodule quasialgebra.

Proof. This is elementary and follows from the equivalence of the comodule categories under twisting. See also [3] in the module version. ∎

There is a parallel theory with all arrows reversed. Thus, H can be a quasi-Hopf algebra, with associative product and Δ coassociative up to conjugation by an invertible 3-cocycle $\phi \in H^{\otimes 3}$ [1]. In this case the modules of H form a monoidal category and, in the quasitriangular case, a braided one. In this case we work with H-module quasialgebras and their twistings by $F \in H \otimes H$.

When the theory is developed in this comodule form, it is an easy matter to specialize to the following class of examples: let $H = kG$ the group algebra of a group. This has coproduct, etc.,

$$\Delta x = x \otimes x, \qquad \epsilon x = 1, \qquad Sx = x^{-1}, \qquad \forall x \in G$$

forming a Hopf algebra. However, for any point-wise invertible group cocycle $\phi\colon G \times G \times G \to k$ in the sense

$$\phi(y, z, w)\phi(x, yz, w)\phi(x, y, z)$$
$$= \phi(x, y, zw)\phi(xy, z, w), \qquad \phi(x, e, y) = 1 \tag{9}$$

extended linearly to $kG^{\otimes 3}$, we can regard (kG, ϕ) as a dual quasi-Hopf algebra. Group inversion provides an antipode with $\alpha = \epsilon, \beta(x) = 1/\phi(x, x^{-1}, x)$. Finally, a dual quasitriangular structure is possible only when G is Abelian and corresponds to invertible $\mathscr{R}: G \times G \to k$ such that

$$\mathscr{R}(xy, z) = \mathscr{R}(x, z)\mathscr{R}(y, z)\frac{\phi(z, x, y)\phi(x, y, z)}{\phi(x, z, y)},$$
$$\mathscr{R}(x, yz) = \mathscr{R}(x, z)\mathscr{R}(x, y)\frac{\phi(y, x, z)}{\phi(y, z, x)\phi(x, y, z)} \tag{10}$$

for all $x, y, z \in G$.

A special case is when ϕ is a coboundary,

$$\phi(x, y, z) = \frac{F(x, y)F(xy, z)}{F(y, z)F(x, yz)}, \qquad \mathscr{R}(x, y) = \mathscr{R}_0(x, y)\frac{F(x, y)}{F(y, x)},$$
$$\beta(x) = \frac{F(x^{-1}, x)}{F(x, x^{-1})} \tag{11}$$

for any invertible F obeying $F(x, e) = 1 = F(e, x)$ for all $x \in G$ (a 2-cochain) and any invertible bicharacter $\mathscr{R}_0$. This is the twisting of the group algebra $(kG, \mathscr{R}_0)$ regarded as a dual quasitriangular Hopf algebra [5] with trivial initial ϕ_0.

Next, a coaction of kG means precisely a G-grading, where $\beta(v) = v \otimes |v|$ on homogeneous elements of degree $|v|$. Hence the notion of an H-comodule quasialgebra in this case becomes:

DEFINITION 2.3. A G-graded quasialgebra is a G-graded vector space A, a product map $A \otimes A \to A$ preserving the total degree and associative in the sense

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)\phi(|a|, |b|, |c|), \qquad \forall a, b, c \in A$$

(of homogeneous degree), for a 3-cocycle ϕ. A G-graded quasialgebra is called *coboundary* if ϕ is a coboundary as in (11).

This is the setting which we will use, with the above as the underlying explanation of the constructions. If G is finite we can equally regard its function algebra $k(G)$ with $\phi \in k(G)^{\otimes 3}$ as a quasi-Hopf algebra $k_\phi(G)$ as in [9] and then view a G-graded quasialgebra as equivalently a $k_\phi(G)$-module quasialgebra. Here the action of $h \in k_\phi(G)$ is $h.v = vh(|v|)$ on homogeneous elements.

COROLLARY 2.4. *$k_F G$ defined as kG with a modified product*

$$x \cdot_F y = xyF(x, y), \qquad \forall x, y \in G$$

is a coboundary G-graded quasialgebra. The degree of $x \in G$ is x, and F is any 2-*cochain on G.*

Proof. This is a special case of the twisting proposition. Here kG coacts on itself by $\beta = \Delta$, i.e., the degree of $x \in G$ is x. We now twist kG to the dual quasi-Hopf algebra $(kG, \phi = \partial F)$. In the process, we also twist kG as a comodule algebra to $k_F G$ as a comodule quasialgebra under this dual quasi-Hopf algebra. ∎

We note an elementary property of $k_F G$.

PROPOSITION 2.5. *With trivial initial bicharacter $\mathscr{R}_0$, the category of G-graded vector spaces is symmetrically braided* (*in the sense* $\Psi^2 = \mathrm{id}$) *and $k_F G$ is braided-commutative in the sense*

$$a \cdot b = \cdot \circ \Psi(a \otimes b), \qquad \forall a, b \in k_F G.$$

Proof. This is immediate from (11) and the definition of Ψ from $\mathscr{R}$. The latter is clearly $\Psi(a \otimes b) = b \otimes a \frac{F(|a|, |b|)}{F(|b|, |a|)}$ for elements of homogeneous degree $|a|, |b|$, which is braided but trivially so in the sense $\Psi^2 = \mathrm{id}$ (i.e., the category of G-graded spaces in this case is symmetric monoidal rather than strictly braided). ∎

3. MORE ABOUT THE QUASIALGEBRA $k_F G$

In this section, we will study further properties of the G-graded quasialgebra $k_F G$ beyond the general ones arising from their categorical structure in the preceding section. We assume that G is Abelian, that F is a 2-cochain, and that ϕ is a 3-cocycle.

First of all, we note that $k_F G$ has a natural symmetric bilinear form whereby the basis of group elements is orthonormal. In general, the associated quadratic function on $k_F G$ will not be multiplicative (a quadratic character).

PROPOSITION 3.1. *If $k_F G$ admits a quadratic character then F^2 is a coboundary and $\phi^2 = 1$. If the Euclidean norm quadratic function defined by $q(x) = 1$ for all $x \in G$ is multiplicative* (*making $k_F G$ a composition algebra*) *then $F^2 = 1$.*

Proof. (For all discussions of quadratic forms we suppose that k has characteristic not 2.) Given a quadratic character $q\colon k_F G \to k$, we have $q(x \cdot y) = q(F(x, y)xy) = F^2(x, y)q(xy) = q(x)q(y)$ on $x, y \in G$; i.e., $F^2(x, y) = q(x)q(y)/q(xy)$ is a coboundary in the group cohomology. In general, if $F^2 = \partial q$ we still need to specify a bilinear form with diagonal

q, so the converse is not automatic. If we take the canonical quadratic function associated to G as an orthonormal basis, we will have $q(x \cdot_F y) = F^2(x, y) = F^2(x, y)q(x)q(y)$ for all $x, y \in G$, so if this is multiplicative then $F^2 = 1$. ∎

This will be the case for some of the Cayley algebras in the next section, as well as for many other examples, and is the reason that F, ϕ typically have values ± 1 in these cases.

Also, we already know from the construction in Corollary 2.4 that

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)\phi(x, y, z), \qquad \phi(x, y, z) = \frac{F(x, y)F(xy, z)}{F(x, yz)F(y, z)}$$

and hence that $k_F G$ is associative iff $\phi = 1$. Also, recall that $F(x, e) = F(e, x) = 1$ (where $e \in G$ is the group identity) is part of the definition of a cochain, and

$$\phi(e, x, y) = \phi(x, e, y) = \phi(x, y, e) = 1$$

is part of the definition of a cocycle (the middle one implies the other two); in particular it holds for our coboundary ϕ.

Likewise, we know from Proposition 2.5 that $k_F G$ is braided-commutative with respect to the braiding

$$\Psi(x \otimes y) = \mathscr{R}(x, y) y \otimes x, \qquad \mathscr{R}(x, y) = \frac{F(x, y)}{F(y, x)}, \qquad \forall x, y \in G.$$

Hence it is commutative in the usual sense iff F is symmetric. This is also clear from the form of the product in $k_F G$ since G itself is Abelian. More interesting for us,

Definition 3.2. We say that $k_F G$ is *altercommutative* if Ψ is given by $\mathscr{R}$ of the form

$$\mathscr{R}(x, y) = \begin{cases} 1 & \text{if } x = e \text{ or } y = e \text{ or } x = y \\ -1 & \text{otherwise} \end{cases}$$

for all $x, y \in G$.

Note that $\mathscr{R}(x, x) = \mathscr{R}(x, e) = \mathscr{R}(e, x) = 1$ for any $k_F G$, so the content here is the value -1 in the remaining "otherwise" case. Also note that an altercommutative $k_F G$ can never be commutative unless $G = \mathbb{Z}_2$. The condition is somewhat similar to the notion of a "supercommutative" algebra. One also has (more familiar) cases for the breakdown of associativity, such as the notion of an alternative algebra. We have,

PROPOSITION 3.3. *$k_F G$ is an alternative algebra if and only if*

$$\phi^{-1}(y,x,z) + \mathcal{R}(x,y)\phi^{-1}(x,y,z) = 1 + \mathcal{R}(x,y)$$
$$\phi(x,y,z) + \mathcal{R}(z,y)\phi(x,z,y) = 1 + \mathcal{R}(z,y)$$

for all $x, y, z \in G$. In this case,

$$\phi(x,x,y) = \phi(x,y,y) = \phi(x,y,x) = 1$$

for all $x, y \in G$.

Proof. It is enough to consider the conditions of an alternative algebra on our basis elements, $x, y, z \in G$, i.e.,

$$(x\cdot y)\cdot z - x\cdot(y\cdot z) + (y\cdot x)\cdot z - y\cdot(x\cdot z) = 0$$
$$(x\cdot y)\cdot z - x\cdot(y\cdot z) + (x\cdot z)\cdot y - x\cdot(z\cdot y) = 0.$$

This translates at once into the two equations

$$\begin{aligned} &F(x,y)F(xy,z) + F(y,x)F(yx,z) \\ &\qquad = F(y,z)F(x,yz) + F(x,z)F(y,xz) \\ &F(x,z)F(xz,y) + F(x,y)F(xy,z) \\ &\qquad = F(z,y)F(x,zy) + F(y,z)F(x,yz) \end{aligned}$$

for all $x, y, z \in G$ when we put the product of $k_F G$ in terms of the associative product in G. Dividing through then gives the equations in terms of $\phi, \mathcal{R}$ as stated.

Also, setting $x = y$ in the first equation gives us (for characteristic of k not 2) $\phi(x,x,z) = 1$. Setting $y = z$ in the second equation likewise gives us $\phi(x,y,y) = 1$. Given these, setting $x = z$ in either gives $\phi(x,y,x) = 1$. Actually, it is known that the condition of being an alternative algebra is equivalent to

$$(a\cdot a)\cdot b = a\cdot(a\cdot b), \qquad (a\cdot b)\cdot b = a\cdot(b\cdot b)$$

for all a, b in the algebra, which more immediately implies $\phi(x,x,y) = \phi(x,y,y) = 1$ on basis elements. (Given these, the same two equations applied to $a = x + y, b = z$ in the first case and $a = x, b = y + z$ in the second case provide the full equations for an alternative algebra on basis elements x, y, z and hence imply that $(a\cdot b)\cdot a = a\cdot(b\cdot a)$ holds as well, as usual). ∎

Next we consider involutions. Since we have a special basis of $k_F G$, it is natural to consider involutions diagonal in this basis.

LEMMA 3.4. *$k_F G$ admits an involution which is diagonal in the basis G iff $\mathscr{R} = \partial s$ (a group coboundary) for some 1-cochain $s\colon G \to k^*$ with $s^2 = 1$. In this case, one has $\mathscr{R}(x, y) = \mathscr{R}(y, x)$ and $\phi(x, y, z) = \phi(z, y, x)^{-1}$ for all $x, y, z \in G$.*

Proof. Consider the endomorphism σ of the vector space $k_F G$ of the form,

$$\sigma(x) = s(x)x, \qquad \forall x \in G,$$

extended linearly, for some function $s\colon G \to k$. For an involution we need $\sigma^2 = \mathrm{id}$, $s(1) = 1$, and $\sigma(a \cdot b) = \sigma(b)\sigma(a)$, $\forall a, b$ in our algebra. It is enough to consider these on the basis elements. Then clearly the first two correspond to

(i) $s(e) = 1$ and $s^2(x) = 1$ for all $x \in G$.

For the second condition,

$$\sigma(x \cdot y) = s(xy)x \cdot y = s(xy)F(x, y)xy$$

while

$$\sigma(y) \cdot \sigma(x) = s(y)y \cdot s(x)x = s(x)s(y)F(y, x)xy.$$

Equality for all x, y corresponds to

(ii) $\frac{s(x)s(y)}{s(xy)} = \frac{F(x,y)}{F(y,x)}$ for all $x, y \in G$.

We interpret this as stated, where the right-hand side is $\mathscr{R}$ corresponding to the braiding Ψ. In view of (i), it implies that $\mathscr{R}^2(x, y) = 1$ and hence that $\mathscr{R}$ is symmetric. It also implies that $\partial\mathscr{R} = 1$ in the group cohomology, which is the condition on ϕ stated since $\mathscr{R}(x, y) = F(x, y)/F(y, x)$ for all $x, y \in G$. ∎

In fact, we will be particularly interested in involutions with the property

$$a + \sigma(a), a \cdot \sigma(a) \in k1$$

(a multiple of the identity) for all a in the algebra. Let us call this a *strong involution*. We also suppose from now on that G is finite.

PROPOSITION 3.5. *$k_F G$ admits a diagonal strong involution σ if and only if*

(i) *$G \simeq (\mathbb{Z}_2)^n$ for some n,*

(ii) *$\sigma(e) = e$, $\sigma(x) = -x$ for all $x \neq e$, and*

(iii) *$k_F G$ is altercommutative in the sense of Definition 3.2.*

Proof. Consider an endomorphism of the diagonal form $\sigma(x) = s(x)x$. A general element is $a = \sum_{x \in G} \alpha_x x$. We have of course

$$\sigma(a) = \sum_{x \in G} \alpha_x s(x) x;$$

hence

$$a + \sigma(a) = \sum_{x \in G} \alpha_x (1 + s(x)) x \in k1$$

for all coefficients α_x if and only if $s(x) = -1$ for all $x \neq e$. Here the group identity $e \in G$ is the basis element $1 \in k_F G$. Since we also need $\sigma(1) = 1$ for an involution, these two conditions hold iff

(ii′) $s(e) = 1$ and $s(x) = -1$ for all $x \neq e$.

Next, consider a basis element $x \in G$, then

$$x \cdot \sigma(x) = s(x) x \cdot x = s(x) F(x, x) x^2 \in k1$$

implies that $x^2 = e$; i.e., every element of G has order 2. Hence G (which is a finite Abelian group) is isomorphic to $(\mathbb{Z}_2)^n$ for some n. Next, consider $x + y$ for basis elements $x \neq y$. Then

$$\begin{aligned} (x + y) \cdot \sigma(x + y) &= (x + y) \cdot (s(x)x + s(y)y) \\ &= (s(x)F(x, x) + s(y)F(y, y))e \\ &\quad + (s(x)F(x, y) + s(y)F(y, x))xy \in k1 \end{aligned}$$

tell us that

(iii′) $s(x)F(y, x) + s(y)F(x, y) = 0$ for all $x \neq y$.

When $x = e$ or $y = e$ this is empty. Otherwise, given (ii′) it is equivalent to $\mathscr{R}(x, y) = -1$ for all $x \neq y$, $x \neq e$, and $y \neq e$, which is the altercommutativity condition.

Conversely, given these facts, a general element $a = \sum_{x \in G} \alpha_x x$ obeys

$$\begin{aligned} a \cdot \sigma(a) &= \sum_{x, y \in G} \alpha_x \alpha_y s(y) x \cdot y = \sum_{x, y \in G} \alpha_x \alpha_y F(x, y) s(y) xy \\ &= \sum_{z \in G} z \left(\sum_{x, y \in G} \delta_{xy, z} \alpha_x \alpha_y s(y) F(x, y) \right) \in k1 \end{aligned}$$

since the terms with $z \neq e$ have contribution only when $x \neq y$, and in this case the terms cancel pairwise due to condition (iii′). Hence if these conditions hold, we have the stated properties for σ. They clearly imply the ones in the preceding lemma as well. ∎

On the other hand, given $k_F G$ we have a natural function $s(x) = F(x, x)$ and consider now the particular endomorphism corresponding to this.

PROPOSITION 3.6. *If $\sigma(x) = F(x, x)x$ for all $x \in G$ defines a strong involution and $|G| \neq 2$ in k, then the algebra $k_F G$ is simple.*

Proof. Let I be an ideal of $k_F G$ different from $k_F G$, and $a = \sum_{x \in G} \alpha_x x$ be an element of I. Then for each $y \neq e \in G$,

$$a \cdot y = \sum_{x \in G} \alpha_x F(x, y) xy \in I, \qquad y \cdot a = \sum_{x \in G} \alpha_x F(y, x) xy \in I.$$

Adding these together and using Proposition 3.5 in the form of the conditions (ii′) and (iii′), we see that

$$\begin{aligned} a \cdot y + y \cdot a &= \sum_{x \neq y} \alpha_x (F(y, x) + F(x, y)) xy + \alpha_y 2F(y, y)e \\ &= 2\alpha_e y - 2\alpha_y e \in I. \end{aligned}$$

Then $(a \cdot y + y \cdot a) \cdot y = -2(\alpha_e e + \alpha_y y) \in I$ as well. Since this is for all $y \neq e$, we have $\sum_{y \neq e} \alpha_y y + (|G| - 1)\alpha_e e \in I$ and hence $\alpha_e e \in I$ provided $|G| - 2 \neq 0$ in k. In this case $\alpha_e = 0$ and $-\alpha_y e + \alpha_e y = -\alpha_y e \in I$ tell us that $\alpha_y = 0$ for all y; i.e., $a = 0$. Hence $I = \{0\}$. ∎

We also see from Proposition 3.5 that σ a strong involution restricts us to $G \cong (\mathbb{Z}_2)^n$.

COROLLARY 3.7. *If $G \cong (\mathbb{Z}_2)^n$ and F obeys $F(x, x) = -1$ and $F(x, y) = -F(y, x)$ for all $x \neq e$, $y \neq e$, and $x \neq y$, then $k_F G$ is an altercommutative algebra with diagonal strong involution and is simple when $n \geq 2$.*

Proof. From $\mathscr{R}(x, y) = F(x, y)/F(y, x)$ it is clear that the $k_F G$ are altercommutative (and hence noncommutative for $n \geq 2$). From this, the choice $G = (\mathbb{Z}_2)^n$ and $F(x, x) = -1$ for $x \neq e$, we conclude by Proposition 3.5 that the algebras $k_F G$ admit the diagonal strong involution $\sigma(x) = F(x, x)x$ and hence by Proposition 3.6 that they are simple when $n \geq 2$. ∎

Also in this context, we have a partial converse to Proposition 3.1.

PROPOSITION 3.8. *If $G \cong (\mathbb{Z}_2)^n$ then the Euclidean norm quadratic function defined by $q(x) = 1$ for all $x \in G$ makes $k_F G$ a composition algebra if and only if*

(i) $F^2(x, y) = 1$ *for all* $x, y \in G$.

(ii) $F(x, xz)F(y, yz) + F(x, yz)F(y, xz) = 0$, *for all* $x, y, z \in G$ *with* $x \neq y$.

In this case the conditions in Proposition 3.6 *hold*: *i.e.*, $\sigma(x) = F(x,x)x$ *for all* $x \in G$ *is a strong involution and* $k_F G$ *is simple if* $|G| \neq 2$. *Moreover*, $k_F G$ *is alternative*.

Proof. Suppose that q is multiplicative. $q(x \cdot y) = q(x)q(y)$ on all basis elements $x, y \in G$ is $F^2 = 1$, as we know already from Proposition 3.1. The next case $q((x+y)\cdot z) = q(x+y)q(z)$ on basis elements $x, y, z \in G$ with $x \neq y$ does not yield any new condition (both sides are 2). Now consider the elements $x, y, z, w \in G$ with $x \neq y$, $z \neq w$, but $xz = yw$. Because every element of G is of order 2, this also means $xw = yz$. Because G is a group, $xz \neq xw$, however. Hence

$$\begin{aligned}
&q((x+y)\cdot(z+w))\\
&\quad = q\big(xz(F(x,z)+F(y,w)) + xw(F(x,w)+F(y,z))\big)\\
&\quad = (F(x,z)+F(y,w))^2 + (F(x,w)+F(y,z))^2\\
&\quad = 4 + 2(F(x,z)F(y,w) + F(x,w)F(y,z))
\end{aligned}$$

while $q(x+y)q(z+w) = 4$. This is the second condition stated after writing $w = xyz$ and renaming xz to z.

Conversely, assuming these conditions and given $a = \sum_x \alpha_x x$ and $b = \sum_y \beta_y y$, we have

$$a \cdot b = \sum_{z\in G} z\Big(\sum_{x\in G} \alpha_x \beta_{xz} F(x, xz)\Big)$$

since every element of G has order 2. Hence

$$\begin{aligned}
q(a\cdot b) &= \sum_z \Big(\sum_x \alpha_x \beta_{xz} F(x,xz)\Big)^2\\
&= \sum_z \sum_x \sum_y \alpha_x \alpha_y \beta_{xz} \beta_{yz} F(x,xz)F(y,yz).
\end{aligned}$$

In this sum the diagonal part where $x = y$ contributes

$$\sum_z \sum_x \alpha_x^2 \beta_{xz}^2 = q(a)q(b)$$

since $F^2 = 1$, while the remaining contribution from $x \neq y$ is

$$\sum_z \sum_{x\neq y} \alpha_x \alpha_y \beta_{xz} \beta_{yz} F(x,xz)F(y,yz), \qquad (*).$$

By condition (ii) this is equal to

$$
\begin{aligned}
-\sum_{x\neq y}\sum_{z}\alpha_x\alpha_y\beta_{xz}\beta_{yz}F(x,yz)F(y,xz)\\
=-\sum_{x\neq y}\sum_{w}\alpha_x\alpha_y\beta_{yw}\beta_{xw}F(x,xw)F(y,yw)
\end{aligned}
$$

where we change the order of summation and change variables to $w = xyz$. But this has the same form as our original expression for $(*)$ but with a minus sign; hence this is zero.

Next, we observe that the condition (ii) can be broken down equivalently as conditions

(ii.a) $F(x, xy) + F(x, y) = 0$ for all $x, y \in G$ with $x \neq e$.

(ii.b) $F(x, yz)F(y, xz) + F(x, z)F(y, z) = 0$ for all $x, y, z \in G$ with $x \neq e$, $y \neq e$, and $x \neq y$.

The first of these is (ii) in either of the cases $x = e, y \neq e$ or $x \neq e, y = e$ (followed by a relabeling), while the second is the remaining case $x \neq e$, $y \neq e, x \neq y$ after making use of (ii.a) to substitute $F(x, xz)$ and $F(y, yz)$.

In this case, (ii.a) implies the condition (ii′) of Proposition 3.6. On the other hand, $z = e$ in the original form of the present condition (ii) gives us

$$F(x,x)F(y,y) + F(x,y)F(y,x) = 0, \qquad \forall x \neq y$$

which, given $F^2 = 1$, implies the condition (iii′) of Proposition 3.6. Hence σ is strongly involutive.

Finally, also in this case, the equations of an alternative algebra in terms of F (see the proof of Proposition 3.3) reduce to the following. If $x = e$ or $y = e$, the first equation is trivial. Otherwise, the case $x \neq e, y \neq e$, and $x = y$ reduces to $\phi(x, x, z) = 1$, which in our present case where $G \cong (\mathbb{Z}_2)^n$ reduces to

$$F(x,x) = F(x,xz)F(x,z).$$

This holds because the left-hand side is -1 by the condition (ii′) of Proposition 3.6 and the right-hand side is $-F(x, z)^2 = -1$ by (ii.a) and $F^2 = 1$. The remaining case is $x \neq e, y \neq e$, and $x \neq y$. In this case the altercommutativity property in Proposition 3.6 reduces the first equation for an alternative algebra to

$$F(y,z)F(x,yz) + F(x,z)F(y,xz) = 0.$$

Since $F^2 = 1$, this is equivalent to (ii.b).

On the other hand, under the assumptions of Proposition 3.6, the conditions (ii.a) and (ii.b) are equivalent to

(ii.c) $F(x, y) + F(xy, y) = 0$ for all $x, y \in G$ with $y \neq e$.

(ii.d) $F(x, y)F(x, z) + F(xy, z)F(xz, y) = 0$ for all $x, y, z \in G$ with $y \neq e, z \neq e$, and $y \neq z$.

(Or one can obtain them directly from our original condition (ii).) We use these versions in a similar analysis for the content of the second of the conditions in Proposition 3.3 for an alternative algebra. ∎

We have written the proof of the last part of the proposition in a reversible way. Hence we also conclude,

COROLLARY 3.9. *If $\sigma(x) = F(x, x)x$ for all $x \in G$ is a strong involution, and $F^2 = 1$, then the following are equivalent*:

(i) *$k_F G$ is an alternative algebra, and*

(ii) *$k_F G$ is a composition algebra.*

The conditions in Proposition 3.8 and the corollary are evidently highly restrictive, because if k has characteristic different from 2 it is known that we have only the following composition algebras with the Euclidean norm: k, the algebra with basis $1, v, v^2 = -1$, the algebra over k with the product of quaternions, and the algebra over k with the product of octonions. If k is algebraically closed these algebras are isomorphic to $k, k \oplus k, M_2(k)$ (2×2 matrices), and Zorn's algebra of vectorial matrices [10]. The latter is then the only finite-dimensional simple alternative algebra that is not associative. On the other hand, the diagonal strong involution conditions in Proposition 3.6 are definitely weaker and hold for the entire family of Cayley algebras, as we will see in the next section.

Finally, for completeness, we include a slight generalization of Proposition 3.8.

PROPOSITION 3.10. *Let $k_F G$ be an algebra that admits a strong diagonal involution $\sigma(x) = s(x)x$. Then the nondegenerate form $n(x) = x \cdot \sigma(x)$ makes $k_F G$ a composition algebra if and only if*

(i) $s(xy)F(x, y)^2 F(xy, xy) = s(x)s(y)F(x, x)F(y, y)$, *for all* $x, y \in G$.

(ii) $F(x, xz)F(y, yz)F(z, z)s(z) + F(x, yz)F(y, xz)F(xyz, xyz)s(xyz) = 0$, *for all $x, y \in G$ with $x \neq y$.*

Proof. Let A be a quasialgebra $k_F G$ that admits a strong diagonal involution $\sigma(x) = s(x)x$ and let us consider the form $n(x) = x \cdot \sigma(x)$. For all $x, y \in G$, if the form $n(x)$ admits composition we have $n(x \cdot y) =$

$n(x)n(y)$ and then $n(x \cdot y) = F^2(x, y)F(xy, xy)s(xy) = n(x)n(y) = s(x)s(y)F(x, x)F(y, y)$. On the other hand, for two elements of the algebra $k_F G$, $a = \sum_{x \in G} \alpha_x x$ and $b = \sum_{y \in G} \beta_y y$ we have $n(a) = \sum_{x \in G} \alpha_x^2 s(x)F(x, x)$ and $n(b) = \sum_{y \in G} \beta_y^2 s(y)F(y, y)$. But we know that

$$a \cdot b = \sum_{z \in G} z \Big(\sum_{x \in G} \alpha_x \beta_{xz} F(x, xz) \Big)$$

so

$$\begin{aligned} n(a \cdot b) &= \sum_{z \in G} \Big[\sum_{x \in G} \alpha_x \beta_{xz} F(x, xz) \Big]^2 s(z) F(z, z) \\ &= \sum_{x, y, z \in G} \alpha_x \beta_{xz} F(x, xz) \alpha_y \beta_{yz} F(y, yz) s(z) F(z, z), \end{aligned}$$

and like in the proof of Proposition 3.8 the result follows after comparing the last expression with

$$n(a)n(b) = \Big(\sum_{x \in G} \alpha_x^2 s(x) F(x, x) \Big) \Big(\sum_{y \in G} \beta_y^2 s(y) F(y, y) \Big).$$

∎

4. CAYLEY ALGEBRAS

In this section, we show that the complex number algebra, the quaternion algebra, the octonion algebra, and the higher Cayley algebras are all G-graded quasialgebras of the form $k_F G$ for suitable G and F, which we construct. We recall that these algebras can be constructed inductively by the Cayley-Dickson process; we show that this process is compatible with our quasialgebra approach to nonassociative algebras.

Let A be a finite-dimensional (not necessarily associative) algebra with identity element 1 and a strong involution σ, i.e., an involution such that $a + \sigma(a), a \cdot \sigma(a) \in k1$ for all $a \in A$. We have studied this condition in the context of our quasialgebras $k_F G$ in Proposition 3.5. The Cayley-Dickson process says that we can obtain a new algebra $\bar{A} = A \oplus vA$ of twice the dimension (i.e., elements are denoted a, va for $a \in A$) and multiplication defined by

$$(a + vb) \cdot (c + vd) = (a \cdot c + \alpha d \cdot \sigma(b)) + v(\sigma(a) \cdot d + c \cdot b)$$

and with a new strong involution $\bar{\sigma}$

$$\bar{\sigma}(a + vb) = \sigma(a) - vb.$$

The symbol v here is a notational device to label the second copy of A in $\overline{A}$. However, $v \cdot v = \alpha 1$ according to the stated product, so one should think of the construction as a generalization of the idea of complexification when $\alpha = -1$. If $A = k$ and $\alpha = -1$ then $\overline{A} = k[v]$ modulo the relation $v^2 = -1$ will be called the "complex number algebra" over a general field k. As in the preceding section, we suppose k has characteristic not 2.

We start with the cochain version of the Cayley-Dickson construction, motivated by the formulae above.

PROPOSITION 4.1. *Let G be a finite Abelian group and F a cochain on it (so $k_F G$ is a G-graded quasialgebra). For any s: $G \to k^*$ with $s(e) = 1$ we define $\overline{G} = G \times \mathbb{Z}_2$ and on it the cochain $\overline{F}$ and function $\bar{s}$,*

$$\overline{F}(x, y) = F(x, y), \qquad \overline{F}(x, vy) = s(x)F(x, y), \qquad \overline{F}(vx, y) = F(y, x)$$

$$\overline{F}(vx, vy) = \alpha s(x)F(y, x), \qquad \bar{s}(x) = s(x), \qquad \bar{s}(vx) = -1,$$

for all $x, y \in G$. Here $x \equiv (x, e)$ and $vx \equiv (x, \nu)$ denote elements of $\overline{G}$, where $\mathbb{Z}_2 = \{e, \nu\}$ with product $\nu^2 = e$.

If $\sigma(x) = s(x)x$ is a strong involution, then $k_{\overline{F}}\overline{G}$ is the Cayley-Dickson process applied to $k_F G$.

Proof. The only features to be checked for a cochain are that $\overline{F}$ should be pointwise invertible (which is clear from invertibility of s, F) and $\overline{F}(e, vx) = s(e)F(e, x) = 1$ and $\overline{F}(vx, e) = F(e, x) = 1$. Hence we have a new quasialgebra $k_{\overline{F}}\overline{G}$.

This reproduces the product $\cdot$ of the Cayley-Dickson process with respect to $\sigma(x) = s(x)x$, since that is

$$vx \cdot y = v(y \cdot x) = vF(y, x)yx$$

$$vx \cdot vy = \alpha ys(x) \cdot x = \alpha s(x)F(y, x)yx$$

$$x \cdot vy = vs(x)x \cdot y = s(x)F(x, y)xvy$$

in terms of the product of $\overline{G}$ on the right. Moreover, $\bar{s}$ on $\overline{G}$ clearly reproduces the $\overline{\sigma}$ in the Cayley-Dickson procedure as well. ∎

This provides a cochain approach to the Cayley-Dickson process. The last proposition also makes evident that all composition algebras with identity element, over a field of characteristic different from 2, are in fact, quasialgebras $k_F G$. We know that if we start with a field k with characteristic different from 2 we can construct the sequence of algebras $k(\nu)$ (the Cayley-Dickson extension of k with $\alpha = \nu$), $(k(\nu), \beta)$ (the Cayley-Dickson extension of $k(\nu)$ with $\alpha = \beta$), and $((k(\nu), \beta), \gamma)$ (the Cayley-Dickson extension of $(k(\nu), \beta)$ with $\alpha = \gamma$), in all cases admitting a strong diagonal involution such that the associated form admits composition. If A is a composition algebra with identity over a field of characteristic different from 2, it is isomorphic to one of these four classes of algebras. This proves too that the last proposition of Section 3 is a complete characterization of all composition algebras $k_F G$ with identity, in terms of its cochain F. We also know from Proposition 3.5 that all these algebras with strong diagonal involution are altercommutative. More generally,

PROPOSITION 4.2. *The braiding of $k_{\overline{F}}\overline{G}$ is $\forall x, y \in G$,*

$$\overline{\mathscr{R}}(x,y) = \mathscr{R}(x,y), \qquad \overline{\mathscr{R}}(x,vy) = s(x), \quad \overline{\mathscr{R}}(vx,y) = s(y)^{-1},$$
$$\overline{\mathscr{R}}(vx,vy) = \frac{s(x)}{s(y)}\mathscr{R}(y,x). \tag{12}$$

Hence $k_{\overline{F}}\overline{G}$ is altercommutative iff $k_F G$ is altercommutative and $s(x) = -1$ for all $x \in G$ and $x \neq e$. In this case $\bar{s}$ has the same property as well.

Proof. The form of $\overline{\mathscr{R}}$ follows at once from $\overline{F}$ in Proposition 4.1. The second part is then immediate from the definition of altercommutative. ∎

As in Section 3, we are particularly interested in the canonical involution defined by $s(x) = F(x,x)$ and in the Cayley-Dickson extension with $\alpha = -1$. In that case $\overline{F}$ is determined from F alone. Let us call this choice the *standard Cayley-Dickson process* in our cochain approach.

COROLLARY 4.3. *If $F(x,x) = -1$ for all $x \in G$ and $x \neq e$ then the same holds for $\overline{F}$ under the standard Cayley-Dickson process. Moreover, in this case,*

(i) *$\bar{s}$ has the standard form on $\overline{G}$.*
(ii) *If $F^2 = 1$ then $\overline{F}^2 = 1$ as well.*
(iii) *If $k_F G$ is altercommutative then so is $k_{\overline{F}}\overline{G}$.*

Proof. We have

$$\bar{\sigma}(x) = \overline{F}(x,x)x = F(x,x)x = \sigma(x),$$
$$\bar{\sigma}(vx) = \overline{F}(vx,vx)vx = -F(x,x)^2 vx = -vx$$

since $F^2(x,x) = 1$. The first two parts are then immediate. The third part is a special case of the preceding proposition. ∎

In particular, the standard complex, quaternion, octonion, etc., algebras are all of this form given by iterating the standard Cayley-Dickson process. To describe their cochains, we consider the special case where $G = (\mathbb{Z}_2)^n$

and F is of the form

$$F(x, y) = (-1)^{f(x, y)}$$

for some $\mathbb{Z}_2$-valued function f on $G \times G$ (which is a natural supposition for the class with $F^2 = 1$).

COROLLARY 4.4. *If $G = (\mathbb{Z}_2)^n$ and $F = (-1)^f$ then the standard Cayley-Dickson process has $\overline{G} = (\mathbb{Z}_2)^{n+1}$ and $\overline{F} = (-1)^{\bar{f}}$. We use a vector notation $\vec{x} = (x_1, \ldots, x_n) \in (\mathbb{Z}_2)^n$ where $x_i \in \{0, 1\}$ (and the group $\mathbb{Z}_2$ is now written additively). Then*

$$\bar{f}\big((\vec{x}, x_{n+1}), (\vec{y}, y_{n+1})\big) = f\big(\vec{x}, \vec{y}\big)(1 - x_{n+1}) + f\big(\vec{y}, \vec{x}\big)x_{n+1} + y_{n+1}f(\vec{x}, \vec{x}) + x_{n+1}y_{n+1}.$$

Proof. From the above, we have clearly

$$\bar{f}(x, y) = f(x, y), \qquad \bar{f}(x, vy) = f(x, y) + f(x, x),$$
$$\bar{f}(vx, y) = f(y, x), \qquad \bar{f}(vx, vy) = 1 + f(x, x) + f(y, x).$$

We then convert this to a vector notation where each copy of $\mathbb{Z}_2$ in G is the additive group of $\mathbb{Z}_2$. We then make use of the *product* in $\mathbb{Z}_2$ to express whether a term is included or not (thus $x_{n+1}y_{n+1}$ contributes 1 iff both $x_{n+1} = 1$ and $y_{n+1} = 1$, etc.) ∎

Iterating this now generates the f for the quaternions, octonions, etc.:

PROPOSITION 4.5. (i) *The complex number algebra has this form with*

$$G = \mathbb{Z}_2, \qquad f(x, y) = xy, \qquad x, y \in \mathbb{Z}_2$$

where we identify G as the additive group $\mathbb{Z}_2$ but also make use of its product.

(ii) *The quaternion algebra is of this form with*

$$\overline{G} = \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \bar{f}\big(\vec{x}, \vec{y}\big) = x_1y_1 + (x_1 + x_2)y_2$$

where $\vec{x} = (x_1, x_2) \in \overline{G}$ is a vector notation.

(iii) *The octonion algebra is of this form with*

$$\overline{\overline{G}} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \qquad \bar{\bar{f}}\big(\vec{x}, \vec{y}\big) = \sum_{i \le j} x_iy_j + y_1x_2x_3 + x_1y_2x_3 + x_1x_2y_3.$$

(iv) *The 16-onion algebra is of the form $\overline{\overline{\overline{G}}} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and*

$$\bar{\bar{\bar{f}}}\big(\vec{x}, \vec{y}\big) = \sum_{i \le j} x_iy_j + \sum_{i \ne j \ne k \ne i} x_ix_jy_k + \sum_{\text{distinct } i, j, k, l} x_ix_jy_ky_l + \sum_{i \ne j \ne k \ne i} x_iy_jy_kx_4.$$

We are now able to apply our various criteria in the last section for the structure of the algebras of the form $k_F G$ to this construction and to all these algebras. Note in all these cases (and for the whole 2^n-onion family generated in this way) f has a bilinear part defined by the bilinear form

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & & & \vdots \\ 0 & \cdots & 1 & 1 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

For the complex number and quaternion algebras this is the only part, which implies that F is a bicharacter and hence, in particular, is a group coboundary $\phi = 1$; i.e., these algebras are associative. The f for the octonions has this bilinear part, which does not change associativity, plus a cubic term which contributes to ϕ. Explicitly,

$$\phi\left(\vec{x}, \vec{y}, \vec{z}\right) = (-1)^{|\vec{x}\vec{y}\vec{z}|}$$

in terms of the determinant of the three vectors as a matrix. The 16-onion has additional cubic and quartic terms, etc. This makes the origin of the breakdown of associativity for the higher members of the family particularly clear. It is also clear that if one takes only the quadratic parts of these cochains,

$$G = (\mathbb{Z}_2)^n, \qquad f\left(\vec{x}, \vec{y}\right) = \sum_{i \le j} x_i y_j$$

then one has for $k_F G$ the associative versions of these algebras, namely their associated Clifford algebras (here with negative signature).

In the remainder of this section we suppose that the quasialgebra $k_F G$ admits a diagonal involution $\sigma(x) = s(x)x$ in the basis G. We shall now denote by $k_{\overline{F}}\overline{G}$ the generalized Cayley-Dickson procedure with respect to this (with general $\alpha \neq 0$). It is easy to see that $\bar{s}$ provides a diagonal involution on it; i.e., this property is preserved under the procedure.

PROPOSITION 4.6. *If s defines a diagonal involution, then the associativity cocycle $\overline{\phi}$ of $k_{\overline{F}}\overline{G}$ is given by*

$$\overline{\phi}(x, y, z) = \phi(x, y, z), \qquad \overline{\phi}(vx, y, z) = \mathscr{R}(y, z)\phi(x, y, z)$$
$$\overline{\phi}(x, vy, z) = \mathscr{R}(y, z)\mathscr{R}(xy, z)\phi(x, y, z),$$

$$\begin{aligned}
\bar{\phi}(x, y, vz) &= \mathscr{R}(x, y)\phi(x, y, z)\\
\bar{\phi}(vx, vy, z) &= \mathscr{R}(xy, z)\phi(x, y, z),\\
\bar{\phi}(vx, y, vz) &= \mathscr{R}(y, z)\mathscr{R}(x, y)\phi(x, y, z)\\
\bar{\phi}(x, vy, vz) &= \mathscr{R}(x, yz)\phi(x, y, z),\\
\bar{\phi}(vx, vy, vz) &= \mathscr{R}(xy, z)\mathscr{R}(x, y)\phi(x, y, z)
\end{aligned}$$

$\forall x, y, z \in G$.

Proof. We use the definition of $\bar{\phi}$ as coboundary of $\bar{F}$, the form of this in Proposition 4.1, and $F(x, y) = \mathscr{R}(x, y)F(y, x)$ from Definition 3.2. We also use Lemma 3.4 which tells us that $s^2 = 1$, $\mathscr{R}(x, y) = s(x)s(y)/s(xy)$, and $\phi(x, y, z)\phi(z, y, x) = 1$. ∎

COROLLARY 4.7. *Suppose that s defines a diagonal involution. Then $k_{\bar{F}}\bar{G}$ is associative if and only if $k_F G$ is associative and commutative.*

Proof. We already know that $k_F G$ is commutative iff F is symmetric, which means iff $\mathscr{R} = 1$. So in this case if $\phi = 1$ then $\bar{\phi} = 1$. So if $k_F G$ is associative and commutative, $k_{\bar{F}}\bar{G}$ is associative. Conversely, if $\bar{\phi} = 1$ then by restriction, $\phi = 1$ so $k_F G$ is associative. Moreover, $1 = \bar{\phi}(x, y, v) = \mathscr{R}(x, y)\phi(x, y, e) = \mathscr{R}(x, y)$ tells us that $k_F G$ is commutative. ∎

For our final result we need the further condition that $k_F G$ has trivial centre $k1 = \langle e \rangle$. This is equivalent to the statement that all basis elements $x \neq e$ are not central. Since $k_F G$ is braided-commutative with respect to $\mathscr{R}$, the condition is equivalent to e the only basis element x such that $\mathscr{R}(x, y) = 1$ for all $y \in G$ (such an element x is called "bosonic"). For example, the trivial centre condition holds if $k_F G$ is altercommutative and $|G| > 2$.

PROPOSITION 4.8. *Suppose that s defines a diagonal involution and $k_F G$ has trivial centre. Then $k_{\bar{F}}\bar{G}$ is alternative iff $k_F G$ is associative and $s(x) = -1$ for all $x \in G$ and $x \neq e$.*

Proof. The condition for $k_{\bar{F}}\bar{G}$ being alternative is expressed via Proposition 3.3 in terms of $\bar{\mathscr{R}}$ and $\bar{\phi}$. These in turn are given by (12) and Proposition 4.6. We enumerate the various cases of the two conditions. For example,

$$\bar{\phi}(x, vy, z) + \bar{\mathscr{R}}(z, vy)\bar{\phi}(x, z, vy) = 1 + \bar{\mathscr{R}}(z, vy), \qquad \forall x, y, z \in G$$

reduces to

$$\mathscr{R}(y, z)\mathscr{R}(yx, z)\phi(x, y, z) + s(z)\mathscr{R}(x, z)\phi(x, z, y) = 1 + s(z)$$

which, using (10) and facts about $\mathscr{R}$, ϕ from Lemma 3.4, reduces to

$$\phi(x,z,y)\mathscr{R}(x,z)(\phi(y,x,z)+s(z)) = 1 + s(z), \qquad \forall x,y,z \in G.$$

This clearly holds if $\phi = 1$ and $s(z) = -1$ for all $z \neq e$ and similarly for all other cases of the alternativity conditions.

Conversely, if $k_{\overline{F}}\overline{G}$ is alternative then the above condition holds. Setting $y = e$ implies that $(\mathscr{R}(x,z) - 1)(1 + s(z)) = 0$ for all $x, z \in G$. Under the assumption of no central basis element other than e, we conclude that $s(z) = -1$ for all $z \neq e$. The condition above then reduces to $\phi(y,x,z) = 1$ for all $z \neq e$. Hence $\phi = 1$. ∎

For example, if s defines a strong diagonal involution then we know from Proposition 3.5 that $k_F G$ is altercommutative and that $s(x) = -1$ for all $x \neq e$. So in this setting, $k_{\overline{F}}\overline{G}$ is alternative iff $k_F G$ is associative.

5. NEW QUASIALGEBRAS

It is clear that there are many examples of quasialgebras $k_F G$ according to the group G and the cochain F. In this section we will consider examples where $F^2 = 1$; we will see that even in this case we can obtain very different types of algebras. We assume that the characteristic of k is zero.

Motivated by Proposition 3.8, we first consider examples where the cochain F in the basis G is a Hadamard matrix of dimension $n = |G|$ (that is, a matrix F with entries ± 1 such that $F^t F = nI$ where t is transpose and I is the identity matrix) normalized such that the first row and column (corresponding to e) have entries 1.

LEMMA 5.1. *If the cochain F forms a Hadamard matrix then for all $x, y \in G$, $x \neq y$ there is a bijection $p = p_{x,y}\colon G \to G$ such that*

$$p(p(z)) = z, \qquad F(x,z)F(y,z) + F(x,p(z))F(y,p(z)) = 0, \qquad \forall z \in G.$$

Proof. As the cochain F forms a Hadamard matrix, we have $\sum_{z \in G} F(x,z)F(y,z) = 0$ for all $x \neq y$. Since the values are ± 1, the cardinality of the subset $A \subseteq G$ formed by the elements $z \in G$ such that $F(x,z)F(y,z) = 1$ is therefore the same as the cardinality of the complementary subset $B \subseteq G$ formed by the elements $z \in G$ such that $F(x,z)F(y,z) = -1$. We choose any bijection ρ between these subsets and define $p_{x,y}(z) = \rho(z)$ for $z \in A$ and $p_{x,y}(z) = \rho^{-1}(z)$ for $z \in B$. ∎

One may partially characterize the resulting algebras in terms of the allowed bijections p. For example, if $G = (\mathbb{Z}_2)^n$ and if p can be chosen so that $p_{x,y}(z) = xyz$ (product in G) then it follows from Proposition 3.8 that $k_F G$ is necessarily a 2^n-dimensional composition algebra for the Euclidean norm. More generally, we can consider G a power of $\mathbb{Z}_{4n}$ and require existence of a bijection of the form $p_{x,y}(z) = x^{2n}y^{2n}z$.

EXAMPLE 5.2. If $G = \mathbb{Z}_4$, there is a unique F forming a symmetric Hadamard matrix with 1 on the diagonal and admitting bijections of the form $p_{x,y}(z) = x^2y^2z$ for all $x^2y^2 \neq e$. In this case $k_F G$ is a commutative, associative, and nonsimple algebra.

Proof. We suppose that for all $x^2y^2 \neq e$, $F(x, z)F(y, z) + F(x, x^2y^2z)F(y, x^2y^2z) = 0$ for all $z \in G$. Taking the standard basis e, e_1, e_2, e_3 of $\mathbb{Z}_4$, we can enumerate the cases $F(e_1, z) + F(e_1, e_2 z) = 0$, $F(e_3, z) + F(e_3, e_2 z) = 0$, $F(e_1, z)F(e_2, z) + F(e_1, e_2 z)F(e_2, e_2 z) = 0$, and $F(e_2, z)F(e_3, z) + F(e_2, e_2 z)F(e_3, e_2 z) = 0$ for all $z \in G$. These equations and the other assumptions force one to

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

The resulting multiplication table of $k_F G$ is then

$$\begin{matrix} e & e_1 & e_2 & e_3 \\ e_1 & e_2 & -e_3 & -e \\ e_2 & -e_3 & e & -e_1 \\ e_3 & -e & -e_1 & e_2 \end{matrix}.$$

This is easily seen to be commutative and associative. It has a nontrivial ideal spanned by $\{e_1 + e, e_2 - e_3, e_1 + e_2\}$. ∎

It is known that Hadamard matrices necessarily have dimension 2 or $4n$. However, we can consider the class of algebras $k_F G$ of dimensions 3 or $4n + 1$ where the matrix elements of the cochain are of the form

$$F = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & & & \\ \vdots & & F_0 & \\ 1 & & & \end{bmatrix},$$

where F_0 is a Hadamard matrix.

EXAMPLE 5.3. Let $k_F\mathbb{Z}_3$ be a quasialgebra with cochain of the above form with F_0 a Hadamard matrix of dimension 2. Then $k_F\mathbb{Z}_3$ is a nonassociative simple algebra (commutative or altercommutative).

Let $k_F\mathbb{Z}_5$ be a quasialgebra with cochain of the above form with F_0 the Hadamard matrix of dimension 4 with maximal excess (so $F_0 = J - 2I$ where J has all entries 1). Then $k_F G$ is a simple, commutative, nonassociative algebra.

Proof. If F_0 is an Hadamard matrix of order 2 we have only two different possibilities for $k_F\mathbb{Z}_3$, with the multiplication Table I

$$\begin{array}{ccc} e & e_1 & e_2 \\ e_1 & -e_2 & e \\ e_2 & e & e_1 \end{array}$$

or the multiplication Table II

$$\begin{array}{ccc} e & e_1 & e_2 \\ e_1 & e_2 & -e \\ e_2 & e & e_1 \end{array}.$$

Let $I \neq k_F\mathbb{Z}_3$ be an ideal of $k_F\mathbb{Z}_3$ and let $a = \sum_{x\in\mathbb{Z}_3}\alpha_x x$ be an element of I. Then $(a\cdot e_1)\cdot e_2 \in I$. If $k_F\mathbb{Z}_3$ is defined by Table I, this implies that $\alpha_{e_1} = 0$ (if not, $e_1 \in I$ and $I = k_F\mathbb{Z}_3$). Similarly, if $k_F\mathbb{Z}_3$ is defined by Table II, it implies that $\alpha_e = 0$ (if not $e \in I$ and $I = k_F\mathbb{Z}_3$). Multiplying a by e_i and reusing these results, one may conclude that I must be zero in either case. On the other hand, it is clear that the algebra defined by Table I is commutative and the one defined by Table II is altercommutative. The result for $k_F\mathbb{Z}_5$ is a special case of the next lemma. ∎

LEMMA 5.4. *Let G be a finite Abelian group and $F(x, y) = (-1)^{\delta_{x,y}}$ for $x, y \neq e$. If G has an element of order > 3 then $k_F G$ is a simple, commutative, nonassociative algebra.*

Proof. These algebras are clearly commutative. If $y \in G$ is an element of order > 3 then $y^3\cdot(y\cdot y) = -y^3\cdot y^2 = -y^5 = -y^4\cdot y = -(y^3\cdot y)\cdot y$ so the algebra is nonassociative. Here y^2, y^3 are powers taken in the group G. Also for such an element, consider an ideal $I \neq k_F G$ and let $a = \sum_{x\in G}\alpha_x x$ be an element of I. Then $y\cdot a = \sum_x \alpha_x F(y,x)yx = \sum_{x\neq y}\alpha_x yx - \alpha_y y^2$ is an element of I. Multiplying by y^2 we obtain that the element $y^2\cdot(\sum_{x\neq y}\alpha_x yx - \alpha_y y^2) = \sum_{x\neq y}\alpha_x y^3 x + \alpha_y y^4$ is an element of I. Multiplying further by y^{-3} and assuming that y does not have order 7, we obtain that the element $y^{-3}\cdot(\sum_{x\neq y}\alpha_x y^3 x + \alpha_y y^4) = \alpha_y y + \sum_{x\neq y, x\neq y^{-6}}\alpha_x x - \alpha_{y^{-6}} y^{-6}$ is an element of I. Thus $\alpha_{y^{-6}} y^{-6} \in I$ and hence $\alpha_{y^{-6}} = 0$ (otherwise $y^{-6} \in I$ implies each element of G is in I). If

$y^7 = e$ then we obtain instead that the element $-\alpha_y y + \sum_{x \neq y} \alpha_x x$ is an element of I and hence that $\alpha_y = 0$. In either case, multiplying a by the elements of G and reusing the result implies that $I = 0$. ∎

Other natural classes of examples come from the theory of fields. Let p be an odd prime and $q = p^r$ for integer $r \geq 1$. Let χ be the character defined in the finite field k_q of order q by

$$\chi(0) = 0, \qquad \chi(x) = \begin{cases} 1 & \text{if } x \neq 0,\ x \text{ a square} \\ -1 & \text{if } x \neq 0,\ x \text{ not a square}. \end{cases}$$

The multiplicative group $G = k_q - \{0\}$ and the cochain $F(x, y) = \chi(x)\chi(y)$ when $x, y \neq e$ define a natural class of algebras $k_F G$ which are commutative and nonassociative and admit the diagonal involution $s = \chi$. Further classes of examples (to be considered elsewhere) include the generalization of the octonions based on Galois sequences in [11, Chap. 2]. For example, over $\mathbb{C}$, one may take $G = \mathbb{Z}_3 \times \mathbb{Z}_3$ and F of the form $F = e^{(2\pi i/3)f}$, in contrast to the $F^2 = 1$ case considered above.

6. AUTOMORPHISM QUASI-HOPF ALGEBRAS

We return to the general setting of Section 2, where G is equipped with a 3-cocycle ϕ forming a dual quasi-Hopf algebra.

We let A be a finite-dimensional G-graded quasialgebra in the sense of Definition 2.3 and introduce a general construction for its comeasuring or "automorphism" dual quasi-Hopf algebra. We let $\{e_i\}$ be a basis of A with homogeneous degrees, denoted $|e_i| = |i| \in G$. We let $e_i \cdot e_j = \sum_k c_{ij}^k e_k$ define the structure constants of A in this basis. Also, we consider the group $G \times G$ with cocycle

$$\phi((a, g), (b, h), (c, f)) = \frac{\phi(g, h, f)}{\phi(a, b, c)}.$$

PROPOSITION 6.1. *Associated to a G-graded quasialgebra A is a dual quasibialgebra $M_1(A)$ (the comeasuring dual quasibialgebra) defined as the free $G \times G$-quasialgebra generated by $\{1, t^i_j\}$ where $|t^i_j| = (|i|, |j|)$ and $|1| = (e, e)$, modulo the additional relations*

$$\sum_a c_{ij}^a t_a^k = \sum_{a,b} c_{ab}^k t_i^a \cdot t_j^b.$$

We define Δ, ϵ *as*

$$\Delta t^i_j = \sum_a t^i_a \otimes t^a_j, \qquad \epsilon\left(t^i_j\right) = \delta^i_j$$

extended multiplicatively, and extend ϕ *to a linear functional* $\phi\colon M_1(A)^{\otimes 3} \to k$ *by*

$$\phi\left(t^{i_1}_{p_1} \cdots t^{i_\alpha}_{p_\alpha}, t^{j_1}_{q_1} \cdots t^{j_\beta}_{q_\beta}, t^{k_1}_{r_1} \cdots t^{k_\gamma}_{r_\gamma}\right)$$
$$= \delta^{i_1}_{p_1} \cdots \delta^{i_\alpha}_{p_\alpha} \delta^{j_1}_{q_1} \cdots \delta^{j_\beta}_{q_\beta} \delta^{k_1}_{r_1} \cdots \delta^{i_\gamma}_{p_\gamma} \phi\left(|i_1| \cdots |i_\alpha|, |j_1| \cdots |j_\beta|, |k_1| \cdots |k_\gamma|\right).$$

Proof. Since $G \times G$-graded spaces form a monoidal category, we define the free tensor algebra on the vector space spanned by basis $\{t^i_j\}$ in the usual way in a monoidal category. This means iterated tensor products in the generators, which we understand as nested to the right. The product is the tensor product composed with the appropriate associativity morphism. The $G \times G$-degree is multiplicative. In our case the result is the algebra generated by $1, t^i_j$ and the associativity rule

$$\left(\left(t^{i_1}_{p_1} \cdots t^{i_\alpha}_{p_\alpha}\right) \cdot \left(t^{j_1}_{q_1} \cdots t^{j_\beta}_{q_\beta}\right)\right) \cdot \left(t^{k_1}_{r_1} \cdots t^{k_\gamma}_{r_\gamma}\right)$$
$$= \left(t^{i_1}_{p_1} \cdots t^{i_\alpha}_{p_\alpha}\right) \cdot \left(\left(t^{j_1}_{q_1} \cdots t^{j_\beta}_{q_\beta}\right) \cdot \left(t^{k_1}_{r_1} \cdots t^{k_\gamma}_{r_\gamma}\right)\right)$$
$$\times \frac{\phi\left(|p_1| \cdots |p_\alpha|, |q_1| \cdots |q_\beta|, |r_1| \cdots |r_\gamma|\right)}{\phi\left(|i_1| \cdots |i_\alpha|, |j_1| \cdots |j_\beta|, |k_1| \cdots |k_\gamma|\right)}$$

where the degree $|t^{i_1}_{p_1} \cdots t^{i_\alpha}_{p_\alpha}| = (|i_1| \cdots |i_\alpha|, |p_1| \cdots |p_\alpha|)$ does not depend on the nesting of the products in the expression.

On this free quasiassociative algebra we define Δ, ϵ as shown. They are extended to products as algebra maps for the nonassociative product. It is easy to see that Δ, ϵ are compatible with the quasiassociativity, and that the extended ϕ as shown makes the free quasiassociative algebra into a dual quasibialgebra $\tilde{M}_1$ in the sense of Section 2. That the extended ϕ is a cocycle reduces to ϕ a group cocycle. The quasiassociativity axiom for a dual quasibialgebra reduces to the $G \times G$-quasiassociativity.

Next, it is easy to verify that the quotient by the relations shown is consistent with the G-quasiassociativity of our algebra A. In terms of its structure constants, the latter is

$$\sum_a c^a_{ij} c^b_{ak} = \sum_a c^a_{jk} c^b_{ia} \phi(|i|, |j|, |k|).$$

Then,

$$\begin{aligned}\left(t^i_p\cdot t^j_q\right)&\cdot t^k_r\phi(|i|,|j|,|k|)c^a_{jk}c^b_{ia}\\ &=\left(t^i_p\cdot t^j_q\right)\cdot t^k_r c^a_{ij}c^b_{ak}\\ &=c^c_{pq}t^a_c\cdot t^k_r c^b_{ak}=c^c_{pq}c^d_{cr}t^b_d\\ &=c^c_{qr}c^d_{pc}t^b_d\phi(|p|,|q|,|r|)=c^c_{qr}t^i_p\cdot t^a_c c^b_{ia}\phi(|p|,|q|,|r|)\\ &=t^i_p\cdot\left(t^j_q\cdot t^k_r\right)\phi(|p|,|q|,|r|)c^a_{jk}c^b_{ia}\end{aligned}$$

as required. We use the summation convention for upper-lower indices.

Finally, we verify that these relations are compatible with Δ. Thus,

$$\Delta\left(c^k_{ab}t^a_i\cdot t^b_j\right)=c^k_{ab}t^a_p\cdot t^b_q\otimes t^p_i\cdot t^p_j=c^b_{pq}t^k_b\otimes t^p_i\cdot t^q_j=c^a_{ij}t^k_b\otimes t^b_a=c^a_{ij}\Delta t^k_a$$

as required. Compatibility with ϵ is trivial. It is also clear that ϕ restricts to the quotient by virtue of the G-grading of the algebra A. For example, $\phi(t^i_j, t^a_p t^b_q c^k_{ab}, t^r_s)=\delta^i_j\delta^r_s\phi(|i|,|p|\,|q|,|r|)c^k_{pq}$ while $\phi(t^i_j, c^a_{pq}t^k_a, t^r_s)=\delta^i_j\delta^r_s\phi(|i|,|k|,|r|)c^k_{pq}$, but $c^k_{pq}=0$ unless $|p|\,|q|=|r|$. Thus, the quotient of $\tilde{M}_1$ by the relations shown defines a dual quasibialgebra M_1. ∎

If G is in addition equipped with a quasibicharacter $\mathscr{R}$ (making (kG,ϕ) into a dual quasitriangular quasi-Hopf algebra) then there is a natural braiding in the category of G-graded spaces as explained in Section 2. In our case, we extend $\mathscr{R}$ to a quasibicharacter on $G\times G$ by

$$\mathscr{R}((a,g),(b,h))=\frac{\mathscr{R}(g,h)}{\mathscr{R}(a,b)}.$$

PROPOSITION 6.2. *If $\mathscr{R}$ is a quasibicharacter on G, the comeasuring dual quasibialgebra $M_1(A)$ has a natural quotient $M_1(\mathscr{R},A)$ with the additional relation of braided-commutativity as a $G\times G$-quasialgebra. Then $M_1(\mathscr{R},A)$ is a dual quasitriangular dual quasibalgebra with $\mathscr{R}$ extended as a linear functional on $M_1(\mathscr{R},A)^{\otimes 2}$ by*

$$\mathscr{R}\left(t^{i_1}_{p_1}\cdots t^{i_\alpha}_{p_\alpha},t^{j_1}_{q_1}\cdots t^{j_\beta}_{q_\beta}\right)=\delta^{i_1}_{p_1}\cdots\delta^{i_\alpha}_{p_\alpha}\delta^{j_1}_{q_1}\cdots\delta^{j_\beta}_{q_\beta}\mathscr{R}\left(|i_1|\cdots|i_\alpha|,|j_1|\cdots|j_\beta|\right).$$

Proof. Explicitly, braided-commutativity as a $G\times G$-graded algebra is

$$\begin{aligned}\left(t^{j_1}_{q_1}\cdots t^{j_\beta}_{q_\beta}\right)&\cdot\left(t^{i_1}_{p_1}\cdots t^{i_\alpha}_{p_\alpha}\right)\mathscr{R}\left(|p_1|\cdots|p_\alpha|,|q_1|\cdots|q_\beta|\right)\\ &=\mathscr{R}\left(|i_1|\cdots|i_\alpha|,|j_1|\cdots|j_\beta|\right)\left(t^{i_1}_{p_1}\cdots t^{i_\alpha}_{p_\alpha}\right)\cdot\left(t^{j_1}_{q_1}\cdots t^{j_\beta}_{q_\beta}\right)\end{aligned}$$

for the braiding Ψ determined by $\mathcal{R}$ on $G \times G$ as shown. This is the braided-commutativity property of a dual quasitriangular dual quasibialgebra in the sense of Section 2, with $\mathcal{R}$ defined as stated. That this $\mathcal{R}$ is well-defined on the free quasiassociative algebra $\tilde{M}_1$ is clear. That it descends to the quotient by the relations of M_1 follows by the G-grading of our algebra A as for ϕ in the preceding proof. That it is well-defined on $M_1(\mathcal{R}, A)$ itself requires repeated use of the quasibicharacter property and is omitted. ∎

Moreover, both $M_1(A)$ and hence $M_1(\mathcal{R}, A)$ coact on A:

PROPOSITION 6.3. *M_1 coacts on A by β: $e_i \mapsto e_a \otimes t_i^a$ and β is an algebra map.*

Proof. The definition of the coaction is consistent with the relations of A

$$\beta(e_i \cdot e_j) = \beta(e_i)\beta(e_j) = e_a \cdot e_b \otimes t_i^a \cdot t_j^b = c_{ab}^k e_k \otimes t_i^a \cdot t_j^b$$
$$= c_{ij}^a e_k \otimes t_a^k = \beta\left(c_{ij}^a e_a\right)$$

by virtue of the relations of M_1. ∎

In fact, it should be clear from the proof that the relations of M are the minimum relations such that a coaction of this form extends as an algebra map. In the case where ϕ is trivial we recover in fact the dual (arrows-reversed) version of the measuring bialgebra $M(A, A)$ in [7], and this is the motivation behind our construction. Some further recent applications of this comeasuring bialgebra construction in the associative (not quasiassociative) case appear in [12].

Before turning to examples, we note that $M_1(A)$ and $M_1(\mathcal{R}, A)$ have natural further quotients. Thus,

PROPOSITION 6.4. *The diagonal quotient $M_D(A)$ of $M_1(A)$ by the relations $t_j^i = 0$ when $i \neq j$ is an associative bialgebra with generators t_i and relations*

$$c_{ij}^k\left(t_k - t_i t_j\right) = 0, \qquad \Delta(t_i) = t_i \otimes t_i, \qquad \epsilon(t_i) = 1.$$

It can also be viewed as a dual quasibialgebra with

$$\phi\left(t_{i_1} \cdots t_{i_\alpha}, t_{j_1} \cdots t_{j_\beta}, t_{k_1} \cdots t_{k_\gamma}\right) = \phi\left(|i_1| \cdots |i_\alpha|, |j_1| \cdots |j_\beta|, |k_1| \cdots |k_\gamma|\right).$$

The diagonal quotient $M_D(\mathcal{R}, A)$ of $M_1(\mathcal{R}, A)$ is the commutative quotient of $M_D(A)$ and can be viewed as a dual quasitriangular dual quasibialgebra with

$$\mathcal{R}\left(t_{i_1} \cdots t_{i_\alpha}, t_{j_1} \cdots t_{j_\beta}\right) = \mathcal{R}\left(|i_1| \cdots |i_\alpha|, |j_1| \cdots |j_\beta|\right).$$

Proof. This is elementary. The quasiassociativity and braided-commutativity of $M_1(A)$, $M_1(\mathcal{R}, A)$ clearly reduce in the diagonal case to usual associativity and commutativity. The coproduct becomes group-like. ∎

Finally, we have not required yet that A is unital. When it is, we choose our basis so that $e_0 = 1$. In this case we let $\{e_i\}$ denote the remaining basis elements.

PROPOSITION 6.5. *When $e_0 = 1$, the unit of A, we define $M_0(A)$ as the quotient of $M_1(A)$ by $t^0_0 = 1$ and $t^i_0 = t^0_i = 0$. This forms a dual quasibialgebra with relations and coproduct*

$$c^a_{ij} t^k_a = c^k_{ab} t^a_i t^b_j, \qquad c^0_{ij} = c^0_{ab} t^a_i t^b_j, \qquad \Delta t^i_j = t^i_a \otimes t^a_j.$$

Similarly, $M_0(\mathcal{R}, A)$ remains dual quasitriangular and preserves this form.

Proof. We set $t^i_0 = 0 = t^0_i$ and denote $t^0_0 = c$. Note that $c^j_{i0} = c^j_{0i} = \delta^j_i$ and $c^0_{00} = 1$. Therefore, the relations of $M_1(A)$ become (for all labels not 0) the relations as stated and the additional relations

$$t^i_j c = t^i_j = c t^i_j, \qquad c^2 = c.$$

In view of these latter relations, it is natural to set $c = 1$. Moreover, the matrix coproduct and counit are clearly compatible with the quotient. One has $\Delta c = c \otimes c$, so this is consistent with $c = 1$ as well. Moreover, since $|e_0| = e$, the identity in G, it is clear that setting $c = 1$ is consistent with the definition of ϕ, $\mathcal{R}$ on M_1. Likewise, their delta-function form is consistent with $t^i_0 = 0 = t^0_i$. Hence $M_0(A)$ and $M_0(\mathcal{R}, A)$ inherit these structures and are dual quasibialgebras.

Note also that the coaction of M_1 becomes $\beta(e_0) = e_0 \otimes 1$ and $\beta(e_i) = e_a \otimes t^a_i$. The relations of M_0 are such that the bilinear form on the span of $\{e_i\}$ defined by c^0_{ij} is preserved. ∎

Given our basis, we can identify $A/k1$ with the span of $\{e_i\}$ for $i \neq 0$, and $B(e_i, e_j) = c^0_{ij}$ is a natural bilinear form on it. We see that our reduced comeasuring dual quasi-quantum groups $M_0(A)$, $M_0(\mathcal{R}, A)$ preserve this. Also, the two relations for the t^i_j imply that

$$c^d_{ab} c^0_{dc} \left(t^a_i \cdot t^b_j \right) \cdot t^c_k = c^0_{dc} c^a_{ij} t^d_a \cdot t^c_k = c^a_{ij} c^0_{ak}$$

so that the trilinear form $c^a_{ij} c^0_{ak}$ is also preserved in a certain sense. Finally, we have the further diagonal quotients of $M_0(A)$ and $M_0(\mathcal{R}, A)$.

COROLLARY 6.6. *For F a cochain on G and basis G of $A = k_F G$, the diagonal quotient $M_{D0} = (kG, \phi)$ as a dual quasi-Hopf algebra.*

Proof. In this basis $c_{ij}^k = 1$ iff $ij = k$ in G and zero otherwise. Hence the relations of M_D are $t_i t_j = t_k$ for $k = ij$ and empty for $k \neq ij$. For M_{D0} we further identify $t_e = 1$ as in the group algebra. Finally, $|t_i| = i$ so we obtain kG, ϕ as a dual quasi-Hopf algebra. When G is commutative we obtain a commutative algebra and $M_{D0} = M_{D0}(\mathscr{R}, k_F G)$. ∎

We now compute these constructions for the complex numbers and for the quaternions, as real 2- and 4-dimensional algebras. More generally, we work over a general ground field of characteristic not 2.

EXAMPLE 6.7. When $A = k[i]$ (where $i^2 = -1$), the comeasuring bialgebra $M_1(k[i])$ is generated by 1 and a matrix of generators $\binom{a\,b}{c\,d}$, with the relations

$$a^2 - c^2 = a = d^2 - b^2, \qquad ac + ca = c, \qquad -c = bd + db,$$
$$ab - cd = b = ba - dc, \qquad ad + cb = d = bc + da.$$

This has a natural bialgebra quotient of the form $\binom{c\ \ s}{-s\ c}$ with

$$\Delta c = c \otimes c - s \otimes s, \qquad \Delta s = s \otimes c + c \otimes s,$$
$$c^2 - s^2 = c, \qquad sc + cs = s.$$

The quotient $M_0 = M_0(\mathscr{R}) = M_{D0}$ is $k\mathbb{Z}_2$ (as generated by d), and its coaction is $\beta(1) = 1 \otimes 1, \beta(i) = i \otimes d$.

Proof. We write out the eight relations for M_0 using the structure constants of $k[i]$. The quotient M_0 is already diagonal and commutative. Hence by the preceding corollary, it gives $kG = k\mathbb{Z}_2$. Its coaction on $k[i]$ is the canonical nontrivial one corresponding to the G-grading. Note that evaluating with the nontrivial character of $\mathbb{Z}_2$ gives the canonical automorphism $i \to -i$. ∎

The intermediate quotient here is the "trigonometric bialgebra": the coproduct has the same form as the addition rules for the sine and cosine function. Whereas it is usually considered as a coalgebra [7], we obtain here a natural algebra structure forming a bialgebra. It too coacts on $k[i]$ by our constructions as the push out of the universal coaction of M_1.

We also note that when $A = k\mathbb{Z}_2$, the comeasuring bialgebra $M_1(k\mathbb{Z}_2)$ has the same form as in the preceding example but with all minus signs replaced by $+$, see [12]. The quotient of the form $\binom{a\,b}{b\,a}$ can then be diagonalized as $g^{\pm} = a \pm b$ and becomes the bialgebra

$$g^{\pm} g^{\pm} = g^{\pm}, \qquad \Delta g^{\pm} = g^{\pm} \otimes g^{\pm}, \qquad \epsilon g^{\pm} = 1$$

of two mutually noncommuting projectors $g^{\pm}$. This is an infinite-dimensional algebra with every element of the form either $g^+g^-g^+ \cdots$ or $g^-g^+g^- \cdots$ (alternating). One may make a similar diagonalization $g^{\pm} = c \pm is$ for the trigonometric bialgebra in the case when $i = \sqrt{-1} \in k$.

PROPOSITION 6.8. *When $A = \mathbb{H}$ the quaternion algebra over k, the comeasuring bialgebra $M_0(\mathbb{H})$ has generators 1 and three vectors of generators $\vec{t}_j = (t^i_j)$, $i = 1,2,3$ and relations*

$$\vec{t}_1 \times \vec{t}_2 = \vec{t}_3, \qquad \vec{t}_1 \times \vec{t}_1 = 0, \qquad + cyclic, \qquad \vec{t}_i \cdot \vec{t}_j = \delta_{ij}.$$

Here $\times$ is the vector cross product and $\cdot$ is the vector dot product. For $\mathscr{R}$ trivial, the quotient $M_0(\mathscr{R})$ is defined by the additional relations that the generators commute.

Proof. We choose the standard basis (where $e_0 = 1$ and e_i, $i = 1,2,3$ have the relations $e_1 e_2 = e_3$ and $e_1^2 = -1$ and their cyclic permutations). In this case the structure constants are $c^k_{ij} = \epsilon_{ijk}$, the totally antisymmetric tensor with $\epsilon_{123} = 1$, and $c^0_{ij} = -\delta_{ij}$, the standard Euclidean metric. The relations of M_0 then become

$$\delta_{ab} t^a_i t^b_j = \delta_{ij}, \qquad t^a_i t^b_j \epsilon_{abc} = t^c_a \epsilon_{ija}.$$

For $M_0(\mathscr{R})$ it is enough to note that we can choose $\mathscr{R}(|i|, |j|) = 1$ and hence that the t^i_j mutually commute. In this case we have $\det(t) = 1$; i.e., $M_0(\mathscr{R})$ is a quotient of $k[SL_3]$. ∎

The relations in M_0 here are asymmetric, a reflection of their role as coacting from the right on $\mathbb{H}$. If we consider also the left-handed versions of our constructions, we have a joint quotient where

$$t^i_a t^j_b \delta^{ab} = \delta^{ij}$$

is added. In this case we have a Hopf algebra with $St^i_j = t^j_i$ and the corresponding $M_0(\mathscr{R})$ is the group coordinate ring $k[SO_3]$. Since the above $M_0(\mathscr{R})$ is universal among commutative bialgebras coacting on $\mathbb{H}$, it must project onto the group coordinate ring of the classical automorphism group.

The corresponding computation for the octonions yields for M_0 a dual quasi-Hopf algebra with nontrivial ϕ. Its detailed form is somewhat more complex than the quaternion case and will be considered elsewhere; however, on general grounds we know that it projects, for example, on to the group coordinate ring $k[G_2]$ (the classical automorphism group of the octonions).

7. QUASIASSOCIATIVE LINEAR ALGEBRA

In this section we use our categorical approach to octonions to provide the natural "quasiassociative" setting for the basic linear algebra associated to them. We define the natural notion of "representation." We also provide the definition of V^* for any finite-dimensional G-graded vector space, and the associated endomorphism quasialgebra $V \otimes V^*$. These constructions are the specialization to the G-graded quasialgebra setting of standard constructions for braided categories.

Thus, the notion of representations, indeed of all linear algebra and quantum group constructions, makes sense in any braided category (see [13]). One writes all constructions as compositions of morphisms, inserting the associator Φ as necessary. For example, in the case of the category of (kG, ϕ)-comodules, we clearly have:

DEFINITION 7.1. A representation or "action" of a G-graded quasialgebra A is a G-graded vector space V and a degree-preserving map $\triangleright: A \otimes V \to V$ such that

$$(ab) \triangleright v = \phi(|a|, |b|, |v|) a \triangleright (b \triangleright v), \qquad 1 \triangleright v = v$$

on elements of homogeneous degree. Here $|a \triangleright v| = |a|\,|v|$.

This is the obvious polarization of the quasiassociativity of the product of A. Clearly, a quasialgebra acts on itself by the product map (the regular representation).

Next, we recall that an object V in a braided category is called "rigid" if there is an object V^* and morphisms

$$\mathrm{ev}: V^* \otimes V \to \underline{1}, \qquad \mathrm{coev}: \underline{1} \to V \otimes V^*$$

such that

$$(\mathrm{id} \otimes \mathrm{ev}) \Phi_{V, V^*, V} (\mathrm{coev} \otimes \mathrm{id}) = \mathrm{id},$$

$$(\mathrm{ev} \otimes \mathrm{id}) \Phi^{-1}_{V^*, V, V^*} (\mathrm{id} \otimes \mathrm{coev}) = \mathrm{id}$$

holds. In the case of the comodule category of a dual quasi-Hopf algebra, these maps exist whenever V is finite-dimensional (see [5] for the explicit formulae; cf. [1]). For the dual quasi-Hopf algebra kG, ϕ, i.e., for the category of G-graded vector spaces, these maps are given by

$$\mathrm{ev}\left(f^i \otimes e_j\right) = \delta^i_j, \qquad \mathrm{coev}(1) = \sum_i e_i \otimes f^i \phi^{-1}\left(|i|, |i|^{-1}, |i|\right)$$

in terms of a basis of V with degree $|e_i| \equiv |i| \in G$ and its usual dual; i.e., ev can be taken as the usual evaluation, V^* as the usual dual, but coev is modified by the group 3-cocycle. Here $|f^i| = |i|^{-1}$ so that ev, coev are degree preserving.

PROPOSITION 7.2. *If V is rigid then* $\mathrm{End}(V) = V \otimes V^*$ *becomes a G-graded quasialgebra. The product map is*

$$(v \otimes f)(w \otimes h) = v \otimes h\langle f, w\rangle \frac{\phi(|v|, |f|, |w|\,|h|)}{\phi(|f|, |w|, |h|)}.$$

Moreover, finite-dimensional representations V of a quasialgebra A are in $1-1$ *correspondence with quasialgebra maps $A \to$* $\mathrm{End}(V)$.

Proof. This is proven by commuting diagrams exactly as one would prove these statements in linear algebra, only inserting the associator Φ whenever needed to change bracketing. Thus (in any monoidal category) one finds

$$(\mathrm{id} \otimes (\mathrm{ev} \otimes \mathrm{id})) \circ \left(\mathrm{id} \otimes \Phi^{-1}_{V^*, V, V^*}\right) \circ \Phi_{V, V^*, V \otimes V^*} \colon (V \otimes V^*) \otimes (V \otimes V^*) \to V \otimes V^*$$

as the natural product on $\mathrm{End}(V) = V \otimes V^*$. Its action on V is the map

$$(\mathrm{id} \otimes \mathrm{ev}) \circ \Phi_{V, V^*, V} \colon (V \otimes V^*) \otimes V \to V.$$

This pulls back under a G-graded algebra map $\rho \colon A \to \mathrm{End}(V)$ to an action of A. Conversely, if A acts on V then

$$\rho = (\triangleright \otimes \mathrm{id}) \circ \Phi^{-1}_{A, V, V^*} \circ (\mathrm{id} \otimes \mathrm{coev}).$$

We then specialize to the case of G-graded quasialgebras using the form of Φ in terms of the 3-cocycle ϕ. ∎

In lieu of all the commutative diagrams here, we will prove this more explicitly in a concrete form for our particular setting. First, we identify $V \otimes V^*$ with matrices in the usual way relative to our basis, i.e.,

$$\alpha = \sum \alpha^i_j E^j_i, \qquad E^j_i = e_i \otimes f^j$$

as a definition of the components of $\alpha \in V \otimes V^*$. Then the preceding proposition translates into the following proposition. We write $n = \dim(V)$ and $|i| \in G$ as the further data provided by V, which we also use.

PROPOSITION 7.3. *Let $|i| \in G$ for $i = 1, \ldots, n$ be a choice of grading function. Then the usual $n \times n$ matrices M_n with the new product*

$$(\alpha \cdot \beta)^i_j = \sum_k \alpha^i_k \beta^k_j \frac{\phi\left(|i|, |k|^{-1}, |k|\,|j|^{-1}\right)}{\phi\left(|k|^{-1}, |k|, |j|^{-1}\right)}, \qquad \forall \alpha, \beta \in M_n$$

form a G-graded quasialgebra $M_{n,\phi}$, where $|E^j_i| = |i|\,|j|^{-1} \in G$ is the degree of the usual basis element of M_n. An action of a G-graded quasialgebra in the n-dimensional vector space with grading $|i|$ is equivalent to an algebra map ρ: $A \to M_{n,\phi}$.

Proof. The product suggested by the preceding proposition is

$$E^j_i \cdot E^l_k = \delta^j_k E^l_i \frac{\phi\left(|i|, |j|^{-1}, |j|\,|l|^{-1}\right)}{\phi\left(|j|^{-1}, |j|, |l|^{-1}\right)},$$

which yields the formula shown for $\alpha = \sum \alpha^i_j E^j_i$, etc. This product is quasiassociative since

$$\left(E^j_i \cdot E^l_k\right) \cdot E^n_m = \delta^j_k \delta^l_m E^n_i \frac{\phi\left(|i|, |m|^{-1}, |m|\,|n|^{-1}\right) \phi\left(|i|, |k|^{-1}, |k|\,|l|^{-1}\right)}{\phi\left(|m|^{-1}, |m|, |n|^{-1}\right) \phi\left(|k|^{-1}, |k|, |l|^{-1}\right)}$$

$$E^j_i \cdot \left(E^l_k \cdot E^n_m\right) \phi\left(|E^j_i|, |E^l_k|, |E^n_m|\right)$$
$$= \delta^l_m \delta^j_k \frac{\phi\left(|i||j|^{-1}, |k||l|^{-1}, |m||n|^{-1}\right) \phi\left(|k|, |m|^{-1}, |m||n|^{-1}\right) \phi\left(|i|, |k|^{-1}, |k||n|^{-1}\right)}{\phi\left(|m|^{-1}, |m|, |n|^{-1}\right) \phi\left(|k|^{-1}, |k|, |n|^{-1}\right)}$$

which are equal since

$$\phi\left(|i|\,|k|^{-1}, |k|\,|m|^{-1}, |m|\,|n|^{-1}\right)$$
$$= \frac{\phi\left(|k|^{-1}, |k|, |n|^{-1}\right) \phi\left(|i|, |k|^{-1}, |k|\,|m|^{-1}\right) \phi\left(|i|, |m|^{-1}, |m|\,|n|^{-1}\right)}{\phi\left(|k|^{-1}, |k|, |m|^{-1}\right) \phi\left(|k|, |m|^{-1}, |m|\,|n|^{-1}\right) \phi\left(|i|, |k|^{-1}, |k|\,|n|^{-1}\right)}$$

by repeated use of the cocycle property of ϕ.

Note also that the grading function $|i|$ is equivalent to specifying a G-graded vector space $V = \{e_i\}$ with grading $|e_i| = |i|$. An action of a G-graded quasialgebra G is equivalent to structure constants $v^j_{\alpha i}$ such that

$$c^{\gamma}_{\alpha\beta} v^j_{\gamma i} = v^k_{\beta i} v^j_{\alpha k} \phi\left(|\alpha|, |\beta|, |i|\right), \qquad v^j_{0i} = \delta^j_i$$

where $\{x_\alpha\}$ (say) is a basis of A with $x_0 = 1$ and $x_\alpha \triangleright e_i = v^j_{\alpha i} e_j$. The corresponding map $A \to M_{n,\phi}$ is

$$\rho(x_\alpha)^i_j = \frac{v^i_{\alpha j}}{\phi\left(|j|, |j|^{-1}, |j|\right) \phi\left(|i|\,|j|^{-1}, |j|, |j|^{-1}\right)}.$$

That ρ is an algebra map (from the definition stated) is

$$c^{\gamma}_{\alpha\beta} v^{i}_{\gamma j} \frac{1}{\phi\big(|j|,|j|^{-1},|j|\big)\phi\big(|i|\,|j|^{-1},|j|,|j|^{-1}\big)}$$

$$= v^{i}_{\alpha a} v^{a}_{\beta j} \frac{\phi\big(|i|,|a|^{-1},|a|\,|j|^{-1}\big)}{\begin{array}{r}\phi\big(|a|^{-1},|a|,|j|^{-1}\big)\phi\big(|a|,|a|^{-1},|a|\big)\phi\big(|i|\,|a|^{-1},|a|,|a|^{-1}\big)\\ \times\phi\big(|j|,|j|^{-1},|j|\big)\phi\big(|a|\,|j|^{-1},|j|,|j|^{-1}\big)\end{array}}.$$

Since the structure maps are degree preserving, we know that $|a| = |\beta|\,|j|, |i| = |\alpha|\,|a|$ for nonzero terms on the right-hand side. That ρ is an algebra map is then equivalent to $v^{j}_{\alpha i}$, an action in view of the identity

$$\begin{aligned}&\phi\big(|\alpha|\,|\beta|,|j|,|j|^{-1}\big)\phi\big(|\alpha|\,|\beta|\,|j|,|j|^{-1}|\beta|^{-1},|\beta|\big)\\ &\quad= \phi\big(|\beta|,|j|,|j|^{-1}\big)\phi\big(|\beta|\,|j|,|\beta|^{-1}|j|^{-1},|\beta|\,|j|\big)\\ &\qquad\times \phi\big(|\alpha|,|\beta|\,|j|,|\beta|^{-1}|j|^{-1}\big)\phi\big(|\beta|^{-1}|j|^{-1},|\beta|\,|j|,|j|^{-1}\big)\\ &\qquad\times \phi\big(|\alpha|,|\beta|,|j|\big)\end{aligned}$$

which holds by repeated use of the 3-cocycle property. ∎

For example, the left regular representation of a quasialgebra on itself provides a representation $\rho\colon A \to M_{n,\phi}$, where $n = \dim(A)$. For the octonions, for example, we have a representation in 8×8 quasimatrices.

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