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# Growth in Gaussian elimination for weighing matrices, $W(n, n - 1)$

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## Abstract

We consider the values for large minors of a skew-Hadamard matrix or conference matrix  $W$  of order  $n$  and find that maximum  $n \times n$  minor equals to  $(n - 1)^{n/2}$ , maximum  $(n - 1) \times (n - 1)$  minor equals to  $(n - 1)^{(n/2)-1}$ , maximum  $(n - 2) \times (n - 2)$  minor equals to  $2(n - 1)^{(n/2)-2}$ , and maximum  $(n - 3) \times (n - 3)$  minor equals to  $4(n - 1)^{(n/2)-3}$ . This leads us to conjecture that the growth factor for Gaussian elimination (GE) of completely pivoted (CP) skew-Hadamard or conference matrices and indeed any CP weighing matrix of order  $n$  and weight  $n - 1$  is  $n - 1$  and that the first and last few pivots are  $(1, 2, 2, 3$  or  $4, \dots, n - 1$  or  $(n - 1)/2, (n - 1)/2, n - 1)$  for  $n > 14$ . We show the unique  $W(6, 5)$  has a single pivot pattern and the unique  $W(8, 7)$  has at least two pivot structures. We give two pivot patterns for the unique  $W(10, 9)$ . © 2000 Elsevier Science Inc. All rights reserved.

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## 1. Introduction

Let  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ . We reduce  $A$  to upper triangular form by using Gaussian elimination (GE) operations. Let  $A^{(k)} = [a_{ij}^{(k)}]$  denote the matrix obtained after the

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first  $k$  pivoting operations, so  $A^{(n-1)}$  is the final upper triangular matrix. A diagonal entry of that final matrix will be called a pivot. Matrices with the property that no exchanges are actually needed during GE with complete pivoting are called completely pivoted (CP) or feasible. Let  $g(n, A) = \max_{i,j,k} |a_{ij}^{(k)}|/|a_{11}^{(0)}|$  denote the growth associated with GE on a CP  $A$  and  $g(n) = \sup\{g(n, A) | A \in \mathbb{R}^{n \times n}\}$ . The problem of determining  $g(n)$  for various values of  $n$  is called the growth problem.

The determination of  $g(n)$  remains a mystery. Wilkinson [7] proved that

$$g(n) \leq [n 2^3 3^{1/2} \dots n^{1/n-1}]^{1/2} = f(n).$$

In Table 1, there are values of  $f(n)$  for representative values of  $n$ .

The above bound is certainly not sharp and the true upper bound is much smaller. Wilkinson [8,9] noted that there were no known examples of matrices for which  $g(n) > n$ . Cryer [2] conjectured that “ $g(n, A) \leq n$ , with equality iff  $A$  is a Hadamard matrix”. This was proved to be untrue in [6].

A Hadamard matrix  $H$  of order  $n \times n$  is an orthogonal matrix with elements  $\pm 1$  and  $HH^T = nI$ .

The problem is quite different if partial pivoting is allowed and Wilkinson [9, p. 212] gives an example of a matrix of order  $n$  and elements  $0, \pm 1$  and growth factor  $2^{n-1}$ .

It is easy to see that  $g(1) = 1$  and  $g(2) = 2$  for all  $n > 2$ . By using algebraic methods, it was proved [1,2] that  $g(3) = 2.25, g(4) = 4$  and  $g(5) \leq 4\frac{17}{18}$ .

One of the curious frustrations of the growth problem is that it is quite difficult to construct any examples of  $n \times n$  matrices,  $A$ , other than Hadamard for which  $g(n, A)$  is even close to  $n$ . Wilkinson [9] has remarked that in real-world problems,  $g(n, A)$  has never been observed to be very large. Cryer [2] did numerical experiments in which he computed  $g(n, A)$ , doing complete pivoting on  $n \times n$  matrices,  $A$ , with entries chosen randomly from the interval  $[-1, 1]$  and for sizes up to  $n = 8$ . He had to generate over 50,000  $3 \times 3$  examples before finding one with  $g(3, A) > 2$ . Also the largest  $g(n, A)$  he obtained by testing 10,000 random matrices for sizes up to  $n = 8$  was 2.8348.

Thus, in order to obtain matrices with large growth, sophisticated numerical optimization techniques must be applied [6]. By using such methods, matrices with growth larger than  $n = 13, 14, 15, 16, 18, 20, 25$  were specified, and thus the conjecture that  $g(n, A) \leq n$  is false. Table 2 summarizes the growth size attained for various values of  $n$  [2,4].

The matrices that give rise to the growth factors of Table 2 are often extremely sensitive to small perturbations in their entries in that tiny perturbations to a complete

Table 1

$n$	10	20	50	100	200	1000
$f(n)$	19	67	530	3300	26,000	7,900,000

Table 2

$n$	13	14	15	16	18	20	25
Growth size	13.0205	14.5949	16.1078	18.0596	20.45	24.25	32.99

elimination matrix rarely results in another such matrix. This makes it rather difficult to specify matrices which give rise to large growth.

If a Hadamard matrix,  $H$ , of order  $n$  can be written as  $H = I + S$  where  $S^T = -S$  then  $H$  is called *skew-Hadamard*.

A  $(0, 1, -1)$  matrix  $W = W(n, k)$  of order  $n$  satisfying  $WW^T = kI_n$  is called a *weighing matrix of order  $n$  and weight  $k$*  or simply a *weighing matrix*. A  $W(n, n)$ ,  $n \equiv 0 \pmod{4}$ , is a Hadamard matrix of order  $n$ . A  $W = W(n, k)$  for which  $W^T = -W$  is called a *skew-weighing matrix*. A  $W = W(n, n-1)$  satisfying  $W^T = W$ ,  $n \equiv 2 \pmod{4}$ , is called a *symmetric conference matrix*. Conference matrices cannot exist unless  $n-1$  is the sum of two squares: thus they cannot exist for orders 22, 34, 58, 70, 78, 94. For more details and construction of weighing matrices the reader can refer the book by Geramita and Seberry [5].

We have now studied, by computer, the pivots and growth factors for  $W(n, n-1)$ ,  $n = 6, 10, 14, 18, 26, 30, 38, 42, 50, 54, 62, 74$  constructed by two circulant matrices and for  $n = 8, 12, 16, 20, 28, 36, 44, 52, 60, 68, 76, 84, 92, 100$  constructed by four circulant matrices and obtained the results in Tables 3 and 4.

Wilkinson's initial conjecture seems to be connected with Hadamard matrices. Interesting results in the size of pivots appears when GE is applied to CP skew-Hadamard and weighing matrices of order  $n$  and weight  $n-1$ . In these matrices, the growth is also large, and experimentally, we have been led to believe it equals  $n-1$  and special structure appears for the first few and last few pivots. These results give rise to new conjectures that can be posed for this category of matrices.

**Conjecture** (*The growth conjecture for weighing matrices  $W(n, n-1)$* ). Let  $W = W(n, n-1)$  be a CP weighing matrix. Reduce  $W$  by GE. Then

- (i)  $g(n, W) = n-1$ .
- (ii) The three last pivots are equal to  $n-1$  or  $(n-1)/2, (n-1)/2, n-1$ .
- (iii) Every pivot before the last has magnitude at most  $n-1$ .
- (iv) The first four pivots are equal to 1, 2, 2, 3 or 4, for  $n > 14$ .

**Notation.** Write  $A$  for a matrix of order  $n$  whose initial pivots are derived from matrices with CP structure. Write  $A(j)$  for the absolute value of the determinant of the  $j \times j$  principal submatrix in the upper left-hand corner of the matrix  $A$  and  $A[j]$  for the absolute value of the determinant of the  $(n-j) \times (n-j)$  principal submatrix in the bottom right-hand corner of the matrix  $A$ . Throughout this paper when we have used  $i$  pivots we then find all possible values of the  $A(n-i)$  minors. Hence, if any minor is CP it must have one of these values. The magnitude of the pivots appearing after the application of GE operations on a CP matrix  $W$  is given by

$$p_j = W(j)/W(j-1), \quad j = 1, 2, \dots, n, \quad W(0) = 1. \quad (1)$$

We use  $W(j)$ ,  $W[j]$  similarly. We also use the following results

**Lemma 1** [3]. *Let  $A$  be an orthogonal matrix of order  $n$  satisfying  $AA^T = kI_n$ . Then*

$$A(j) = k^{j-(n/2)} A[n - j].$$

**Corollary 1.** *If  $A$  is an  $n \times n$  weighing matrix of weight  $k = n - 1$ , then the  $k$ th pivot from the end is*

$$p_{n+1-j} = \frac{kA[j - 1]}{A[j]}.$$

## 2. The first four pivots

**Lemma 2.** *Let  $W$  be a CP weighing matrix,  $W(n, n - 1)$ , of order  $n \geq 6$  then if GE is performed on  $W$ , the first three pivots are 1, 2, and 2.*

**Proof.** We note that in the upper left-hand corner of a CP weighing matrix,  $W(n, n - 1)$ , of order  $n \geq 6$  the following submatrices can always occur:

$$\begin{bmatrix} 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix},$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 0 \\ 1 & 1 & - \end{bmatrix}.$$

Thus, the first three pivots, using Eq. (1), are

$$p_1 = 1, \quad p_2 = 2 \quad \text{and} \quad p_3 = 2. \quad \square$$

**Proposition 1.** *Let  $W$  be a CP weighing matrix,  $W(n, n - 1)$ , of order  $n \geq 8$ . Then if GE is performed on  $W$ , the first four pivots are 1, 2, 2, 3 or 4.*

**Proof.** The first three pivots are given in Lemma 2. Now in the upper left-hand corner of a CP weighing matrix,  $W(n, n - 1)$ , of order  $n \geq 8$  the following submatrices can always occur:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & - & - & 1 \\ 1 & 1 & - & - \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 & 0 & - \\ 1 & - & - & - \\ 1 & - & 1 & 1 \\ 1 & 1 & - & 1 \end{bmatrix}.$$

The fourth pivots for  $n \geq 8$ , using Eq. (1), are

$$p_4 = 4 \text{ or } 3. \quad \square$$

### 3. Exact calculations

We assume that row and column permutations have been carried out. So we have a CP skew-Hadamard or CP conference matrix  $A$  in the initial steps from which we can calculate the maximum minors  $A(n)$ ,  $A(n - 1)$ ,  $A(n - 2)$  and  $A(n - 3)$ . We explore the use of a variation of a clever proof used by combinatorialists to find the determinant of a matrix satisfying  $AA^T = (k - \lambda)I + \lambda J$ , where  $I$  is the  $v \times v$  identity matrix,  $J$  the  $v \times v$  matrix of ones and  $k, \lambda$  are integers to simplify our proofs.

**Proposition 2.** *Let  $A$  be a skew-Hadamard or conference matrix of order  $n$ . Then the  $(n - 1) \times (n - 1)$  minors are:  $A(n - 1) = 0, (n - 1)^{(n/2)-1}$ .*

**Proof.** Since  $AA^T = (n - 1)I$  and  $\det(A) = (n - 1)^{n/2}$ , the  $(n - 1) \times (n - 1)$  matrix  $B$  formed by deleting the first row and column of  $A$  satisfies  $\det BB^T = (n - 1)^{n-2}$  or zero accordingly as the  $(1, 1)$  element of  $A$  is non-zero or zero. Hence  $\det B = (n - 1)^{(n/2)-1}$  or zero and we have the result.  $\square$

**Proposition 3.** *Let  $A$  be a skew-Hadamard or conference matrix of order  $n$ . Then the  $(n - 2) \times (n - 2)$  minors are  $A(n - 2) = 0, 2(n - 1)^{(n/2)-2}$ .*

**Proof.** There are six cases: they have upper left-hand corner

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

These have determinants  $2, \pm 1, \pm 1, 0$  respectively. We use the lower right-hand principal minor,  $C$ , of order  $n - 2$  to calculate  $CC^T$  for each case. We find the second case, where the determinant is  $(n - 1)^{(n/2)-2}$ , is not CP as there must be  $-2s$  after the first step of GE. Hence the maximum determinant of  $C$  is  $2(n - 1)^{(n/2)-2}$ .  $\square$

**Lemma 3.** *The possible values for the determinants of  $3 \times 3$  matrices with entries  $0, \pm 1$  where there is at most one zero in each row and column are  $0, 1, \pm 2, \pm 3$  and  $\pm 4$ .*

**Proof.** For matrices of the required type, up to equivalence, we have these four cases

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & \pm 1 & \pm 1 \\ 1 & \pm 1 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & \pm 1 & \pm 1 \\ 1 & \pm 1 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \pm 1 \\ 1 & \pm 1 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \pm 1 \\ 1 & \pm 1 & 0 \end{bmatrix}.$$

We used a computer to search all the possibilities and found that for no zeros the determinant can be 0 or 4, for one zero the determinant can be 0, 2 or 4, for two zeros the determinant can be 1 or 3, and for three zeros the determinant can be 0 or 2.  $\square$

We now proceed to study  $A(n - 3)$ .

**Proposition 4.** *Let  $A$  be a skew-Hadamard or conference matrix of order  $n$ . Then the  $(n - 3) \times (n - 3)$  minors are  $A(n - 3) = 0$ ,  $2(n - 1)^{(n/2)-3}$ , or  $4(n - 1)^{(n/2)-3}$  for  $n \equiv 0 \pmod{4}$  and  $2(n - 1)^{(n/2)-3}$ , or  $4(n - 1)^{(n/2)-3}$  for  $n \equiv 2 \pmod{4}$ .*

**Proof.** We first note that the submatrices

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 0 \\ 1 & 1 & - \end{bmatrix}$$

always occur in any skew-Hadamard or conference matrix of order  $> 6$ . We first consider the upper left-hand corner

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 1 \\ 1 & 1 & - \end{bmatrix},$$

which corresponds to a CP matrix with pivots 1, 2, 2.

We assume the CP matrix is in the form below where for ease of comprehension we have written the elements  $a, b, c, d, p, q, s$  in the top  $6 \times 6$  matrix although they will not appear there in the CP matrix.

$$\left[ \begin{array}{ccc|cccccc} & & & & \overbrace{1 \dots 1}^u & \overbrace{1 \dots 1}^v & \overbrace{1 \dots 1}^w & \overbrace{1 \dots 1}^x \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & 1 & 0 & q & 1 & \dots & 1 & - & \dots & - \\ 1 & 1 & - & p & s & 0 & 1 & \dots & 1 & - & \dots & - \\ \hline 0 & a & c & & & & & & & & & \\ 1 & 0 & b & & & & & & & & & \\ 1 & d & 0 & & & & & & & & & \\ 1 & 1 & 1 & & & & & & & & & \\ \vdots & \vdots & \vdots & & & & & & & & & \\ 1 & 1 & 1 & & & & & & & & & \\ 1 & 1 & - & & & & & & & & & \\ \vdots & \vdots & \vdots & & & & & & & & & \\ 1 & 1 & - & & & & & & & & & \\ 1 & - & 1 & & & & & & & & & \\ \vdots & \vdots & \vdots & & & & & & & & & \\ 1 & - & 1 & & & & & & & & & \\ 1 & - & - & & & & & & & & & \\ \vdots & \vdots & \vdots & & & & & & & & & \\ 1 & - & - & & & & & & & & & \end{array} \right].$$

We use the orthogonality of the matrix  $A$ , and the order  $n$ , to obtain constraints for all the variables  $a, b, c, d, p, q, s, u, v, w, x$  in terms of each other and  $n$ , the original order. We then calculate  $CC^T$  and then these constraints are solved by either Matlab or a simple, but tedious, calculation to obtain the values for the minors as 0 and  $4(n - 1)^{(n/2)-3}$  for  $n \equiv 0(\text{mod } 4)$  and  $4(n - 1)^{(n/2)-3}$  for  $n \equiv 2(\text{mod } 4)$ .

We now consider the second case with upper left-hand corner,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & - & 0 \\ 1 & 1 & - \end{bmatrix},$$

which also corresponds to a CP matrix with pivots 1, 2, 2.

We proceed, as before, to obtain the three values 0,  $2(n - 1)^{(n/2)-3}$ ,  $4(n - 1)^{(n/2)-3}$  for  $n \equiv 0(\text{mod } 4)$  and the two non-zero values  $2(n - 1)^{(n/2)-3}$ ,  $4(n - 1)^{(n/2)-3}$  as the only determinants for  $n \equiv 2(\text{mod } 4)$ .  $\square$

**Theorem 1.** *When GE is applied on a CP skew-Hadamard or conference matrix  $W$  of order  $n$  the last three pivots in backward order are  $n - 1$ ,  $(n - 1)/2$ , and  $(n - 1)/2$  or  $n - 1$ .*

**Proof.** The last three pivots are given by

$$p_n = \frac{W(n)}{W(n - 1)}, \quad p_{n-1} = \frac{W(n - 1)}{W(n - 2)}, \quad p_{n-2} = \frac{W(n - 2)}{W(n - 3)}.$$

Since

$$\begin{aligned} W(n) &= (n - 1)^{n/2}, \\ W(n - 1) &= (n - 1)^{(n/2)-1}, \\ W(n - 2) &= 2(n - 1)^{(n/2)-2}, \\ W(n - 3) &= 2(n - 1)^{(n/2)-3} \quad \text{or} \quad 4(n - 1)^{(n/2)-3}, \end{aligned}$$

the values of the three last pivots are  $n - 1$ ,  $(n - 1)/2$ , and  $(n - 1)/2$  or  $n - 1$ , respectively.  $\square$

#### 4. Numerical calculations

**Lemma 4.** *The maximum determinant of all  $n \times n$  matrices with elements  $\pm 1$  or 0, where there is at most one zero in each row and column is:*

Order	Maximum determinant	Possible determinantal values
$2 \times 2$	2	0, 1, 2
$3 \times 3$	4	0, 1, 2, 3, 4
$4 \times 4$	16	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16
$5 \times 5$	48	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 32, 36, 40, 48

**Remark.** In fact, we found that considering all  $5 \times 5$  matrices with elements  $\pm 1$  and no more than one zero per row and column, if the matrix had no zeros the determinant could be 0, 16, 32 or 48; had exactly one zero the determinant could be 0, 8, 16, 24, 32 or 40; had exactly two zeros the determinant could be 0, 4, 8, 12, 16, 20, 24, 28, 32 or 36; had exactly three zeros the determinant could be 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30 or 36; had exactly four zeros the determinant could be 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25 or 27; had exactly five zeros the determinant could be 0, 2, 4, 6, 8, 10, 12, 18, 20, 22.

Considering all  $4 \times 4$  matrices with elements  $\pm 1$  and no more than one zero per row and column, if the matrix had no zeros the determinant could be 0, 8, 16; had exactly one zero the determinant could be 0, 4, 8, 12; had exactly two zeros the determinant could be 0, 2, 4, 6, 8, 10; had exactly three zeros the determinant could be 1, 3, 5, 7, 9; had exactly four zeros the determinant could be 1, 3, 5, 9.

Considering all  $3 \times 3$  matrices with elements  $\pm 1$  and no more than one zero per row and column, if the matrix had no zeros the determinant could be 0, 4; had exactly one zero the determinant could be 0, 2, 4; had exactly two zeros the determinant could be 1, 3; had exactly three zeros the determinant could be 0, 2.

Considering all  $2 \times 2$  matrices with elements  $\pm 1$  and no more than one zero per row and column, if the matrix had no zeros the determinant could be 0,  $\pm 2$ ; had exactly one zero the determinant could be  $\pm 1$ ; had exactly two zeros the determinant could be  $\pm 1$ .

**Lemma 5.**  $W(4) = 10$  for a  $W(6, 5)$ .

**Proof.** Every  $4 \times 4$  subdeterminant of  $W(6, 5)$  must contain two zeros. Hence, its determinant can only be 0, 2, 4, 6, 8, or 10. We show that the first four non-zero values are not possible in a  $W(6, 5)$ .

Without any loss of generality we assume that the  $4 \times 4$  subdeterminant has first row and column comprising only +1s. Because we are dealing with a weighing matrix the second row and column must contain two 1s and two -1s.

We denote the vectors  $(1, -, -)$ ,  $(1, -, 1)$  and  $(-, -, 1)$  as  $a_1$ ,  $a_2$  and  $a_3$ , respectively. We denote the  $2 \times 2$  submatrices

$$\begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix},$$

by  $b_1$  and  $b_2$ , respectively, where  $x$  and  $y$  are both 1 or  $-1$ .

Calculation shows that the  $4 \times 4$  matrix with second row and column comprising  $a_1$  and  $a_1^T$  can be completed by both  $b_1$  and  $b_2$ , but are equivalent under permutation of rows and columns, to the matrix  $A_1$  below.

Furthermore calculations show that the  $4 \times 4$  matrix with second row and column  $a_i$  and  $a_j^T$ , and completion matrix of shape  $b_k$ , give only two inequivalent matrices,  $A_2$  and  $A_3$ , under row and column permutations.



$(a_2, a_2^T, b_1)$ ,  $(a_3, a_3^T, b_1)$  and  $(a_3, a_2^T, b_2)$  are equivalent to  $A_1$ .

$(a_2, a_2^T, b_2)$ ,  $(a_3, a_3^T, b_2)$  and  $(a_3, a_2^T, b_1)$  are equivalent to  $A_2$ .

Now, writing  $x$  for  $\pm 1$ , and using the orthogonality conditions for the  $W(6, 5)$ , we have

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & - & - \\ 1 & - & x & 0 \\ 1 & - & 0 & \bar{x} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & 0 & - \\ 1 & - & - & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & - & 1 & - \\ 1 & 1 & - & 0 \\ 1 & - & 0 & 1 \end{bmatrix}.$$

Now  $A_1$  has determinant 0.  $A_2$  and  $A_3$  have determinant 10. This gives the result. □

**Lemma 6.** *The unique pivots of the  $W(6, 5)$  are  $\{1, 2, 2, \frac{5}{2}, \frac{5}{2}, 5\}$ .*

**Proof.** We use the determinants of  $W(1) = 1$ ,  $W(2) = 2$ ,  $W(3) = 4$ ,  $W(4) = 10$ ,  $W[1] = 1$ ,  $W[2] = 2$ .

Hence the pivot pattern is given by

$$p_1 = 1, \quad p_2 = \frac{W(2)}{W(1)} = 2, \quad p_3 = \frac{W(3)}{W(2)} = 2,$$

$$p_4 = \frac{W(4)}{W(3)} = \frac{5}{2}, \quad p_5 = 5 \frac{W[1]}{W[2]} = \frac{5}{2}, \quad p_6 = 5 \frac{W[0]}{W[1]} = 5. \quad \square$$

**Lemma 7.** *The pivots of the  $W(8, 7)$  are  $\{1, 2, 2, 4, \frac{7}{4}, \frac{7}{2}, \frac{7}{2}, 7\}$  or  $\{1, 2, 2, 3, \frac{7}{3}, \frac{7}{2}, \frac{7}{2}, 7\}$ .*

**Proof.** From Lemma 2 and Proposition 1, we have that

$$p_1 = 1, \quad p_2 = 2, \quad p_3 = 2, \quad p_4 = 4 \text{ or } 3.$$

From Theorem 1, we also have that

$$p_8 = 7, \quad p_7 = \frac{7}{2} \quad \text{and} \quad p_6 = \frac{7}{2}.$$

Since  $\prod_{i=1}^8 p_i = \det W(8, 7) = 7^4$  the only values that  $p_5$  can take are  $\frac{7}{4}$  or  $\frac{7}{3}$ . □

**Remark.** The following matrices have pivot patterns 1, 2, 2, 4,  $\frac{7}{4}$ ,  $\frac{7}{2}$ ,  $\frac{7}{2}$ , 7 and 1, 2, 2, 3,  $\frac{7}{3}$ ,  $\frac{7}{2}$ ,  $\frac{7}{2}$ , 7, respectively.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & - & - & 1 & 1 & - & 0 & - \\ 1 & - & 1 & - & - & 0 & 1 & - \\ 1 & 1 & - & - & 0 & 1 & - & - \\ 1 & - & 0 & 1 & - & 1 & - & 1 \\ 1 & 1 & - & 0 & - & - & 1 & 1 \\ 0 & 1 & 1 & 1 & - & - & - & - \\ 1 & 0 & 1 & - & 1 & - & - & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 0 & - & - & 1 & 1 & - \\ 1 & - & - & - & 1 & 1 & 0 & 1 \\ 1 & - & 1 & 1 & 0 & 1 & - & - \\ 1 & 1 & - & 1 & - & 0 & - & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & - & 1 & - & - & 0 \\ 1 & - & 1 & 0 & - & - & 1 & 1 \\ 1 & 0 & - & 1 & 1 & - & 1 & - \end{bmatrix}.$$

**Lemma 8.** The pivots of the  $W(10, 9)$  can be  $\{1, 2, 2, 3, 3, 4, \frac{9}{4}, \frac{9}{2}, \frac{9}{2}, 9\}$  or  $\{1, 2, 2, 4, 3, 3, \frac{9}{4}, \frac{9}{2}, \frac{9}{2}, 9\}$ .

**Proof.** The  $W(10, 9)$  is unique up to permutation of rows and columns and multiplication of rows and columns by  $-1$ . We have found two CP  $W(10, 9)$  which have difference pivot patterns showing the sensitivity of the pivots to permutations of rows and columns.

The following matrices have pivot patterns  $\{1, 2, 2, 3, 3, 4, \frac{9}{4}, \frac{9}{2}, \frac{9}{2}, 9\}$ , and  $\{1, 2, 2, 4, 3, 3, \frac{9}{4}, \frac{9}{2}, \frac{9}{2}, 9\}$ , respectively.

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - & 1 & 0 \\ 1 & - & - & 1 & 0 & - & - & 1 & 1 & - \\ 1 & 1 & - & - & - & 1 & 0 & - & 1 & - \\ 1 & - & 1 & 1 & - & 1 & 1 & 0 & - & - \\ 1 & 1 & 1 & 1 & - & - & - & - & 0 & 1 \\ 0 & - & 1 & - & - & 1 & - & 1 & 1 & 1 \\ 1 & 0 & - & - & - & - & 1 & 1 & - & 1 \\ 1 & - & - & 0 & 1 & 1 & - & - & - & 1 \\ 1 & 1 & 1 & - & 1 & 0 & - & 1 & - & - \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & - & - & 1 & 1 & - & 1 & 0 & - & - \\ 1 & - & 1 & - & - & - & 0 & 1 & - & 1 \\ 1 & 1 & - & - & 1 & - & - & 1 & 1 & 0 \\ 1 & 1 & - & 1 & - & 0 & - & - & - & 1 \\ 1 & - & - & - & 0 & 1 & 1 & - & 1 & 1 \\ 1 & 1 & 1 & 0 & - & - & 1 & - & 1 & - \\ 0 & - & 1 & 1 & 1 & - & - & - & 1 & 1 \\ 1 & - & 0 & 1 & - & 1 & - & 1 & 1 & - \\ 1 & 0 & 1 & - & 1 & 1 & - & - & - & - \end{bmatrix} \quad \square$$

We calculated the values of all the large minors of the unique  $W(12, 11)$ . These are given in the next table. We also calculated all the minors for one of the  $W(20, 19)$  and found exactly the same results as those in the table.

Minor	Minimum non-zero determinant	All determinants
$W(n - 1)$	$m = (n - 1)^{n/2-1}$	$0, m$
$W(n - 2)$	$m = (n - 1)^{n/2-2}$	$0, m, 2m$
$W(n - 3)$	$m = (n - 1)^{n/2-3}$	$0, m, 2m, 3m, 4m$
$W(n - 4)$	$m = (n - 1)^{n/2-4}$	$0, m, 2m, 3m, 4m, 6m, 8m, 9m, 10m, 12m, 16m$

Tables 3 and 4 give us the pivot patterns calculated by a computer for the first few  $W(n, n - 1)$  for both  $n \equiv 2(\text{mod } 4)$  and  $n \equiv 0(\text{mod } 4)$ . Although our theory predicts that the third last pivot could be  $n - 1$  or  $(n - 1)/2$ , in both these tables only the value  $(n - 1)/2$  has been observed.

Table 3

$n$	Growth	Pivot pattern
6	5	$(1, 2, 2, \frac{5}{2}, 5, 5)$
10	9	$(1, 2, 2, 3, 3, 4, \frac{9}{4}, \frac{9}{2}, 9)$ or $(1, 2, 2, 4, 3, 3, \frac{9}{4}, \frac{9}{2}, 9)$
14	13	$(1, 2, 2, 3, \frac{10}{3}, \frac{17}{5}, 3.2941, 3.9464, 3.8235, \frac{13}{5}, \frac{13}{2}, \frac{13}{2}, 13)$
14	13	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, \frac{13}{3}, \frac{13}{3}, \frac{13}{4}, \frac{13}{2}, 13)$
18	17	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, \frac{11}{3}, \frac{148}{33}, 3.7432, 4.5415, 3.9054, \frac{17}{75}, \frac{17}{1073}, \frac{17}{5}, \frac{17}{4}, \frac{17}{2}, \frac{17}{2}, 17)$
26	25	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{25}{6}, 5, \frac{13}{3}, 6.7308, 5.2857, 4.3919, 4.9231, 5.9375, 6.0855, 5.5743, \frac{25}{1173}, \frac{25}{3}, \frac{25}{4}, \frac{25}{2}, 25)$
30	29	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, \frac{34}{9}, 3.9412, \dots, \frac{29}{3}, \frac{29}{4}, \frac{29}{2}, 29)$
38	37	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{13}{3}, 4.4615, \dots, \frac{37}{4}, \frac{37}{873}, \frac{37}{3}, \frac{37}{4}, \frac{37}{2}, 37)$
42	41	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{37}{9}, 4.4595, \dots, \frac{41}{873}, \frac{41}{1073}, \frac{41}{3}, \frac{41}{4}, \frac{41}{2}, 41)$
46	45	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, \frac{34}{9}, 4.2353, \dots, \frac{45}{4}, \frac{45}{6}, \frac{45}{4}, \frac{45}{2}, \frac{45}{4}, \frac{45}{2}, 45)$
50	49	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{40}{9}, \frac{40}{9}, \frac{40}{9}, \frac{49}{873}, \frac{49}{3}, \frac{49}{4}, \frac{49}{2}, 49)$
54	53	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, \frac{34}{9}, 4.4706, \dots, \frac{53}{6}, \frac{53}{4}, \frac{53}{2}, \frac{53}{4}, \frac{53}{2}, 53)$
62	61	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, \frac{34}{9}, 4.2353, \frac{37}{9}, \dots, \frac{61}{1073}, \frac{61}{572}, \frac{61}{4}, \frac{61}{2}, 61)$
74	73	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{37}{9}, 4.9730, \dots, \frac{73}{6}, \frac{73}{4}, \frac{73}{4}, \frac{73}{2}, \frac{73}{2}, 73)$

Table 4

$n$	Growth	Pivot pattern
8	7	$(1, 2, 2, 4, \frac{7}{4}, \frac{7}{2}, 7, 7)$ or $(1, 2, 2, 3, \frac{7}{3}, \frac{7}{2}, 7, 7)$
12	11	$(1, 2, 2, 3, \frac{10}{3}, \frac{17}{5}, \frac{11}{7}, \frac{11}{17}, \frac{11}{57}, \frac{11}{4}, \frac{11}{2}, 11)$ or $(1, 2, 2, 4, 3, \frac{10}{3}, \frac{11}{10}, \frac{11}{4}, \frac{11}{2}, 11, 11)$
16	15	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, \frac{31}{9}, 4.0806, 3.913, 4.5455, \frac{15}{10}, \frac{15}{7}, \frac{15}{4}, \frac{15}{2}, 15)$
20	19	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, \frac{34}{9}, 4.1176, 4.2857, 4.1, 4.6341, \frac{19}{25}, \frac{19}{16}, \frac{19}{4}, 5.2778, \frac{19}{10}, \frac{19}{7}, \frac{19}{4}, \frac{19}{2}, 19)$
28	27	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, \frac{34}{9}, 4.2353, \frac{38}{9}, 4, 4.9474, 4.6755, 4.9966, 4.8074, 5.5416,$ $5.5093, 6.0458, 5.4375, 6.5172, 6.6623, 6.0395, 7.9412, \frac{27}{10}, \frac{27}{3}, \frac{27}{4}, \frac{27}{2}, 27, 27)$
36	35	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, \frac{34}{9}, 3.8824, \dots, \frac{35}{3}, \frac{35}{4}, \frac{35}{2}, 35)$
44	43	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{37}{9}, 5.0270, \dots, \frac{43}{6}, \frac{43}{4}, \frac{43}{2}, \frac{43}{4}, \frac{43}{2}, 43)$
52	51	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{37}{9}, 4.7703, \dots, \frac{51}{25}, \frac{51}{4}, \frac{51}{4}, \frac{51}{2}, \frac{51}{4}, \frac{51}{2}, 51)$
60	59	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{38}{9}, 5.1579, \dots, \frac{59}{10}, \frac{59}{3}, \frac{59}{4}, \frac{59}{2}, 59)$
68	67	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{38}{9}, 5.1579, \dots, \frac{67}{6}, \frac{67}{4}, \frac{67}{4}, \frac{67}{2}, \frac{67}{4}, \frac{67}{2}, 67)$
76	75	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{40}{9}, \dots, \frac{75}{10}, \frac{75}{3}, \frac{75}{4}, \frac{75}{2}, 75)$
84	83	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{37}{9}, 5.1351, \dots, \frac{83}{6}, \frac{83}{4}, \frac{83}{4}, \frac{83}{2}, \frac{83}{4}, \frac{83}{2}, 83)$
92	91	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{40}{9}, \dots, \frac{91}{6}, \frac{91}{4}, \frac{91}{4}, \frac{91}{2}, \frac{91}{4}, \frac{91}{2}, 91)$
100	99	$(1, 2, 2, 3, \frac{10}{3}, \frac{18}{5}, 4, 4, \frac{38}{9}, 5.4737, \dots, \frac{99}{6}, \frac{99}{4}, \frac{99}{4}, \frac{99}{2}, \frac{99}{4}, \frac{99}{2}, 99)$

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