Integer Generalized Inverses of Incidence Matrices*

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ABSTRACT

Graphical procedures are used to characterize the integral $\{1\}$- and $\{1, 2\}$-inverses of the incidence matrix A of a digraph, and to obtain a basis for the space of matrices X such that $AXA=0$. These graphical procedures also produce the Smith canonical form of A and a full rank factorization of A using matrices with entries from $\{-1, 0, 1\}$. It is also shown how the results on incidence matrices of oriented graphs can be used to find generalized inverses of matrices of unoriented bipartite graphs.

I. INTRODUCTION

Let A be an $m\times n$ complex matrix, and consider the equations

$$AXA=A, \tag{1}$$

$$XAX=X, \tag{2}$$

$$(AX)^*=AX, \tag{3}$$

$$(XA)^*=XA, \tag{4}$$

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where X is an $n\times m$ complex matrix and * denotes the conjugate transpose. The unique matrix X satisfying all four equations is known as the Moore-Penrose inverse of A and is denoted by $A^{\dagger}$. Next let $A\{i, j,\dots, l\}$ denote the set of matrices X which satisfy equations (i), $(j),\dots,(l)$ from among the equations (1–4). A matrix $X\in A\{i, j,\dots, l\}$ is called an $\{i, j,\dots, l\}$-inverse of A. The integral matrices from this set form the collection integral-$A\{i, j,\dots, l\}$ and are called integral $\{i, j,\dots, l\}$-inverses of A. The reader can see [2] or [4] for various properties of generalized inverses.

In [7] Ijiri gave a method of calculating $A^{\dagger}$ for an incidence matrix A of a digraph. In this case $A^{\dagger}$ is never integral. In this paper we discuss $\{1\}$- and $\{1,2\}$-inverses of A, as these are the classes which contain integral matrices. A characterization of the inverse of an $(m-1)\times(m-1)$ nonsingular submatrix of A given by Resh [8] is extended to obtain a particular integral $\{1, 2\}$-inverse of A. Graphical procedures are used to characterize integral-$A\{1\}$ and integral-$A\{1,2\}$, and to obtain a basis for the space of $n\times m$ matrices X such that $AXA=0$. These graphical procedures also produce the Smith canonical form of A and a full rank factorization of A using matrices with entries from $\{-1, 0, 1\}$.

In the last section we show how the results on incidence matrices of oriented graphs can be used to find generalized inverses for incidence matrices of unoriented bipartite graphs. When the results are simplified to the case of the reduced incidence matrix of a complete bipartite graph, one obtains techniques related to the simplex method for the transportation problem.

II. PRELIMINARIES ON INCIDENCE MATRICES

We first introduce the basic ideas of incidence matrices. For a more complete discussion the reader can see [9]. Let $\mathcal{G}=(V, E)$ be a graph whose vertices and edges have been labeled as $V=\{v_1,\dots, v_m\}$ and $E=\{e_1,\dots, e_n\}$. Suppose further that the edges have been given some arbitrary orientation by means of an incidence function f, which associates with each edge an ordered pair of vertices. The incidence matrix of $\mathcal{G}$ is an $m\times n$ matrix $A=(a_{ij})$ with columns associated with the edges of $\mathcal{G}$ and rows associated with the vertices of $\mathcal{G}$. If $f(e_j)=(v_i, v_k)$ then $a_{ij}=1$, $a_{kj}=-1$, and the other entries of the jth column are 0. We assume $\mathcal{G}$ is connected so that the rank of A is $m-1$. The columns of an $(m-1)\times(m-1)$ nonsingular submatrix of A correspond to the edges of a spanning tree of $\mathcal{G}$, that is, a connected subgraph which contains all vertices of $\mathcal{G}$ and which has no cycles.

A cannot have an integral $\{1,3\}$-inverse, and hence $A^{\dagger}$ is not integral, since the number of nonzero rows is greater than the rank of A; see [6, Theorem 2]. Similarly it will follow from the results below that A will have an integral $\{1,4\}$-inverse if and only if $n=m-1$. This is the case only when the graph $\mathcal{G}$ is a tree. Also, when $n=m-1$, A has full column rank and $A\{1\}=A\{1,2\}=A\{1,4\}=A\{1,2,4\}$. Thus we will concern ourselves only with $\{1\}$- and $\{1,2\}$-inverses.

Choose some spanning tree $\mathcal{T}$ of $\mathcal{G}$ and some fixed vertex v_r called the reference vertex. We assume that the edges of $\mathcal{G}$ have been ordered so that the $m-1$ edges of $\mathcal{T}$ come first and that the vertices have been ordered so that the reference vertex is v_m. In this case A may be partitioned as

$$A=\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

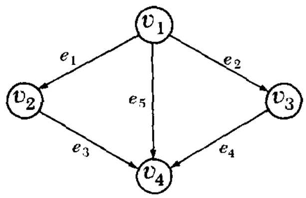
where A_1 is an $(m-1)\times(m-1)$ nonsingular matrix, called the reduced incidence matrix of $\mathcal{T}$. The matrix (A_1, A_2) is called the reduced incidence matrix of $\mathcal{G}$. Each column of A contains exactly one 1 and one -1, so that the sum of the rows of A is 0, and any row is the negative of the sum of the other rows. Thus $A_3=-eA_1$ and $A_4=-eA_2$, where e is the $1\times(m-1)$ row matrix $e=(1,\dots,1)$.

An alternating sequence $p=(u_0, a_1, u_1,\dots, a_k, u_k)$ of distinct vertices u_i and edges a_i such that $f(a_i)=(u_{i-1}, u_i)$ or (u_i, u_{i-1}) for $i=1,2,\dots,k$ is called a path in $\mathcal{G}$. If the vertices are distinct except for $u_0=u_k$, then p is called a cycle. If $f(a_i)=(u_{i-1}, u_i)$ we say that a_i has the same orientation as p; otherwise it has opposite orientation.

Associated with any path or cycle p is a column vector C_p of length n, with the entries of C_p determined in the following way: the ith entry of C_p is 0 if e_i, the ith edge of $\mathcal{G}$, does not belong to p; otherwise the ith entry is 1 or -1 depending on whether e_i has the same or opposite orientation as p. If all edges of p belong to the spanning tree $\mathcal{T}$, we also associate with p a "short" column vector c_p of length $m-1$ determined in the same fashion.

The matrices A_1^{-1} and $A_1^{-1}A_2$ are characterized in terms of short column vectors associated with paths in $\mathcal{T}$. From [8, p. 131] the jth column of A_1^{-1} is the short column vector of the unique path in $\mathcal{T}$ from v_j to the reference vertex v_m. Now define the $(m-1)\times(n-m+1)$ matrix N in the following way: For $j=1,2,\dots,n-m+1$, e_{j+m-1} does not belong to $\mathcal{T}$ and is called a link of $\mathcal{T}$. If $f(e_{j+m-1})=(v_s, v_t)$, let p_j be the unique path in $\mathcal{T}$ from v_s to v_t and define the jth column of N to be the short column vector associated with p_j. According to Seshu and Reed [9, p. 93], $N=A_1^{-1}A_2$, so that $A_2=A_1N$. A cycle obtained by adding the jth link e_{j+m-1} to the path of p_j in $\mathcal{T}$ determined by the end vertices of e_{j+m-1} is called a fundamental cycle. The column vectors associated with the fundamental cycles are the columns of $\begin{pmatrix} -N \\ I \end{pmatrix}$, where I is an $(n-m+1)$-square identity matrix.

EXAMPLE.



$$A=\begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 \end{pmatrix}, \quad A_1=\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$A_2=\begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_1^{-1}=\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}, \quad N=\begin{pmatrix} 1 & 1 \\ -1 & 0 \\ 1 & 1 \end{pmatrix}.$$

III. INTEGRAL {1}- AND {1,2}-INVERSES OF AN INCIDENCE MATRIX

We continue to use the notation of Section II. Now define the $n\times m$ matrix B as

$$B=\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

The characterization of A_1^{-1} given by Resh [8] can be extended to characterize B, namely, the jth column of B is the column vector associated with the unique path in $\mathfrak{T}$ from v_j to the reference vertex. This is easy to see for the particular ordering of the edges of $\mathcal{G}$ and choice of reference vertex being used. The entries of the column vector for the empty path from v_m to v_m are 0, so the mth column of B contains only 0's. The other columns of B have 0's in the last $n-m+1$ rows associated with the links of $\mathfrak{T}$, and in the first $m-1$ rows these columns agree with the short column vectors defining A_1^{-1}.

In Theorem 1 below we specialize a result from [3] to our incidence matrix A. The proof of Theorem 1 is straightforward and is omitted.

THEOREM 1. *For the incidence matrix*

$$A=\begin{pmatrix} A_1 & A_1N \\ -eA_1 & -eA_1N \end{pmatrix}$$

and the matrix

$$B=\begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

as partitioned above, let

$$S=\begin{pmatrix} A_1^{-1} & 0 \\ e & 1 \end{pmatrix}, T=\begin{pmatrix} I_{m-1} & -N \\ 0 & I_q \end{pmatrix},$$

$$F=\begin{pmatrix} A_1 \\ -eA_1 \end{pmatrix} \quad \text{and} \quad G=\begin{pmatrix} I_{m-1} & N \end{pmatrix},$$

where $q=n-m+1$. *Then*

(i) B *is an integer* $\{1,2\}$*-inverse of* A,
(ii) S *and* T *are unimodular matrices,*
(iii) $SAT=\begin{pmatrix} I_{m-1} & 0 \\ 0 & 0 \end{pmatrix}$, *the Smith canonical form of* A, *and*
(iv) $A=FG$ *is an integral full rank factorization of* A.

As was observed above, the entries in A_1^{-1} and N actually come from the set $\{-1, 0, 1\}$. Thus the entries of B, S, T, F, and G come from the set $\{-1, 0, 1\}$.

For the matrices A and B given above, if $XA=0$ or $AX=0$, then $B+X$ is a $\{1\}$-inverse of A, since $A(B+X)A=ABA+AXA=A$. We consider two fundamental classes of integer $n\times m$ matrices X such that $XA=0$ or $AX=0$. A matrix X in class I has all 0 entries except for one row of 1's. Hence $XA=0$, since the sum of the rows of A is zero. A matrix X in class II is defined as follows: For a fundamental cycle p determined by a link of $\mathfrak{T}$, define X to have all 0 entries except for one column, which is C_p. So $AX=0$ by [9, p. 92]. We are now in a position to give a characterization of integral $\{1\}$-inverses of A.

THEOREM 2. *Integral-$A\{1\}$ is the set of matrices of the form $B+X$ where X is an integral linear combination of matrices from classes* I *and* II.

Proof. It is clear that any matrix of the form $B+X$ is an integer $\{1\}$-inverse of A. Conversely let Y be an integer $\{1\}$-inverse of A. Then by [1, p. 240] there is an integer $n\times m$ matrix Z such that $Y=B+Z-BAZAB$. Let $Z=\sum_{ij}z_{ij}E^{ij}$, where E^{ij} is the $n\times m$ matrix with 1 in the (i,j) position and with 0's elsewhere. Let $X=Z-BAZAB=\sum_{ij}z_{ij}(E^{ij}-BAE^{ij}AB)$. Thus the result holds if we show that each

$$X^{ij}=E^{ij}-BAE^{ij}AB=E^{ij}-\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}E^{ij}\begin{pmatrix} I & 0 \\ -e & 0 \end{pmatrix}$$

is an integral linear combination of matrices from classes I and II. If $j\neq m$, then

$$X^{ij}=\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}E^{ij}-\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}E^{ij}=\begin{pmatrix} 0 & -N \\ 0 & I \end{pmatrix}E^{ij},$$

which has all zero entries except for the jth column. This column is either zero or a column of $\begin{pmatrix} -N \\ I \end{pmatrix}$. In the latter case it is the column vector of a fundamental cycle, so X^{ij} is 0 or is in class II. Let

$$H=BA=\begin{pmatrix} I & N \\ 0 & 0 \end{pmatrix}.$$

For $j=m$, $BAE^{im}AB=-(H_i,\ldots,H_i,0)$, where H_i denotes the ith column of H, and $E^{im}=(0,\ldots,0,I_i)$, where I_i denotes the ith column of the $n\times n$ identity matrix. Thus $X^{im}=E^{im}-BAE^{im}AB=(H_i,\ldots,H_i,I_i)$. If $i<m$, then the edge e_i is in $\mathfrak{T}$ and $H_i=I_i$, so that X^{im} is in class I. Otherwise write X^{im} as $X^{im}=(I_i,\ldots,I_i,I_i)-(I_i-H_i,\ldots,I_i-H_i,0)$, where the first matrix is from class I and the second matrix is a sum of matrices from class II, since I_i-H_i is a column of $\begin{pmatrix} -N \\ I \end{pmatrix}$. ∎

COROLLARY 1. *$A\{1\}$ is the class of all matrices of the form $B+X$ where X is a complex linear combination of matrices from classes* I *and* II.

Proof. If Y is a $\{1\}$-inverse of A, then by [2, p. 40] there is an $n\times m$ matrix Z such that $Y=B+Z-BAZAB$. Thus the proof of the corollary is the same as the proof of the theorem. ∎

Note that the number of matrices in classes I and II is $n+m(n-m+1)$. However, in the proof of Theorem 2 the class II matrices which are nonzero in the last column are never needed. Define class II′ to be class II with these matrices eliminated. We are left with $n+(m-1)(n-m+1)=mn-(m-1)^2$ matrices in classes I and II′ which span the subspace of all $n\times m$ matrices X such that $AXA=0$. By [2, p. 78] a minimal spanning set for $\{X: AXA=0\}$ must have $mn-(m-1)^2$ elements. Thus we obtain:

COROLLARY 2. *The matrices in classes* I *and* II′ *form a basis for* $\{X: AXA=0\}$.

We are now ready to characterize all integer $\{1,2\}$-inverses of the incidence matrix A. From [5, Theorem 1], integral-$A\{1,2\}$ is the set of all matrices of the form

$$T\begin{pmatrix} I_{m-1} & Z_2 \\ Z_3 & Z_3Z_2 \end{pmatrix}S,$$

where S and T are the integer matrices of Theorem 1 giving the Smith canonical form of A, and Z_2 and Z_3 are arbitrary conformal integral matrices. Thus integral-$A\{1,2\}$ is the set of all matrices of the form

$$T\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}S+T\left[\begin{pmatrix} 0 & Z_2 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 0 \\ Z_3 & Z_3Z_2 \end{pmatrix}\right]S,$$

or

$$B+\left[\begin{pmatrix} 0 & Z_2 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} -N \\ I \end{pmatrix}Z_3(I \quad Z_2)\right]S,$$

or

$$B+\begin{pmatrix} Z_2 \\ 0 \end{pmatrix}(e \quad 1)+\begin{pmatrix} -N \\ I \end{pmatrix}Z_3\left(A_1^{-1}+Z_2e \quad Z_2\right).$$

Let $Z_3=\sum_{ij}z_{ij}E^{ij}$, where E^{ij} denotes a matrix of the appropriate size with a 1 in position (i,j) and 0's elsewhere, to obtain

$$B+\begin{pmatrix} Z_2 \\ 0 \end{pmatrix}(e \quad 1)+\sum_{ij}z_{ij}\begin{pmatrix} -N \\ I \end{pmatrix}E^{ij}\left(A_1^{-1} \quad 0\right)+\sum_{ij}z_{ij}\begin{pmatrix} -N \\ I \end{pmatrix}E^{ij}Z_2(e \quad 1). \tag{*}$$

This expression can be interpreted to give a procedure for finding any integer $\{1,2\}$-inverse of A. Arbitrary elements of $A\{1,2\}$ can be obtained by letting Z_2 and Z_3 be complex matrices. First we examine two special cases.

LEMMA 1. *An integer $\{1,2\}$-inverse of A is obtained by adding to B any integer linear combination of class* I *matrices with nonzero entries only in the first $m-1$ rows.*

Proof. Let $Z_3=0$ in Equation (*). Thus

$$B+\begin{pmatrix} Z_2 \\ 0 \end{pmatrix}\begin{pmatrix} e & 1 \end{pmatrix}=B+\begin{pmatrix} Z_2 & \dots & Z_2 \\ 0 & \dots & 0 \end{pmatrix}$$

is an integral $\{1,2\}$-inverse of A for any $(m-1)\times 1$ column matrix Z_2. ■

LEMMA 2. *Let z be an integer, let e_j be an edge of $\mathfrak{T}$, and let C_p be the column vector of the fundamental cycle determined by some link of $\mathfrak{T}$. Then an integer $\{1,2\}$-inverse of A is obtained by adding (subtracting) zC_p to each column k of B for which e_j belongs to the path in $\mathfrak{T}$ from v_k to v_m and which has the same (opposite) orientation as this path.*

Proof. Suppose that C_p is determined by the ith link e_{m-1+i}. Let $Z_2=0$ and $Z_3=zE^{ij}$ in Equation (*). Then

$$B+z\begin{pmatrix} -N \\ I \end{pmatrix}E^{ij}\begin{pmatrix} A_1^{-1} & 0 \end{pmatrix}$$

is an integer $\{1,2\}$-inverse of A. Let p_k denote the path in $\mathfrak{T}$ from v_k to v_m which determines the kth column of A_1^{-1}. Now

$$\begin{pmatrix} -N \\ I \end{pmatrix}E^{ij}\begin{pmatrix} A_1^{-1} & 0 \end{pmatrix}=C_p\left(j\text{th row of } A_1^{-1}, 0\right),$$

and the columns of this matrix are 0, C_p, or $-C_p$ when e_j does not belong to p_k, e_j belongs to p_k and has the same orientation as p_k, or e_j belongs to p_k and has the opposite orientation to p_k, respectively. ■

Using these results Equation (*) may be interpreted as a procedure for finding all integer $\{1,2\}$-inverses of A.

THEOREM 3. *All integer* $\{1,2\}$*-inverses of* A *can be obtained from* B *by the following steps*:

Step 1. Choose a column of $m-1$ *integers* $Z_2=(s_1, s_2,\dots, s_{m-1})^t$, *and as in Lemma* 1, *add* $s_j(1, 1,\dots,1)$ *to the* j*th row of* B *for* $j=1,2,\dots,m-1$.

Step 2. Choose a link e_{m-1+i} *and an edge* e_j *of* $\mathfrak{T}$, *choose an integer* z_{ij}, *and proceed as in Lemma 2. If* $s_j=0$ *(equivalently* $E^{ij}Z_2=0$*) go to Step 4. Otherwise go to Step 3.*

Step 3. Add $z_{ij}s_jC_p$ *to every column of* B, *where* C_p *is the column vector of the fundamental cycle determined by the link* e_{m-1+i}.

Step 4. Repeat Steps 2 and 3 for all possible choices of links and edges of $\mathfrak{T}$.

COROLLARY 3. *All* $\{1,2\}$*-inverses of* A *can be obtained from* B *by letting* Z_2 *and each* z_{ij} *be complex in the procedure of Theorem* 3.

When solving a consistent set of equations $Ax=b$ where A is our incidence matrix, it is often convenient to drop the last row (or equation) and work with the reduced incidence matrix $A_r=(A_1\ \ A_1N)$. No information about the graph $\mathcal{G}$ is lost, since one can recover A from A_r; the last row of A is the negative of the sum of the rows of A_r. The result corresponding to Theorem 1 for reduced incidence matrices is:

THEOREM 4. *For the reduced incidence matrix* $A_r=(A_1\ \ A_1N)$, *let*

$$B_r=\begin{pmatrix} A_1^{-1} \\ 0 \end{pmatrix} \quad \text{and} \quad T=\begin{pmatrix} I_{m-1} & -N \\ 0 & I_q \end{pmatrix},$$

where B_r *is* $n\times(m-1)$ *and* $q=n-m+1$. *Then*

(i) B_r *is an integer* $\{1,2\}$*-inverse of* A;

(ii) T *is unimodular and* $A_1^{-1}A_rT=(I_{m-1}\ \ 0)$, *the Smith canonical form of* A.

Since reduced incidence matrices have full row rank, several classes of generalized inverses coincide. That is, $A_r\{1\}=A_r\{1,2\}=A_r\{1,3\}=A_r\{1,2,3\}$. Matrices in these classes are actually the right inverses of A_r. Define the classes I_r and II_r similarly to classes I and II, except take the matrices to be $n\times(m-1)$. As in the proof of Theorem 2, all integral right inverses of A_r can be obtained by adding integral linear combinations of matrices from classes I_r and II_r to B_r. However, in this case the matrices of class I_r are not needed.

THEOREM 5. *Any integral right inverse of A_r can be obtained by adding integral multiples of the column vectors associated with the fundamental cycles of $\mathcal{G}$ to the columns of B_r.*

Proof. By [5, Theorem 1] integral-$A_r\{1,2\}$ is the set of all matrices of the form

$$T\begin{pmatrix} I_{m-1} \\ Z_3A_1 \end{pmatrix}A_1^{-1},$$

where Z_3 is an arbitrary $(n-m+1)\times(m-1)$ integral matrix. Now

$$T\begin{pmatrix} I_{m-1} \\ Z_3A_1 \end{pmatrix}A_1^{-1}=\left[\begin{pmatrix} I_{m-1} & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} 0 & -N \\ 0 & I_q \end{pmatrix}\right]\begin{pmatrix} A_1^{-1} \\ Z_3 \end{pmatrix}$$

$$=B_r+\begin{pmatrix} -N \\ I_q \end{pmatrix}Z_3, \qquad \text{where } q=n-m+1. \qquad \blacksquare$$

COROLLARY 4. *Any right inverse of A_r can be obtained by adding complex multiples of the column vectors associated with the fundamental cycles of $\mathcal{G}$ to the columns of B_r.*

COROLLARY 5. *The matrices in class II_r form a basis for $\{X\colon A_rXA_r=0\}$.*

Proof. There are $(m-1)(n-m+1)=(m-1)n-(m-1)^2$ matrices in class II_r, which by Theorem 5 span $\{X\colon A_rXA_r=0\}$. By [2, p. 78] this is the number of matrices in a minimal spanning set. $\blacksquare$

IV. INCIDENCE MATRICES OF BIPARTITE GRAPHS

In the previous section the definition of the incidence matrix A of a graph $\mathcal{G}$ depended on an orientation given to the edges. The unoriented incidence matrix A_u of $\mathcal{G}$ is defined in the same way as A except all -1 entries are changed to 1. For example,

$$A=\begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

is the incidence matrix of a graph that consists of a single cycle of length 3. The unoriented incidence matrix of this graph is

$$A_u=\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

In general unoriented incidence matrices do not have integral generalized inverses. In this example $\det(A_u)=2$, so the only $\{i, j,\dots, l\}$-inverse of A_u is A_u^{-1}, which is not integral.

The sum of all rows of an unoriented incidence matrix A_u is a row of 2's. Thus as with A one can drop a row of A_u without losing any information. The resulting matrix A_{ur} is the reduced unoriented incidence matrix of $\mathcal{G}$. For simplicity of this presentation we assume that the last row is dropped whenever we write A_r or A_{ur}.

Bipartite graphs are an important class of graphs for which A and A_u have integral $\{1\}$- and $\{1,2\}$-inverses. A graph $\mathcal{G}$ is said to be bipartite if the set of vertices can be partitioned into two subsets such that all edges of $\mathcal{G}$ connect a vertex in one subset with a vertex in the other. The graph is called a complete bipartite graph if all such pairs of vertices are connected by an edge. A transportation matrix is the reduced unoriented incidence matrix of a complete bipartite graph.

Suppose that $\mathcal{G}$ is bipartite and that the vertices have been ordered so that all edges connect a vertex from $V_1=\{v_1, v_2,\dots, v_h\}$ with a vertex in $V_2=\{v_{h+1},\dots, v_n\}$. Furthermore, suppose that the edges of $\mathcal{G}$ have been oriented from V_1 to V_2. Then the incidence matrix A and the unoriented incidence matrix A_u of $\mathcal{G}$ are related by $A=JA_u$, where

$$J=\begin{pmatrix} I_h & 0 \\ 0 & -I_{n-h} \end{pmatrix}.$$

Similarly $A_r=J_rA_{ur}$, where

$$J_r=\begin{pmatrix} I_h & 0 \\ 0 & -I_{n-h-1} \end{pmatrix}.$$

The proof of Theorem 6 from these relations is straightfoward and is omitted.

THEOREM 6. *For incidence matrices of a connected bipartite graph as defined above*:

(i) *X is an integral $\{1\}$-inverse of A (A_r) if and only if XJ (XJ_r) is an integral $\{1\}$-inverse of A_u (A_{ur}).*

(ii) *X is an integral* $\{1,2\}$*-inverse of* A (A_r) *if and only if* XJ (XJ_r) *is an integral* $\{1,2\}$*-inverse of* A_u (A_{ur}).

Theorem 6 can be used to interpret the results of Theorems 1 through 5 for $\{1\}$- and $\{1,2\}$-inverses of the unoriented incidence matrix of a bipartite graph. All that is involved is a change of sign for each column of the generalized inverse corresponding to a vertex of V_2.

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