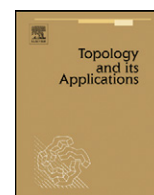Contents lists available at ScienceDirect

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Heegaard splittings of twisted torus knots ☆

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A R T I C L E   I N F O

A B S T R A C T

Little is known on the classification of Heegaard splittings for hyperbolic 3-manifolds. Although Kobayashi gave a complete classification of Heegaard splittings for the exteriors of 2-bridge knots, our knowledge of other classes is extremely limited. In particular, there are very few hyperbolic manifolds that are known to have a unique minimal genus splitting. Here we demonstrate that an infinite class of hyperbolic knot exteriors, namely exteriors of certain "twisted torus knots" originally studied by Morimoto, Sakuma and Yokota, have a unique minimal genus Heegaard splitting of genus two. We also conjecture that these manifolds possess irreducible yet weakly reducible splittings of genus three. There are no known examples of such Heegaard splittings.

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1. Introduction

The only class of hyperbolic manifolds for which there is a complete classification is that of the exteriors of 2-bridge knots. This was done by T. Kobayashi in [5]. In particular there are very few manifolds which are known to have a unique minimal genus Heegaard splitting. We are interested, for reasons which will become clear later, in the following class of knots:

Definition 1.1. The knot $K \subset S^3$ obtained by taking the non-trivial (p,q)-torus knot $K(p,q) \subset S^3$ (embedded on a standard torus $V \subset S^3$), removing a neighborhood of a small unknotted S^1 around m adjacent strands, which we denote by C, and doing a $\frac{1}{s}$-Dehn surgery along C will be called a *twisted torus knot* and setting $r = 2s$, denoted by $T(p,q,m,r)$. Since we always will take $m = 2$ strands we abbreviate to $T(p,q,r)$. (See also [3] and Fig. 5.)

The main theorem we prove is the following:

Theorem 4.9. *Let $K_m = T(p,q,r)$ be a twisted torus knot with $(p,q) = (7,17)$ and $r = 10m - 4$, $m \in \mathbb{Z}$. Then for sufficiently large $m \in \mathbb{Z}$ the knot complement $S^3 - N(K_m)$ has a unique, up to isotopy, minimal genus Heegaard splitting.*

We suspect that the restrictions on the parameters of the knots $T(p,q,r)$ are only a result of the non-trivial proof and thus conjecture:

Conjecture 4.16. *All knot exteriors $E(K)$, where $K = T(p, q, 2, r)$ and K is not μ-primitive, have a unique (minimal) genus two Heegaard splitting.*

A sub-class of twisted torus knots, including those which are covered in Theorem 4.9, was previously studied by Morimoto, Sakuma and Yokota in [11] and were shown to be not μ-primitive (see Definition 2.1). A different sub-class was studied by J. Dean in [3], and contains knots which admit surgeries yielding Seifert fibered spaces over S^2 with three exceptional fibers. This accumulation of information points to twisted torus knots as a very interesting class of knots.

In trying to classify the higher genus Heegaard splittings for these knots one must naturally address the issue of non-minimal genus non-stabilized Heegaard splittings. One of the more tantalizing problems in the current Heegaard theory of 3-manifolds is that it is not known whether manifolds with fewer than two boundary components can possess non-minimal genus Heegaard splittings which are weakly reducible and non-stabilized. (In alternate terminology; non-minimal genus Heegaard splittings which are distance one, where distance is in the sense of Hempel's distance for Heegaard splittings.) The basic problem is that there are currently no known techniques to show that a non-minimal genus weakly reducible Heegaard splitting of a closed manifold is non-stabilized.

In this quest to find such examples it is necessary to find a Heegaard splitting for a knot manifold that is not γ-*primitive* (see definition in Section 2 below), since boundary stabilizations of such splittings are always stabilized (see [9, Theorem 4.6]). In [11] Morimoto, Sakuma and Yokota showed that the knots $K_m = T(7, 17, 10m - 4)$ are not μ-primitive, which is an important special case. Hence the following theorem shows that $E(K_m) = S^3 - N(K_m)$ are candidates for manifolds which could have the desired splittings as above:

Theorem 4.4. *The knots $K_m = T(7, 17, 10m - 4)$ are not γ-primitive for all curves $\gamma \subset \partial S^3 - N(K)$.*

Theorem 4.4 and the theorem of Morimoto, Sakuma and Yokota in [11] provide motivation for the following conjecture:

Conjecture 5.1. *The boundary stabilized genus three Heegaard splitting (V', W') of the unique minimal genus two Heegaard splitting (V, W) of $E(K_m)$, where $K_m = T(7, 17, 2, 10m - 4)$, is non-stabilized.*

We discuss, in Section 5, some of the issues which arise in the attempt to prove that the above boundary stabilized Heegaard splittings are indeed non-stabilized (for some background see [9]).

2. Preliminaries

In this section we recall some definitions and notation used in the paper. We then define the notions of *primitive* and *weakly primitive* curves on the boundary of a knot space:

A *compression body* W of genus g is a compact connected orientable 3-manifold which can be represented as $(S \times [0, 1]) \cup$ {2-handles}, where S is a genus g closed orientable surface and the 2-handles are attached to $S \times \{0\}$. We require that all S^2 components in $(S \times \{0\}) \cup$ {2-handles} be eliminated by attaching 3-balls to them. The connected surface $S \times \{1\}$ will be denoted by $\partial_+ W$ and the not necessarily connected surface $\partial W \smallsetminus \partial_+ W$ will be denoted by $\partial_- W$.

Any compact connected orientable 3-manifold M has a decomposition as $M = V \cup W$ and $V \cap W = \partial_+ V = \partial_+ W = \Sigma$, where V and W are compression bodies. Such a decomposition is called a *Heegaard splitting* and the surface Σ will be called a *Heegaard surface*.

Definition 2.1.

(1) A simple closed curve on the boundary of a compression body W will be called *primitive* if it meets an essential disk $D \subset W$ in a single point. An annulus on the boundary of a compression will be called *primitive* if its core curve is primitive.

(2) Let $K \subset S^3$ be a knot and (V, W) a Heegaard splitting of $S^3 - N(K)$ with $\partial(S^3 - N(K)) \subset V$. Let γ denote a simple closed curve on $\partial_- V$. We say that (V, W) is γ-*primitive* if there is a vertical annulus A in the compression body V such that $\partial A = \gamma \cup \beta$ where $\beta \subset \partial_+ V$ meets an essential disk D of W in a single point. If γ is a meridian for K we say that it is μ-*primitive* (see Fig. 1). An annulus A and a disk D as above will be called an (A, D) *reducing pair*.

(3) If there is a vertical annulus $A \subset V$ as above and an essential disk $D \subset W$ so that $A \cap D = \emptyset$ then we say that γ is *weakly primitive*.

(4) We say that $K \subset S^3$ is γ-*primitive* (*weakly γ-primitive*) if $S^3 - N(K)$ has a minimal genus Heegaard splitting with a primitive γ curve (weakly primitive γ curve). As before if γ is a meridian we will say that K is μ-*primitive* (*weakly μ-primitive*).

3. Heegaard splittings of torus knots

Since a twisted torus knot can be obtained from a torus knot by Dehn surgery in an unknot, we should not be surprised that their Heegaard splittings are related. Hence we would like to discuss those first. It is well known that the exterior
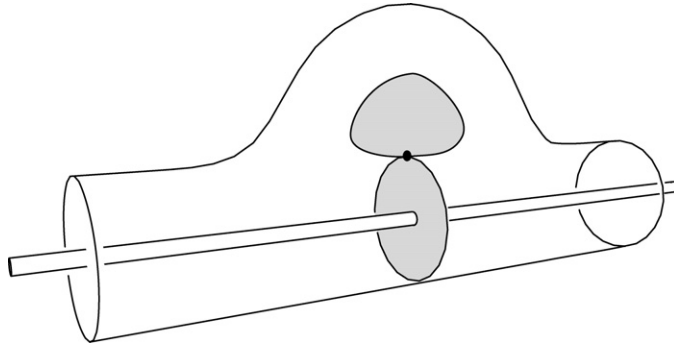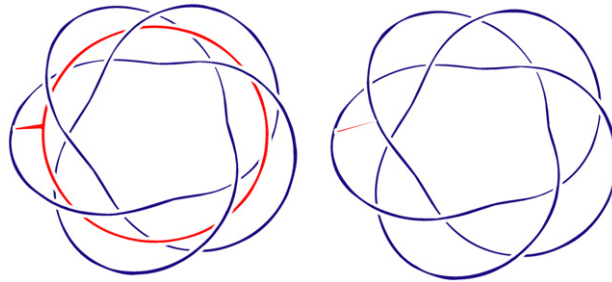
Fig. 1. A primitive meridian.

Fig. 2. Tunnel systems for a torus knot: the inner system and the middle system.

$E(K) = S^3 - N(K)$ of a torus knot $K = K(p, q)$ will have at most three distinct genus two Heegaard surfaces. Generically $E(K)$ will have three distinct such surfaces depending on the values of p and q (see [6]). For the purpose of this work we are interested in what we will call the *middle* Heegaard surface, defined as follows:

Definition 3.1. Embed the knot K in the standard way into an unknotted torus T' in S^3. Let T denote this torus after we have removed a single small disk D which is disjoint from $K \subset T$. Without loss of generality we can think of K as lying in the middle level surface $T_0 = T \times \{0\}$ of a thickened product $T \times [-1, 1]$. By removing a neighborhood of K we obtain a compression body $V = (T \times [-1, 1]) - N(K)$. Note that $\partial_+ V$ is a genus two Heegaard surface. The complementary handlebody W is the union of two solid tori, W_i the inside solid torus and W_o the outside solid torus of the unknotted torus, T'. These solid tori W_i and W_o are attached by a 1-handle, $D \times I$ to form a genus two handlebody. We will say that $\Sigma = \partial_+(T \times [-1, 1]) = \partial W$ is the *middle* Heegaard surface.

Definition 3.2. The Heegaard surface obtained by taking the boundary of a regular neighborhood of the graph $K \cup t_i \cup w_i$ or $K \cup t_o \cup w_o$ will be called the *inner* and *outer* Heegaard surfaces and denoted by Σ_i and Σ_o, respectively. Here t_i is a small arc connecting K to the core curve w_i of W_i and t_o is a small arc connecting K to the core curve w_o of W_o (see Fig. 2).

The remainder of this section will be devoted to proving the theorem below which is of independent interest although it is not directly used in this paper. With the standard choice of (μ, λ) a meridian-longitude pair there is an identification between curves on the boundary of a knot space and a "slope" $r \in \mathbb{Q} \cup \{\infty\}$. When the context is clear we often do not make the distinction between a slope and a curve.

Theorem 3.3. *Let Σ denote the middle Heegaard surface for the non-trivial torus knot $K(p, q)$. The following are equivalent*:

(i) *Σ is μ-primitive (i.e., 1/0-primitive).*
(ii) *Σ is γ-primitive for some s.c.c. $\gamma \subset \partial(S^3 - N(K))$.*
(iii) *There exist $r, s \in \mathbb{Z}$ so that $|ps - rq| = 1$ and either $r = 1$ or $s = 1$.*
(iv) *Σ is isotopic to either the Σ_i or Σ_o Heegaard surfaces.*

Proof. (i) $\Rightarrow$ (ii) Set $\gamma = \mu$. It is a simple closed curve with slope $\frac{1}{0}$.

(iii) $\Rightarrow$ (iv) If there exist such a pair r, s then there is an (r, s)-curve on the torus T which meets the knot $K(p, q)$ in a single point. This curve is a tunnel representing the middle splitting. Furthermore since either $r = 1$ or $s = 1$, when this tunnel is pushed into one of the solid tori, depending on whether $r = 1$ or $s = 1$, it represents a core there. Hence Σ_{mid} is isotopic to either the Σ_i or Σ_o splittings.
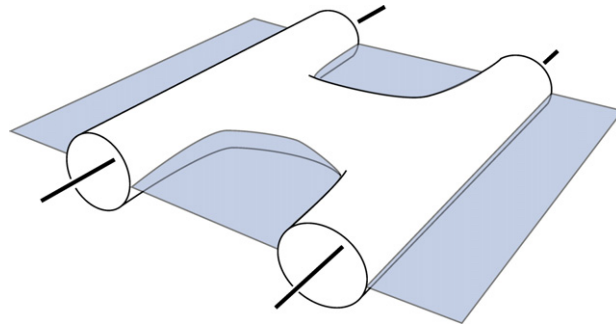
Fig. 3. The disk D_{punct}.
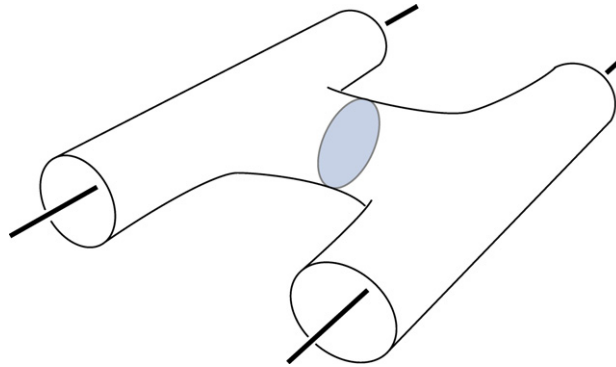
Fig. 4. The disk $D_{p/q}$.

(iv) $\Rightarrow$ (i) Both the Σ_i and Σ_o splittings are μ-primitive (see [6]). If the middle splitting is isotopic to either, it is also μ-primitive.

(ii) $\Rightarrow$ (iii) This is the hard case. The argument follows:

Assume that Σ is γ-primitive, then there is a disk-annulus pair, (D_γ, A_γ) so that $D_\gamma \cap A_\gamma$ is a single point and $D_\gamma \subset W$, $A_\gamma \subset V$.

Define two compressing disks, one for each of the compression bodies V and W. Consider $D_{punct} = D \times \{0\}$, where D is the disk from Definition 3.1 (see Fig. 3). The disk D_{punct} compresses the genus two handlebody W into the upper and lower solid tori. Let $\alpha_{p/q}$ be an arc properly embedded in T so that if we shrink D to a point $\alpha_{p/q}$ becomes a (p,q)-curve on the torus T'. Let $D_{p/q} = \alpha_{p/q} \times [-1,1]$ be the disk obtained by taking the product of $\alpha_{p/q} \subset T$ with the interval $[-1,1]$. This is a non-separating disk that is disjoint from the knot K (see Fig. 4). Arrange A_γ and D_γ to intersect D_{punct} and $D_{p/q}$ minimally, subject to the condition that they meet in a single point.

Claim 3.4. *The annulus A_γ is disjoint from the disk $D_{p/q}$.*

Proof. Assume in contradiction that $A_\gamma \cap D_{p/q} \neq \emptyset$. The intersection cannot contain simple closed curves which are essential on A_γ because this would imply that the torus knot exterior has compressible boundary. By minimality there are no inessential such curves. So all components of intersection are arcs. Since $D_{p/q} \cap \partial X = \emptyset$ these arcs cannot be essential in A_γ. So $A_\gamma \cap D_{p/q}$ is composed of inessential arcs in A_γ.

Consider now an outermost arc of intersection α on A_γ it cuts off a sub-disk E of $A_\gamma - D_{p/q}$ and a sub-disk E' on $D_{p/q}$. The disk $E \cup E'$ cannot be boundary parallel in the compression body by minimality. If it is isotopic to an essential non-separating disk then it is isotopic to $D_{p/q}$ which is the unique such disk. If the parallel region is on the same side as E we can push E through $D_{p/q}$ to reduce intersection. If the parallel region is on the other side, then E cannot be connected to the rest of the annulus to form A_γ. If it is isotopic to a non-separating essential disk in the genus two compression body then it is a band sum of two copies of $D_{p/q}$. However in this case, as before, E cannot be connected to the rest of the annulus to form A_γ. $\square$

Consider now the two disks D_γ and D_{punct} in W. Assume that they intersect and consider an outermost sub-disk of D_γ with respect to its intersection with D_{punct}. We can choose such a sub-disk, called D'_γ, that does not contain the point of intersection with A_γ. This outer sub-disk D'_γ meets D_{punct} in a single arc and since it is essential by minimality it must be a meridional disk for either the upper W_o or lower W_i solid torus. Without loss of generality we will assume that it is
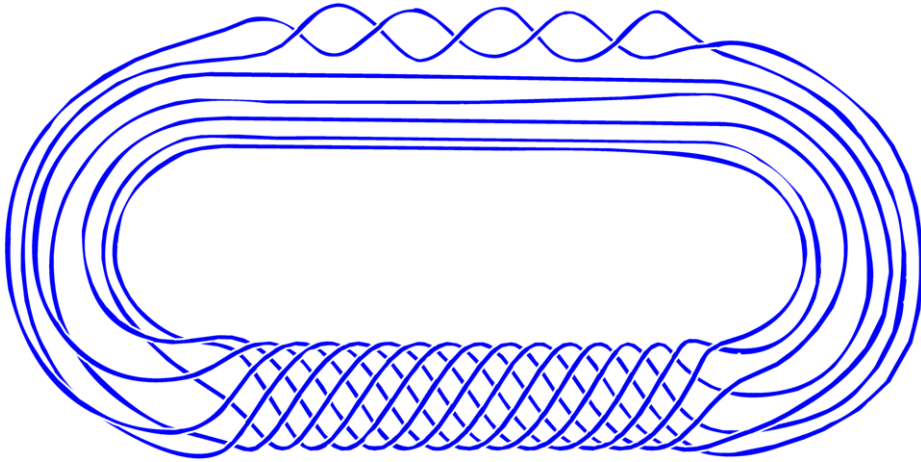
Fig. 5. A knot of Morimoto, Sakuma and Yokota.

a meridional disk for W_o and represents a 0/1 arc on the level torus $T_1 = T \times \{1\}$. If D_γ does not meet D_{punct} then we will take $D'_\gamma = D_\gamma$ instead, and recall that ∂D_γ is a closed curve in T_1 and has a single point of intersection with A_γ. The argument for W_i and $T_0 = T \times \{0\}$ with the slope 1/0 is symmetric.

On the top (or bottom) punctured torus, say T_1 (T_0) we can identify three essential arcs: a_γ a component of $\partial A_\gamma \cap T_1$, $d_\gamma = \partial D'_\gamma \cap T_1$ and $d_{p/q} = \partial D_{p/q} \cap T_1$. If D_γ is disjoint from D_{punct}, then d_γ is actually a closed curve. From our preceding arguments we know that a_γ and d_γ are disjoint in T_1, as are a_γ and $d_{p/q}$. Connect the arcs across the puncture to obtain closed curves: $\widehat{a_\gamma}, \widehat{d_\gamma}$ and $\widehat{d_{p/q}}$. Each pair of the closed curves obtains at most one intersection in the puncture, so we have that $\widehat{a_\gamma}$ and $\widehat{d_\gamma}$ meet at most once, and $\widehat{a_\gamma}$ and $\widehat{d_{p/q}}$ meet at most once. Now, the slope of $\widehat{d_\gamma}$ is 0/1 (or 1/0), the slope of $\widehat{d_{p/q}}$ is p/q and r/s is the slope of the closed curve $\widehat{a_\gamma}$.

If $\widehat{a_\gamma} \cap \widehat{d_{p/q}} = \emptyset$, i.e., $\widehat{a_\gamma}$ and $\widehat{d_{p/q}}$ are parallel and $\widehat{a_\gamma} \cap \widehat{d_\gamma} = \{pt\}$ or $\widehat{a_\gamma} \cap \widehat{d_\gamma} = \emptyset$, i.e., $\widehat{a_\gamma}$ and $\widehat{d_\gamma}$ are parallel and $\widehat{a_\gamma} \cap \widehat{d_{p/q}} = \{pt\}$ then $|p0 - q1| = 1$ (or $|p1 - q0| = 1$) that is $q = 1$ (or $p = 1$) contradicting the fact that $T(p,q)$ is non-trivial. Similarly they cannot all be parallel. If $\widehat{a_\gamma} \cap \widehat{d_\gamma} = \{pt\}$ and $\widehat{a_\gamma} \cap \widehat{d_{p/q}} = \{pt\}$ then $|ps - qr| = 1$ and $|r0 - s1| = 1$, i.e., $s = 1$ (or $|r1 - s0| = 1$, i.e., $r = 1$). Then conclusion (iii) of Theorem 3.3 holds. □

4. Twisted torus knots

In this section we recall the definition of a special sub-class of twisted torus knots $T(7, 17, r)$. This class will play a major role for the rest of the paper.

Definition 4.1. The knot $K \subset S^3$ obtained by taking the (7, 17)-torus knot $T(7, 17)$ in S^3 (embedded on a standard torus $V \subset S^3$), removing a neighborhood of a small unknotted S^1 around 2 adjacent strands, which we denote by C, and doing a $\frac{1}{5m-2}$-Dehn surgery, $m \in \mathbb{Z}$, along C will be denoted by $T(7, 17, 2, 10m - 4)$. As before, we abbreviate to $T(7, 17, 10m - 4)$. (See also [3] and Fig. 5.)

Remark 4.2. All knots in S^3 of the form $T(p, q, r)$, $r \in \mathbb{Z}$, are tunnel number one knots: Given a twisted torus knot $T(p, q, r)$ the underlying torus knot $T(p, q)$ has an unknotting tunnel which is an essential arc on the cabling annulus. We can consider such an arc in the link $L(p, q) = T(p, q) \cup C$. Since C can be isotoped into the Heegaard surface so that the slope determined by the surface on $\partial N(C)$ is 0, then after the surgery on C which gives $T(p, q, r)$, the arc will be an unknotting tunnel for $T(p, q, r)$. Hence we have a genus two Heegaard splitting (V, W) for $E(T(p, q, r))$, where V is the compression body $\overline{N(K \cup t)}$ less a smaller neighborhood of $T(p, q, r)$, and W is the handlebody $S^3 - int(V)$. Since this Heegaard splitting is induced by the middle Heegaard splitting of the underlying torus knot $T(p, q)$ we call it the *middle* Heegaard splitting and denote it by Σ_{mid}.

It is a theorem of Morimoto, Sakuma and Yokota (see [11]), that knots K_m of the form $T(7, 17, 10m - 4)$, $m \in \mathbb{Z}$, are not μ-primitive. Or, in other words, they do not have a (1, 1)-decomposition (see for example [7]).

Proposition 4.3. *The knots $K_m = T(7, 17, 10m - 4)$ are hyperbolic knots for all $m \in \mathbb{Z}$.*

Proof. By Thurston if a knot is atoroidal (simple) it is either a torus knot or hyperbolic. Non-simple tunnel number one knots have been classified by Sakuma and Morimoto (see [10]). In particular, all of their unknotting tunnels have been classified (see Proposition 1.8, Theorems 2.1 and 4.1 of [10]). They all come from a (1, 1)-decomposition for the pair (S^3, K).

So the knots K have a primitive meridian. So do torus knots. But the knots $T(7, 17, 10m - 4)$ do not have a primitive meridian as they are tunnel number super additive by [11]. Hence the knots $T(7, 17, 10m - 4)$ cannot be non-simple and thus are atoroidal. Since they are also not torus knots they must be hyperbolic. $\square$

4.1. The knots $K_m = T(7, 17, 10m - 4)$ are not γ-primitive

It turns out, as will be discussed in Section 5 below, that in order for the knot exteriors of $K_m = T(7, 17, 10m - 4)$ to have weakly reducible and non-stabilized non-minimal Heegaard splittings the knots K_m must have the additional property that they are not γ-primitive for all curves $\gamma \subset \partial S^3 - N(K)$. This is exactly the content of our next theorem:

Theorem 4.4. *The knots $K_m = T(7, 17, 10m - 4)$ are not γ-primitive for any curve $\gamma \subset \partial S^3 - N(K)$ and any $m \in \mathbb{Z}$.*

Proof. By Morimoto, Sakuma and Yokota (see [11]) the knots $K_m = T(7, 17, 10m - 4)$ are not μ-primitive. Assume that γ is a simple closed curve on $\partial E(K)$ which is not a meridian. Since the knots K_m are tunnel number one knots their Heegaard genus $g(S^3 - N(K_m)) = 2$. Hence, if K_m are γ-primitive then γ-surgery on K_m will give genus two manifolds $E(K_m)(\gamma)$ with reducible genus two Heegaard splittings. That is $g(E(K_m)(\gamma)) \leqslant 1$. It follows from [2] that since $\gamma \neq \mu$ we cannot obtain S^3 by γ-surgery, hence we need only consider the case that $g(E(K_m)(\gamma)) = 1$. Therefore in order to prove that K_m are not γ-primitive we need to show that we cannot obtain lens spaces by surgery on K_m.

Consider the following theorem of Ozsvath and Szabo (see [13]):

Theorem 4.5. *If $K \subset S^3$ is a knot which admits surgery yielding a lens space, then the Alexander polynomial $\Delta_K(t)$ of K has the form*:

$$\Delta_K(t) = (-1)^k + \sum_{j=1}^{k} (-1)^{l-j}\left(t^{n_j} + t^{-n_j}\right)$$

where $0 < n_1 < n_2 < \cdots < n_k$ is some increasing sequence of positive integers.

The following theorem regarding the Alexander polynomials of twist knots $T(p, q, 2n)$ denoted by $\Delta_{T(p,q,2n)}(t)$, was proved by H. Morton in [12]:

Theorem 4.6. *Suppose that $0 < s < \frac{q}{3}$ and $s \equiv p^{-1}$ mod q. Then*:

(1) *For all $n \geqslant 2$ the coefficient of t^{ps+2} in $\Delta_{T(p,q,2n)}(t)$ is $\leqslant -2$.*
(2) *For all $n \geqslant 2$ the coefficient of at least one of the terms $t^{ps+1}, t^{ps+2}, t^{ps+3}$ in $t^{2n}\Delta_{T(p,q,-2n)}(t)$ is ± 2.*

Since in our case $n = 5m - 2$ we can apply the theorem for all $m \neq 0$ and conclude that if $\{p, q\} = \{7, 17\}$ then for $s = 5$ we have $5 \equiv 7^{-1}$ mod 17 hence the coefficient of t^{37} must be at most -2 which violates Theorem 4.5.

Though it is not in a symmetric form we can see that there are coefficients which are different from $+1, -1$. Hence by the above Theorem 4.5 we cannot obtain a lens space by surgery and as a result K_m cannot be γ-primitive. $\square$

Example 4.7. Using Hugh Morton's program for computing, the Alexander polynomial for $K_1 = T(7, 17, 6)$ we have:

$$\begin{aligned}
\Delta_{K_1}(T) = &\ T^{102} - T^{101} + T^{95} - T^{94} + T^{88} - T^{87} + T^{85} - T^{84} + T^{81} - T^{80} + T^{78} - T^{77} + T^{74} - T^{73} + T^{71} - T^{70} + T^{68} \\
&- T^{67} + T^{66} - 2T^{65} + 3T^{64} - 3T^{63} + 2T^{62} - T^{60} + T^{59} - 2T^{58} + 3T^{57} - 3T^{56} + 2T^{55} - T^{53} + T^{52} - T^{51} \\
&+ T^{50} - T^{49} + 2T^{47} - 3T^{46} + 3T^{45} - 2T^{44} + T^{43} - T^{42} + 2T^{40} - 3T^{39} + 3T^{38} - 2T^{37} + T^{36} - T^{35} + T^{34} \\
&- T^{32} + T^{31} - T^{29} + T^{28} - T^{25} + T^{24} - T^{22} + T^{21} - T^{18} + T^{17} - T^{15} + T^{14} - T^8 + T^7 - T + 1.
\end{aligned}$$

In particular, the coefficient of t^{37} is -2.

4.2. The knots $T(7, 17, 10m - 4)$ are not weakly γ-primitive

Theorem 4.8. *The knots $K_m = T(7, 17, 10m - 4)$ are not weakly γ-primitive for any simple curve γ on $\partial(S^3 - N(K_m))$ and any $m \in \mathbb{Z}$.*

Proof. Assume in contradiction that the genus two Heegaard splitting of $S^3 - (K_m)$ is weakly γ-primitive. Let (A, D) denote the weak annulus disk pair. Compress the Heegaard surface Σ along the disk D and perform surgery along the annulus A. We obtain either a torus T and an annulus A or a single annulus A, depending on whether the boundary of the disk separates Σ or not. Since the knots K_m are all hyperbolic their exteriors cannot contain essential annuli and tori. Hence
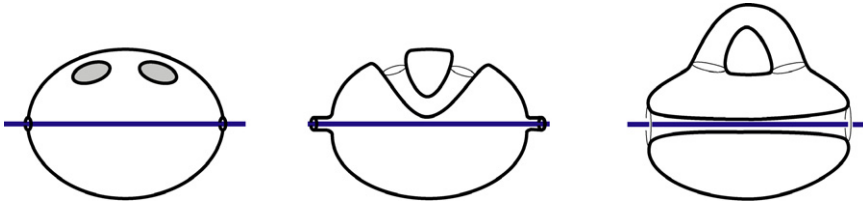
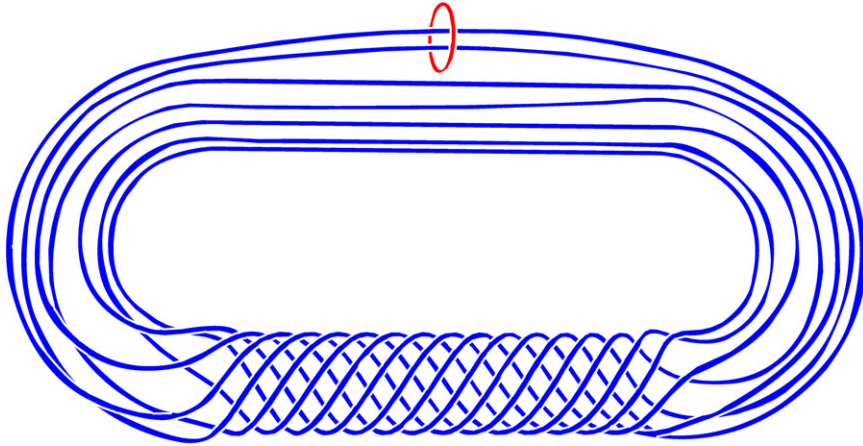**Fig. 6.** A peripheral annulus with disk scars, handle inside/tube outside, and handle outside/tube inside.

Fig. 7. The link $L(p, q)$.

both A and T (if it exists) are boundary parallel. So on the annulus A we see either one or two scars from the compression along the disk. The original surface Σ is obtained from A by attaching a 1-handle to A either on the "inside" or "outside" of A. If the 1-handle is on the "outside" then there is a compressing disk Δ for the annulus A on the "inside" which meets a meridian in a single point. Hence Δ less a collar is a compressing disk for Σ and this would imply that K_m is γ-primitive in contradiction. If the 1-handle is on the "inside" then Δ is a compressing disk for the annulus A meeting a meridian in a single point unless it intersects the 1-handle in an essential way. That is the 1-handle is knotted. But this would mean that Σ is not a Heegaard surface (see Fig. 6). $\square$

4.3. Uniqueness of minimal genus Heegaard splittings

Given a twisted torus knot $T(p, q, r)$ consider the link $L(p, q) = T(p, q) \cup C$, where C is an unknotted simple closed curve encircling two strands of $T(p, q)$ as above (see Fig. 7). The reader can check that the curve C is isotopic into the middle Heegaard surface and that the slope that Σ_{mid} determines on $\partial N(C)$ is the 0 slope. Hence $\frac{1}{s}$-Dehn filling along C is a Dehn twist on Σ_{mid} and thus the Heegaard surface survives the twisting. On the other hand it seems on first glance that C is not isotopic into the other two Heegaard surfaces for $T(7, 17)$. Hence one might expect that the two other genus two Heegaard splittings for $E(T(p, q))$ would be "destroyed" by the $\frac{1}{s}$-Dehn filling along C. In fact we prove an even stronger result:

Theorem 4.9. *Let $K_m = T(p, q, r)$ be a twisted torus knot with $(p, q) = (7, 17)$ and $r = 10m - 4$, $m \in \mathbb{Z}$. Then for sufficiently large $m \in \mathbb{Z}$ the knot complement $S^3 - N(K_m)$ has a unique, up to isotopy, genus two Heegaard splitting.*

We first need some lemmas:

An annulus $A \subset S^3$ will be called *unknotted* if the core of A is unknotted as a curve in S^3 and the linking number of the boundary curves of A is 0. If A^1 and A^2 are unknotted annuli so that $\partial A^1 = \partial A^2$ then if the torus $T^* = A^1 \cup A^2$ is unknotted in S^3, i.e., it bounds two solid tori, then the cores of A^i, $i = 1, 2$, are a meridian curve for one solid torus and a longitude curve for the other. If $T^* = A^1 \cup A^2$ is knotted then it bounds a solid torus on one side and a knot space on the other and the cores of A^i, $i = 1, 2$, are meridian curves for the solid torus.

Lemma 4.10. *The link $L(7, 17) = T(7, 17) \cup C \subset S^3$ is atoroidal.*

Proof. We first claim that $S^3 - N(L(7, 17))$ is irreducible. Let $S \subset S^3 - N(L(7, 17))$ be an essential 2-sphere. The sphere S does not separate C from $T(7, 17)$ since C has linking number 2 with $T(7, 17)$. If S does not separate the two components then obviously it bounds a 3-ball in the component which does not contain C and $T(7, 17)$: Doing $\frac{1}{n}$-surgery on C does not
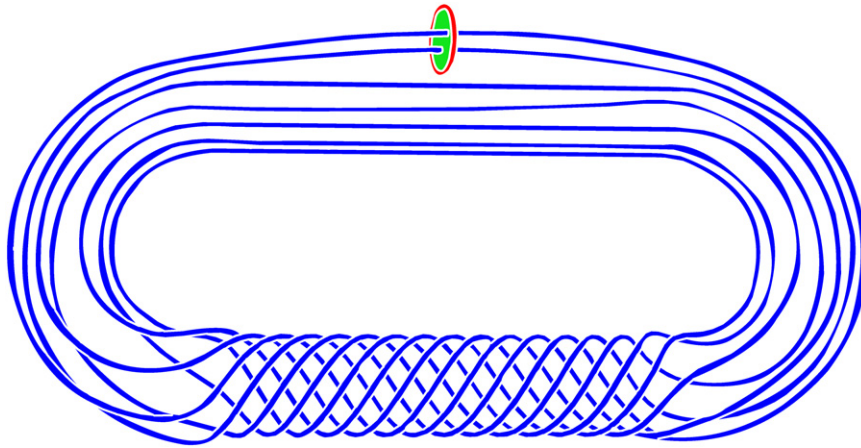
Fig. 8. The link $L(p,q)$ with the twice punctured disk P.

affect either S or the 3-ball and we obtain the exterior of K_m which is irreducible. Hence we have a contradiction to the existence of an essential S.

Assume now that $S^3 - N(L(7,17))$ contains an essential torus T. It follows from Proposition 4.3 that for infinitely many $\frac{1}{n}$-Dehn surgeries on C, $n = 5m - 2$, $m \in \mathbb{Z}$, we obtain a hyperbolic knot $K_m = T(7,17,10m - 4)$ whose complement does not contain essential tori. Hence for those infinitely many surgeries either the torus T compresses or becomes a peripheral torus for K_m.

If T is peripheral in $S^3 - N(K_m)$, for some $m \in \mathbb{Z}$, then the curves C and K_m must be on the same side of T: Otherwise T would be peripheral to the $T(7,17)$ component of $L(7,17)$ and hence would not be essential in $S^3 - N(L(7,17))$. Note that the curve C and the knot $T(7,17)$ cobound a twice punctured disk P (see Fig. 8). If we choose P to intersect T minimally the intersection cannot contain trivial curves on P as this would violate either the minimality of the intersection or the choice of T as essential. Furthermore the intersection $P \cap T$ cannot be empty: As then T would be contained in the complement of a regular neighborhood of $K_m \cup P \cup C$. However $N(K_m \cup P \cup C)$ is homeomorphic to $N(K_m \cup t)$, where t is the unknotting tunnel of K_m. Thus the complement is a genus two handlebody which does not contain incompressible tori. It cannot contain curves which are isotopic to C on P because in this case C can be isotoped onto T where, as T is peripheral, C is either a meridian of K_m or some other curve on T, in which case C is knotted. Both are contradictions. Hence $P \cap T$ is a collection of curves each of which are concentric around one or the other of the two components $\{p_1, p_2\} = \partial P \smallsetminus C$.

Since K_m and C are on the same side of T the intersection must be an even number of concentric curves around each of p_1 and p_2. If there are two or more concentric curves around the same point consider an innermost such pair. They bound annuli A_1 on T and A_2 on P such that the interior of A_2 lies "outside" T. Since T is peripheral the "outside" of T is homeomorphic to the exterior $S^3 - K_m$ which is hyperbolic by Proposition 4.3. Hence A_2 is boundary parallel and there is an isotopy reducing the intersection between P and T. This contradicts the minimality of the intersection $P \cap T$. Hence $P \cap T = \emptyset$ and this is a contradiction as above.

If T is compressible in $S^3 - N(K_m)$ then T compresses for infinitely many $\frac{1}{n}$-surgeries on C. Since $S^3 - N(L(7,17))$ is irreducible then so is $M' = S^3 - N(L(7,17)) - N(T)$. We can now apply Theorem 2.4.4 of [2], which states that in this case either the intersection between the slopes $\Delta(\frac{1}{10m-4}, \frac{1}{10k-4}) \leqslant 1$, $m \neq k$ which is clearly false or that M_1 the component of M' which contains $\partial N(C)$ is homeomorphic to $T^2 \times I$ which is also clearly false since then T would be peripheral to C. The last possibility is that M_1 is a cable space, i.e., it is homeomorphic to the complement of some (p,q)-cable, $p, q \in \mathbb{Z}$, $g.d.c.(p,q) = 1$, $q \geqslant 2$, about the core of a solid torus. In particular M_1 has two boundary components which means that T separates C from $T(7,17)$. Choose T to minimize the intersection $P \cap T$.

The intersection $P \cap T$ cannot be empty as then T would not separate, and it cannot contain inessential curves on P as this would violate either the minimality of the intersection or the choice of T as essential.

Since $P \cap T$ is minimal and $P \cap M_1 \neq \emptyset$ we can assume that $P \cap M_1$ is comprised only of essential annuli in M_1. In particular this implies that $P \cap T$ contains curves parallel to C and hence C is isotopic into T. Note also that T must be unknotted in S^3 since it compresses to the C side after $\frac{1}{0}$-filling on C and also compresses to the $T(7,17)$ side after $\frac{1}{0}$-filling on $T(7,17)$. As C itself is an unknot then after the isotopy onto T it must be either a $(p,1)$ or a $(1,q)$ curve with respect to T. That is, C is a longitude, i.e., it meets a meridian disk for T either on the C side or on the $T(7,17)$ exactly once. However M_1 is a non-trivial cable space hence C cannot meet such a meridian disk on the M_1 side. Thus C must meet a meridian disk for T in a single point on the $T(7,17)$ side. However all the curves of intersection in $P \cap T$ are parallel and there is at least one (the innermost curve) which bounds a twice punctured disk meridian disk on the $T(7,17)$ side, hence C is a meridian there and not a longitude in contradiction. This finishes the proof and we conclude that $S^3 - N(L(7,17))$ is atoroidal. $\square$

Lemma 4.11. *The link $L(7, 17) = T(7, 17) \cup C \subset S^3$ is an-annular.*

Proof. Suppose now that $S^3 - N(L(7, 17))$ contains an incompressible annulus A.

As above, the manifold $S^3 - N(L(7, 17))$ is not a Seifert fibered space over an annulus with a single exceptional fiber. If it was then $\frac{1}{5m-2}$-surgery on the boundary component corresponding to C would yield a Seifert fibered space in contradiction to the fact that K_m is hyperbolic by Proposition 4.3.

There are two possible cases, A joins either distinct components of $\partial(S^3 - N(L(7, 17)))$ or the same one:

If there was an annulus A in $S^3 - N(L(7, 17))$ with one boundary component on $T(7, 17)$ and the other on C, then a regular neighborhood $N(T(7, 17) \cup A \cup C)$ has a torus boundary component T' which is different from $\partial N(C)$ or $\partial N(T(7, 17))$. The torus T' contains a solid torus V on the side away from C and $T(7, 17)$ as $S^3 - N(L(7, 17))$ is atoroidal by Lemma 4.10. Hence $S^3 - N(L(7, 17))$ is the solid torus V glued to itself along A which is a Seifert fibered space over an annulus with a single exceptional fiber, in contradiction.

Assume that $S^3 - N(L(7, 17))$ contains an annulus A with both boundary components on $T(7, 17)$. The boundary of $N(T(7, 17) \cup A)$ contains two tori T' and T'' both different from $\partial N(T(7, 17))$. As $S^3 - N(L(7, 17))$ is atoroidal each of T' and T'' either bounds a solid torus or is peripheral. Neither are peripheral into $T(7, 17)$ and they cannot both bound solid tori as C must be somewhere. Hence one, say T', is peripheral into C. This means that there is an annulus between C and $T(7, 17)$, in contradiction. The case where both boundary components of A are on C is identical. $\quad\square$

Corollary 4.12. *The link $L(7, 17) = T(7, 17) \cup C \subset S^3$ is hyperbolic.*

Proof. Since $S^3 - N(L(7, 17))$ is irreducible, atoroidal and an-annular it follows from Thurston's hyperbolization theorem that it is hyperbolic. $\quad\square$

Lemma 4.13. *Suppose $L = K \cup C$ is a two component tunnel number one link in S^3 with unknotting tunnel τ. Assume that C is the unknot in S^3. Then the Heegaard splitting of $S^3 - N(K)$ induced by τ is μ-primitive.*

Proof. Since C is unknotted then $C \cup \tau$ is a tunnel for K. Furthermore, the complement of $N(K \cup \tau)$ is a genus two handlebody inside the solid torus $V = S^3 - N(C)$ so $(K \cup \tau)$ defines a genus two Heegaard splitting for V. This Heegaard splitting is standard by Casson–Gordon (see [1]). The tunnel system $K \cup \tau$ is a genus two splitting of the solid torus $S^3 - N(C)$, hence it is stabilized. Thus the genus two compression body $(\partial C \times I) \cup \tau \cup K$ contains a non-separating disk D which meets a disk D' for the complementary handlebody in $S^3 - N(C)$ once. But, D must be the cocore of K, because it is the unique non-separating disk in the compression body $(\partial C \times I) \cup \tau \cup K$. Together the meridional annulus $D - N(K)$ and D' demonstrate that K is μ-primitive. $\quad\square$

We are now ready to prove the theorem:

Proof of Theorem 4.9. With the link $L = T(7, 17) \cup C$ there is an associated set of surgeries on C which yield manifolds containing Heegaard surfaces which are not Heegaard splittings for the link exterior. This set is *simple* as in Definition 0.5 of [8]: A subset of $\mathbb{Z} \oplus \mathbb{Z}$ is *very simple* if it is a union of a finite subset $A \subset \mathbb{Z} \oplus \mathbb{Z}$ and a subset of the form $\alpha + n\beta$, $n \in \mathbb{Z}$, where $\{\alpha, \beta\}$ is some basis for $\mathbb{Z} \oplus \mathbb{Z}$. A set is *simple* if it is a finite union of very simple sets.

Let (μ, λ) be the "natural" meridian-longitude pair for $H_1(\partial N(C))$. Consider the "line" $\mathcal{L}_0$ of surgeries containing the slopes $\frac{1}{5m-2}$ (with respect to (μ, λ)). It is precisely the set of slopes that meet the curve of slope 0 once. The intersection of $\mathcal{L}_0$ with the simple set is contained in some ball unless $\mathcal{L}_0$ coincides with one of the lines in the simple set. In this case it is contained in a ball union $\mathcal{L}_0$. Choose $m_0 \in \mathbb{Z}$ bigger in absolute value then the radius of that ball. Set $m \in \mathbb{Z}$ such that $|m| > m_0$.

Let Σ be a genus two Heegaard surface which separates $S^3 - N(K_m)$ into two compression bodies W_1 and W_2 with $\partial N(K) = \partial_- W_1$. By [8, Theorem 0.1] since $L = T(7, 17) \cup C$ is a hyperbolic link and K_m is obtained by surgery on C we can assume that we can isotope C into Σ.

By Theorem 5.1 and Remark 5.3 of [14] the curve C is a core of W_1 or W_2 or the surface $\Sigma - N(C)$ is either incompressible or $\Sigma - N(C)$ compresses to an essential surface. Since Σ is of genus two the latter case would imply that the essential surface is an annulus. This contradicts the fact that K_m is hyperbolic. In the incompressible case the slope of $\partial(\Sigma - N(C))$ determines some line $\mathcal{L}$ in the simple set containing the slope $\frac{1}{5m-2}$. But the slope $\frac{1}{5m-2}$ is in the line $\mathcal{L}_0$. Hence by the choice of m, the lines $\mathcal{L}$ and $\mathcal{L}_0$ coincide. Thus we conclude that the slope of $\partial(\Sigma - N(C))$ is 0. We now have two cases:

(i) The curve C is not isotopic to a core in either compression body, i.e., as above $\Sigma - N(C)$ is essential and furthermore $\partial(\Sigma - N(C))$ is of slope 0 on C.

(ii) The surface Σ is a Heegaard surface for $S^3 - L$, i.e., C is a core in (a) W_1 or (b) W_2.

(I) Assume that the curve C is a core in W_1 (i.e., case (ii)(a)). In this case we satisfy the conditions of Lemma 4.13. We conclude that K_m is μ-primitive which is a contradiction.

(II) Assume that we are in case (i) or case (ii)(b). The pair of pants P in $S^3 - N(K_m)$ which is bounded by the curve C also has slope 0 with respect to C. In these cases the following conditions on C, P and Σ can be satisfied:

(1) There is an embedded annulus A_C between Σ and C meeting $\partial N(C)$ in a curve of slope 0.
(2) Every curve in the intersection $P \cap \Sigma$ is essential in both surfaces.

If C is a core in W_2 then the existence of A_C is obvious and condition (2) is satisfied by Lemma 6 of [18] which guarantees that a strongly irreducible Heegaard surface can be isotoped to meet a properly embedded incompressible surface in essential curves in both.

In case (i) the surfaces $\Sigma - N(C)$ and P are essential so condition (2) is automatically satisfied. If we push C slightly into W_1 or W_2 we satisfy condition (1) because $\Sigma - N(C)$ has slope 0 on $\partial N(C)$.

Choose P and C to minimize the intersection with Σ subject to satisfying conditions (1) and (2). Thus we can assume that $P \cap \Sigma$ is composed of simple closed curves and no arcs. We can assume further that the intersection $P \cap (\Sigma - N(C))$ does not contain curves isotopic to C. The curve C is isotopic to an innermost such curve which satisfies conditions (1) and (2) so that the resulting P has fewer intersections with $\Sigma - N(C)$. We deduce therefore, that we only have simple closed curves concentric around p_1 or p_2 the boundary components of P which are not equal to C.

If there are two or more such concentric curves around p_1 or p_2 then there is a pair such that together they cobound an incompressible annulus $A \subset P$. An innermost such annulus is contained in the handlebody W_2 and is therefore boundary compressible. Let $\partial A = \{\alpha_1, \alpha_2\}$, with α_1 being the interior curve on P, it bounds a vertical annulus in W_1 whose other boundary curve is a meridian for K_m. If A is boundary parallel then either it can be eliminated, thus reducing the intersection, or C is contained in the solid torus determined by the boundary parallelism. In this case C is parallel, using condition (1), on Σ to α_2 and hence to α_1 and thus to a meridian of K_m.

Assume that A is not boundary parallel (the boundary compressing disk D for A may meet C). Now boundary compressing A in Σ gives an essential disk for Σ disjoint from α_1. Hence the knot K_m is weakly γ-primitive. This contradicts Theorem 4.8.

We are left with the possibility that around each of p_1 and p_2 there is at most one curve of intersection of $(\Sigma - N(C)) \cap P$. Since Σ separates and both punctures are in W_1 there must be a single curve of intersection around each puncture.

We conclude therefore that there are exactly two curves of intersection, α_1 around p_1 and α_2 around p_2. This implies that there is an incompressible pair of pants $P' \subset P$ properly embedded in $W_2 - N(C)$ so that $\partial P' = \{\alpha_1, \alpha_2, C\}$. Choose A_C that minimizes the intersection $P' \cap A_C$. There are no arcs of intersection with end points on C as both A_C and P' have slope 0. There are no inessential arcs of intersection on P' with end points on α_1 or α_2. As all the arcs of intersection are inessential on A_C we can cut and paste to create another A_C with fewer intersections. Hence all arcs are essential in P'. Choose an outermost arc β in A_C. Boundary compress P' along that outermost sub-disk in A_C. If the arc β joins α_1 to α_2 then the intersection of P and Σ is reduced by one. If β joins α_1 to itself then boundary compressing will initially increase the number of curves in $P \cap \Sigma$ but create a concentric annulus which can be eliminated as above. This reduces the total number of curves of intersection.

So we conclude that there are no concentric (α type) curves around p_1 or p_2. Using the annulus A_C push the curve C onto Σ. That is $P \cap \Sigma = C$.

An application of the following lemma finishes the proof of the theorem:

Lemma 4.14. *If $P \cap (\Sigma - N(C)) = \emptyset$ then Σ is isotopic to the middle Heegaard splitting Σ_{mid} of $S^3 - N(K_m)$.*

Proof. The unknotted curve $C \subset S^3$ bounds a disk $\Delta \subset S^3$ which contains the pair of pants P so that $C = \partial \Delta \subset \partial P$. The disk Δ is a compressing disk for the solid torus $V = S^3 - N(C)$. If we cut V along Δ we obtain a 2-tangle (B, T) with two marked disks Δ_1 and Δ_2 each containing two points corresponding to $K_m \cap \Delta$.

Let Σ be a genus two Heegaard splitting for $S^3 - N(K_m)$. We can assume by the above discussion that the curve C is contained in Σ as a non-separating curve. Thus cutting V along Δ induces a cutting of Σ along C. Hence the tangle (B, T) contains the twice punctured torus $\Sigma - N(C)$ so that $\partial(\Sigma - N(C)) = \partial \Delta_1 \cup \partial \Delta_2$.

The pair of pants P is boundary compressible in V, as the only non-boundary compressible surfaces in a compression body are vertical annuli and disks. After boundary compressing P we get two vertical annuli with boundary curves C_1 and C_2 on Σ. These vertical annuli are contained in respective disks δ_1 and δ_2 in the component of $B - (\Sigma - N(C))$ which contain the strings t_1, t_2 of T. Note that $\delta_i \cap t_i$ is a single point for each $i = 1, 2$. The set of curves $\{C, C_1, C_2\}$ determines a pair of pants decomposition P_1, P_2 for Σ.

Claim 4.15. *The pair of pants P_i, $i = 1, 2$, is isotopic in $B - N(T)$ to $\Delta_i - N(T)$, $i = 1, 2$.*

Proof. From the construction it follows immediately that one of the pair of pants, say P_2, is isotopic in $B - N(T)$ to P and in particular to $\Delta_2 - N(T)$.

Consider $\Delta_1 \cup (P_1 \cup \delta_1 \cup \delta_2) \subset B$. It is a four times punctured 2-sphere $\widehat{S} \subset (S^3, K_m)$. The sphere $\widehat{S}$ decomposes K_m into two 2-tangles (B_1, T_1) and (B_2, T_2) where $K_m = T_1 + T_2$. Since K_m is a tunnel number one knot it follows from [15] that K_m is doubly prime so it does not contain a Conway sphere and hence $\widehat{S}$ is compressible in $S^3 - N(K_m)$ and in particular it is compressible in either (B_1, T_1) or (B_2, T_2). Note that the compressing disk must separate the strings. This implies by [20]

that at least one of the tangles, say (B_1, T_1), is a rational tangle. If the compressing disk for $\widehat{S}$ meets Δ_1 in a single arc, i.e., (B_1, T_1), is an integer tangle in the terminology of [19], we are done as then we can use the compression disk to guide the isotopy between $(P_1 \cup \delta_1 \cup \delta_2)$ and P. So assume it does not, i.e., the rational tangle is not an integer tangle.

Note that K_m can be decomposed into a non-trivial sum of two 2-tangles as above if and only the underlying torus knot $T(7, 17)$ can. Now consider the cabling annulus $A \subset S^3 - N(T(7, 17))$. When we remove $N(C)$ from $S^3 - N(T(7, 17))$ to obtain $S^3 - N(L(7, 17)$ A is punctured twice. Denote this twice punctured annulus by $\widehat{A}$. The intersection $P \cap \widehat{A}$ consists of three arcs one an essential arc of A and the other two arcs run between ∂A and $\partial N(C)$.

When we cut the solid torus $V = S^3 - N(C)$ along P, $\widehat{A}$ is cut as well. The result is a disk $A' = \widehat{A} - P$. Note that $A' \subset B$. Note also that there are sub-arcs of $\partial A'$ that are on the two strands of the tangle $T \subset B$.

If we choose A that minimizes the intersection $\widehat{S} \cap A$ then the intersection $\widehat{S} \cap A'$ cannot contain simple closed curves. This follows since A does not contain simple closed curves which are homologous to a sum of meridians of $T(7, 17)$ and inessential simple closed curves on $A \cap \widehat{S}$ also bound disks on $\widehat{S}$ and hence can be eliminated since B is irreducible. Thus $\widehat{S} \cap A$ is a collection of arcs and hence $A' \smallsetminus \widehat{S}$ is a collection of disk components.

As the curves C_1 and C_2 are meridional curves and $\widehat{S}$ is embedded, it follows that of all the arcs in $\widehat{S} \cap A'$ exactly two arcs, one on P_1 and one on Δ_1, run between the two different meridional curves on P_1 and Δ_1, respectively. Hence one of the above disk components, say A'', is contained in (B_1, T_1) and runs between the strands of T_1. This is a contradiction as since T_1 is a non-integer rational tangle (B_1, T_1) cannot contain such a disk A''. It follows that T_1 is an integer tangle and P_1 is isotopic into P. $\square$

We claim, however, that in fact P_i, $i = 1, 2$, is isotopic in $B - N(T)$ to $\Delta_i - N(T)$, $i = 1, 2$, respectively. As if say, P_1 is isotopic to Δ_2 then the two pairs of pants P_1 and P_2 are parallel in the rational tangle (B_2, T_2), as above, which must contain P_2 as a sub-disk. Hence Σ would be a genus two surface determining a handlebody component which contains two meridional curves as cores. If this surface is a Heegaard surface then the fundamental group of $E(T(7, 17))$ can be generated by two elements represented by meridians. This contradicts the classification of generating systems of these groups (see [6]).

Hence there is a unique way to tube two copies of P so that the resulting surface is disjoint from Δ. Thus the construction of any twice punctured torus in (B, T) is unique up to isotopy in $B - N(T)$ and are all isotopic to the twice punctured torus since $\Sigma_{mid} - N(C)$. Thus Σ is isotopic to Σ_{mid} and the proof of the lemma is complete. $\square$

This finishes the proof of the theorem. $\square$

In order to prove the uniqueness of the minimal Heegaard splitting in case (I), we used the fact that $K_m = T(7, 17, 2, 10m - 4)$ is not μ-primitive. D. Heath and H.-J. Song prove in [4] that the pretzel knot $P(-2, 3, 7)$ has four non-isotopic tunnels. It is well known that it is μ-primitive. Hence the following conjecture seems plausible:

Conjecture 4.16. *All knot exteriors $E(K)$, where $K = T(p, q, 2, r)$ and K is not μ-primitive, have a unique (minimal) genus two Heegaard splitting.*

5. Boundary stabilization and reducibility

In this section we show how the results above can be put together to obtain candidates for manifolds with a non-minimal genus, weakly reducible and non-stabilized Heegaard splittings.

It is a generally accepted rule amongst those doing research on Heegaard splittings that Heegaard splittings of small genus are easier to handle than those of large genus. Furthermore since we are dealing with questions of *reducibility* there is an advantage to dealing with Heegaard splittings of manifolds with boundary. Having a boundary implies that the Heegaard splitting is composed from either one or two compression bodies. The possibilities for disks inside compression bodies are more restricted then those for handlebodies of the same genus, hence deciding whether a reducing pair of disks exists or not might be more tractable.

Since we are trying to prove a negative, i.e., that a Heegaard splitting is not stabilized, we are forced into a proof by contradiction. Hence the argument can be expected to follow, more or less, the following theme:

Let M be a 3-manifold of genus g. Assume that M has a weakly reducible Heegaard splitting which is stabilized of genus $g + n$, $n \geqslant 1$. Destabilize it to obtain an irreducible Heegaard splitting and somehow obtain a contradiction. If we can find a manifold which has a unique minimal Heegaard splitting and a weakly reducible Heegaard splitting of genus $g + 1$ we would have the additional option of getting a contradiction by showing that the surface we obtain after the destabilization cannot possibly be isotopic to the unique minimal genus Heegaard surface.

To sum up, we are looking for preferably, a tunnel number one knot $K \subset S^3$ so that $E(K)$ has a genus three weakly reducible Heegaard splitting and a unique genus two Heegaard splitting. An obvious place to look for weakly reducible Heegaard splittings is Heegaard splittings which are *amalgamated*.

Consider now the exterior $E(K_m)$ for a knot $K_m = T(7, 17, 10m - 4)$. It has a unique minimal Heegaard splitting (V, W), where V is the compression body, of genus two. Boundary stabilize (V, W) by amalgamating (V, W) with the standard

genus two Heegaard splitting of a collar of $\partial_- V$, which is just $T^2 \times I$ (see [17]). This operation is defined and discussed in detail in [9, Definitions 2.2 and 2.3]. We obtain a weakly reducible genus three Heegaard splitting for $E(K_m)$. By Theorem 4.6 of [9] if a Heegaard splitting of a knot exterior $E(K)$ is γ-primitive for any curve $\gamma \subset \partial E(K)$ then the boundary stabilized Heegaard splitting is a stabilization, i.e., the boundary stabilized Heegaard splitting contains a reducing pair of disks. However, K_m is not μ-primitive by [11] and not γ-primitive ($\gamma \neq \mu$) by Theorem 4.4. Hence the obvious ways for the boundary stabilized Heegaard splitting (V', W') of $E(K_m)$ to be stabilized fail. We state:

Conjecture 5.1. *The boundary stabilized genus three Heegaard splitting (V', W') of the unique minimal genus two Heegaard splitting (V, W) of $E(K_m)$, where $K_m = T(7, 17, 2, 10m - 4)$, is non-stabilized.*

Remark 5.2. If we assume in contradiction that (V', W') is indeed stabilized, then the surface Σ'' obtained by destabilizing the Heegaard surface $\Sigma' = \partial_+ V' = \partial_+ W'$ is ambient isotopic to the Heegaard surface $\Sigma = \partial_+ V = \partial_+ W$.

There is an additional benefit for proving Conjecture 5.1:

It is a well-known theorem of Casson–Gordon (see [1]) that if a *closed* irreducible orientable 3-manifold has a weakly reducible Heegaard splitting then it is Haken. It is a natural question whether this theorem can be extended to manifolds with boundary. In [16] the second author gave the first examples of manifolds with three or more boundary components which have weakly reducible and non-stabilized minimal genus Heegaard splittings so that when the Heegaard surface is weakly reduced the surface obtained is non-essential. This result was improved by the authors, in [9], to manifolds with just two boundary components. It is still an open question if such an example exists for manifolds with a single boundary component.

Thus, Conjecture 5.1 would rule out the possible extension of the Casson–Gordon theorem [1] to manifolds with a single boundary component as follows:

Since (V', W') is of genus three and is weakly reducible, then after weakly reducing we can obtain either an essential 2-sphere or an essential torus. We cannot have an essential 2-sphere in a knot space as they are $K(\pi_1, 1)$'s. Since $E(K_m)$ is hyperbolic by Proposition 4.3 any incompressible torus must be boundary parallel. This rules out the possible extension of a "Casson–Gordon" theorem to manifolds with a single boundary component.

Acknowledgements

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