The Proximate Type and $\lambda$-Proximate Type of an Entire Dirichlet Series with Index-Pair $(p, q)$

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The notion of proximate type for an entire Dirichlet series with index-pair $(p, q)$ is introduced and its existence is proved. In order to study the lower proximate type, the idea has been extended to the case of entire functions of irregular $(p, q)$-growth. The proximate type for a class of entire Dirichlet series under certain conditions is constructed.

Introduction

Let $f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n)$ ($s = \sigma + it$, $0 < \lambda_1 < \lambda_2 < \lambda_3 + \cdots \to \infty$ as $n \to \infty$) be an entire Dirichlet series. Set $M(\sigma) = \max_{\|s\| \leq \sigma} |f(s)|$; $M(\sigma)$ is called the maximum modulus of $f(s)$.

The following notations have been frequently used: $\exp^{[0]} x = \log^{[0]} x = x$; $\exp^{[m]} x = \log^{[m]} x = \exp(\exp^{[m-1]} x)$, $m = \pm 1, \pm 2, \ldots$, and $A_{[q]}(x) = \prod_{k=1}^{q} \log^{[1]} x$.

The concept of $(p, q)$-order and lower $(p, q)$-order of an entire Dirichlet series having index-pair $(p, q)$, $p > q + 1 > 1$, has been recently introduced by Juneja et al. [1]. Thus $f(s)$ is said to be of $(p, q)$-order $\rho$ and lower $(p, q)$-order $\lambda$ if it is of index-pair $(p, q)$ and

$$\lim_{\sigma \to \infty} \sup_{\sigma \leq \sigma} \frac{\log^{[p]} M(\sigma)}{\log^{[q]} \sigma} = \rho(p, q) = \rho$$

and

$$\lim_{\sigma \to \infty} \inf_{\sigma \leq \sigma} \frac{\log^{[q]} M(\sigma)}{\log^{[p]} \sigma} = \lambda(p, q) = \lambda.$$

Definition 1. An entire function $f(s)$ for which $(p, q)$-order and lower $(p, q)$-order are the same is said to be of regular $(p, q)$-growth. Functions which are not of regular $(p, q)$-growth are called of irregular $(p, q)$-growth.

Juneja et al. [2] also defined $(p, q)$-type and lower $(p, q)$-type as follows:

Definition 2. An entire function $f(s)$ having $(p, q)$-order $\rho$ ($b < \rho < \infty$) is said to be of $(p, q)$-type $\tau$ and lower $(p, q)$ type $\nu$ if

$$\nu = \lim_{\sigma \to \infty} \inf_{\sigma \leq \sigma} \frac{\log^{[q]} M(\sigma)}{\log^{[p]} \sigma} = \lambda.$$
DEFINITION 3. An entire function \( f(s) \) of regular \((p, q)\)-growth is said to be of perfectly regular \((p, q)\)-growth if \( \nu = \tau < \infty \).

In studying the growth of an entire Dirichlet series with index-pair \((p, q)\) and \((p, q)\)-order \( \rho \) \((b < \rho < \infty)\), use is made of a comparison function \( \rho(\sigma) \) defined on \((a, 0)\), \( a > 0 \), called the \((p, q)\)-proximate order, or simply proximate order of \( f(s) \), which possesses the following properties:

\[
\rho(\sigma) \text{ is real, continuous and piecewise differentiable for } \sigma > \sigma_0, \quad (0.3)
\]

\[
\rho(\sigma) \to \rho \quad \text{as} \quad \sigma \to \infty \quad (0.4)
\]

\[
M_1(\sigma) \rho'(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty. \quad (0.5)
\]

where \( \rho'(\sigma) \) is either the right or left hand derivative at points where these are different, and

\[
\lim_{\sigma \to \tau} \sup_{\sigma} \frac{\log^{p-1} M(\sigma)}{(\log^{q-1} \sigma)^\rho} = 1. \quad (0.6)
\]

It is evident that \( \rho(\sigma) \) has been linked with the \((p, q)\)-order and \( M(\sigma) \) to give information about the growth of \( f(s) \). Since the proximate order \( \rho(\sigma) \) is not linked with \((p, q)\)-type \( \tau \), it becomes a natural question to the existence of another function which should take into account the \((p, q)\)-type of the function and is closely related with its maximum modulus. In analogy with the proximate order we call this function \( \tau(\sigma) \) as a \((p, q)\)-proximate type or simply proximate type of \( f(s) \) with \((p, q)\).

Here, we define proximate type for an entire Dirichlet series with index-pair \((p, q)\) and then prove its existence. The idea is further extended by defining the \( \lambda \)-proximate type and also establishing its existence. In the end, we show that \( \log M(\sigma) \cdots M(\sigma) / (\log M(\sigma)) \) is a proximate type for a class of entire Dirichlet series.

DEFINITION 4. A real valued function \( \tau(\sigma) \) is said to be a proximate type of an entire function \( f(s) \) with index-pair \((p, q)\), \((p, q)\)-order \( \rho \) \((b < \rho < \infty)\) and \((p, q)\)-type \( \tau \) \((0 < \tau < \infty)\), if, for a given a \((0 < a < \infty)\), \( \tau(\sigma) \) satisfies the following properties:
\[ \tau(\sigma) \text{ is continuous and piecewise differentiable for } \sigma > \sigma_0, \quad (1.1) \]

\[ \tau(\tau) \to \tau \quad \text{as } \sigma \to \infty, \quad (1.2) \]

\[ A_{|q-1|}(\sigma) \tau'(\sigma) \to 0 \quad \text{as } \sigma \to \infty, \quad (1.3) \]

where \( \tau'(\sigma) \) can be interpreted as either \( \tau'(\sigma^-) \) or \( \tau'(\sigma^+) \) when these are unequal, and

\[ \lim \sup_{\sigma \to \infty} \log^{[p-1]}M(\sigma) = a. \quad (1.4) \]

To establish the existence of \((p, q)\)-proximate type for \( f(s) \) we prove

**Theorem 1.** For every entire Dirichlet series \( f(s) = \sum_{n=1}^\infty \sigma_n \exp(ns) \) having index-pair \((p, q)\), \((p, q)\)-order \( \rho \) \((0 < \rho < \infty)\) and \((p, q)\)-type \( \tau \) \((0 < \tau < \infty)\), there exists a \((p, q)\)-proximate type satisfying conditions (1.1) through (1.4).

**Proof.** For a given constant \( a \) \((0 < a < \infty)\), let

\[ S(\sigma) = \log \frac{|log^{[p-1]}M(\sigma)|}{(log^{[q-1]}f)^\rho}. \]

Then in view of (0.2) we have \( \lim \sup_{\sigma \to \infty} S(\sigma) = \tau \). Now two cases arise:

(A) \( S(\sigma) > \tau \) for a sequence of values of \( \sigma \) tending to infinity, or

(B) \( S(\sigma) \leq \tau \) for all large \( \sigma \).

**Case (A).** We set \( Q(\sigma) = \max_{x > \sigma} S(x) \). Since \( S(x) \) is continuous, \( \lim \sup_{x \to \infty} S(x) = \tau \) and \( S(x) > \tau \) for a sequence \( \{x_n\} \) tending to infinity, so \( Q(\sigma) \) exists and is a nonincreasing function, and

\[ \lim_{\sigma \to x} Q(\sigma) = \tau. \quad (1.5) \]

Let \( \sigma_1 \) be a number such that \( \sigma_1 > \exp^{1\sigma_1} \) and \( Q(\sigma_1) = S(\sigma_1) \). Such values will exist for a sequence of values of \( \sigma \) tending to infinity. Next, suppose \( \tau(\sigma_1) = Q(\sigma_1) \) and let \( u_1 \) be the smallest integer not less than \( 1 + \sigma_1 \) such that \( Q(\sigma_1) > Q(u_1) \) and let \( \tau(\sigma) = \tau((\sigma_1) = Q(\sigma_1) \) for \( \sigma_1 < \sigma < u_1 \). Define \( v_1 \) as

\[ \tau(\sigma) > Q(\sigma), \text{ for } u_1 \leq \sigma < v_1. \]
Let \( a_2 \) be the smallest value of \( u \) for which \( u^2 > u \), and \( Q(a_2) = S(a_2) \). If \( a_2 > v_1 \) then, let \( \tau(\sigma) = Q(\sigma) \) for \( v_1 \leq \sigma \leq a_2 \). Since \( Q(\sigma) \) is constant for \( v_1 \leq \sigma \leq a_2 \), therefore \( \tau(\sigma) \) is constant for \( v_1 \leq \sigma \leq a_2 \). We repeat the argument and obtain that \( \tau(\sigma) \) is continuous and differentiable in adjacent intervals for \( \sigma > a_1 \) and so satisfies (1.1). Further,

\[
\tau'(\sigma) = 0 \quad \text{or} \quad \tau'(\sigma) = (-1/A_{aq}(\sigma))
\]

which implies that \( \lim_{\sigma \to \infty} A_{aq-1}(\sigma) \tau'(\sigma) = 0 \). Again, \( \tau(\sigma) \geq Q(\sigma) \geq S(\sigma) \) \( \forall \sigma > a_1 \) and \( \tau(\sigma) = S(\sigma) \) for an infinity values \( \sigma = a_1, a_2, \ldots \), and \( \tau(\sigma) \) is nonincreasing, hence using (1.5) we have \( \lim_{\sigma \to \infty} \tau(\sigma) = \tau \). Thus \( \tau(\sigma) \) satisfies (1.2). Further, since

\[
M(\sigma) = \exp^{p-2}[\alpha \exp\{\log^q \sigma S(\sigma)\}]
\]

for an infinity of \( \sigma \).

\[
M(\sigma) < \exp^{p-2}[\alpha \exp\{\log^q \sigma \tau(\sigma)\}]
\]

for the remaining \( \sigma \).

hence (1.4) is immediate.

Case (B). Let \( S(\sigma) \leq \tau \ \forall \sigma > a_0 \); hence there are again two possibilities.

(B.1) \( S(\sigma) = \tau \) for a sequence of values of \( \sigma \).

(B.2) \( S(\sigma) < \tau \) for all large values of \( \sigma \).

In Case (B.1); we take \( \tau(\sigma) - \tau \ \forall \sigma \). In Case (B.2); we set \( P(\sigma) = \max_{x < \sigma \leq 0} S(x) \), where \( X > \exp^q \) is such that \( S(x) < \tau \ \forall x \geq X \). Since \( P(\sigma) \) chosen in this way is nondecreasing and hence \( \lim_{\sigma \to \infty} P(\sigma) = \tau \).

Take a suitable value of \( a_1 > X \), let us suppose

\[
\tau(\sigma_1) = \tau,
\]

\[
\tau(\sigma) = \tau + \log^q \sigma_1 - \log^q \sigma_1 \quad \text{for} \quad s_1 \leq \sigma \leq \sigma_1,
\]

where \( s_1 (\leq \sigma_1) \) is such that \( P(s_1) = \tau(s_1) \). If \( P(s_1) \neq S(s_1) \), then we take \( \tau(\sigma) = P(\sigma) \) up to the nearest point \( t_1 (\leq s_1) \) at which \( P(t_1) = S(t_1) \). Such values will exist for a sequence of values of \( \sigma \) tending to infinity. \( \tau(\sigma) \) is then constant for \( t_1 \leq \sigma \leq s_1 \). If \( P(s_1) = S(s_1) \), then let \( t_1 = s_1 \), choose \( \sigma_2 (\geq \sigma_1) \) suitably large and let

\[
\tau(\sigma_2) = \tau,
\]

\[
\tau(\sigma) = \tau + \log^q \sigma_2 - \log^q \sigma_2 \quad \text{for} \quad s_2 \leq \sigma \leq \sigma_2,
\]

where \( s_2 (\leq \sigma_2) \) is such that \( P(s_2) = \tau(s_2) \). If \( P(s_2) \neq S(s_2) \), then let \( \tau(\sigma) = P(\sigma) \)
for \( t_2 \leq \sigma \leq s_2 \), \( t_2 (\leq s_2) \) is the point nearest to \( s_2 \) at which \( P(t_2) = S(t_2) \). If \( P(s_2) = S(s_2) \), then let \( t_2 = s_2 \). For \( \sigma < t_2 \), let
\[
\tau(\sigma) = \tau(t_2) + \log^{q+1} t_2 - \log^{q+1} \sigma, \quad x_1 \leq \sigma \leq t_2,
\]
where \( x_1 \) (\( \leq t_2 \)) is the point of intersection of \( y = \tau \) with \( y = \tau(t_2) + \log^{q+1} t_2 - \log^{q+1} \sigma \). Let \( \tau(\sigma) = \tau \) for \( \sigma, t_2 \). It is always possible to choose \( \sigma_2 \) so large that \( \sigma_1 < x_1 \). We repeat the procedure and note that \( \tau(\sigma) \) is continuous and differentiable in adjacent intervals for \( \sigma > t_1 \). Also,
\[
\tau'(\sigma) = 0 \quad \text{or} \quad (1/A_{q-1}(\sigma)) \quad \text{or} \quad (-1/A_{q-1}(\sigma)).
\]
Hence, we have \( \lim_{\sigma \to -\infty} A_{q-1}(\sigma) \tau'(\sigma) = 0 \). Further \( \tau(\sigma) \geq P(\sigma) \geq S(\sigma) \) for \( \sigma \geq t_1 \) and \( \tau(\sigma) = S(\sigma) \) for \( \sigma = t_1, t_2, \ldots \), hence (1.2) and (1.4) follow.

Thus in each case \( \tau(\sigma) \) satisfies (1.1) through (1.4). So it is a proximate type for the entire function \( f(s) \) with index-pair \((p, q)\).

2

It is easy to see that the lower \((p, q)\)-type of an entire Dirichlet series of irregular \((p, q)\)-growth must necessarily be zero. Therefore, the case \( \nu(p, q) > 0 \) is only limited to the study of entire Dirichlet series of regular \((p, q)\)-growth. In such cases, we can similarly define lower \((p, q)\)-proximate type. But, to be more general, let \( \lambda(p, q) \) be such that \( b < \lambda < p < \infty \) and define
\[
\lim_{\sigma \to -\infty} \inf \frac{\log^{p-1} M(\sigma)}{(\log^{q-1} \sigma)^{\lambda}} = \nu_{\lambda}. \tag{2.1}
\]
We call \( \nu_{\lambda} \) the \( \lambda(p, q) \)-type of the entire function \( f(s) \). If \( \rho = \lambda \), then \( \nu_{\lambda} \) is then same as lower \((p, q)\)-type \( \nu \). There exist entire functions for which \( \nu_{\lambda} \) is nonzero and finite. For such functions, we define \( \lambda(p, q) \)-proximate type or simply \( \lambda \)-proximate type as follows:

**Definition 5.** A real valued positive function \( \nu_{\lambda}(\sigma) \) defined on \([a, \infty)\), \( a > 0 \), is said to be a \( \lambda \)-proximate type of an entire Dirichlet series having index-pair \((p, q)\), \((p, q)\)-order \( \rho \), lower \((p, q)\)-order \( \lambda \) (\( b < \lambda \leq \rho < \infty \)) and \( \lambda(p, q) \)-type \( \nu_{\lambda} \) (\( 0 < \nu_{\lambda} < \infty \)) if for a given \( a \) (\( 0 < a < \infty \)), \( \nu_{\lambda}(\sigma) \) has the following properties:

- \( \nu_{\lambda}(\sigma) \) is continuous and piecewise differentiable for \( \sigma > \sigma_0 \). \tag{2.2}
- \( \nu_{\lambda}(\sigma) \to \nu_{\lambda} \quad \text{as} \quad \sigma \to \infty \) \tag{2.3}
- \( A_{q-1}(\sigma) \nu_{\lambda}'(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty \). \tag{2.4}
where \( v'_1(\sigma) \) can be interpreted as either right or left hand derivatives at points where these are different, and

\[
\lim_{\sigma \to \tau} \inf \frac{\log^p \log^{2} M(\sigma)}{\exp\{(\log^{q-1}\sigma)^a v'_1(\sigma)\}} = a. \tag{2.5}
\]

**Theorem 2.** For every entire function \( f(s) \) of \((p, q)-order \rho, \) lower \((p, q)-order \lambda \) \((b < \lambda \leq \rho < \infty)\) and \((p, q)-type \nu_\lambda \) \((0 < \nu_\lambda < \infty)\), there exist a \( \lambda \)-proximate type \( v'_\lambda(\sigma) \) satisfying (2.2) through (2.5).

**Proof.** The proof proceeds exactly on the lines of Theorem 1 for the case \( \tau(\sigma) \) and hence we omit it.

### 3

Now, we construct a proximate type for a class of entire Dirichlet series and for this we prove

**Lemma.** If \( f(s) \) is an entire function of \((p, q)-order \rho \) \((b < \rho < \infty)\) such that

\[
\lim_{\sigma \to \tau} \inf \frac{M'(\sigma) A_{1q} - 2\nu_1(\sigma)}{A_{1p} - 2\nu_1(M(\sigma))(\log^{q-1} \sigma)^{\rho - 1}} = \gamma \tag{3.1}
\]

then

\[
\delta \leq \rho \nu \leq \rho \tau \leq \gamma, \tag{3.2}
\]

where \( M'(\sigma) \) is the derivative of maximum modulus \( M(\sigma) \).

**Proof.** From (3.1), for given \( \varepsilon > 0 \) and \( \sigma > \sigma_\varepsilon \), we have

\[
\delta - \varepsilon < \frac{M'(\sigma) A_{1q} - 2\nu_1(\sigma)}{A_{1p} - 2\nu_1(M(\sigma))(\log^{q-1} \sigma)^{\rho - 1}} < \gamma + \varepsilon
\]

or,

\[
(\delta - \varepsilon) \frac{(\log^{q-1} \sigma)^{\rho - 1}}{A_{1q-2\nu_1}(\sigma)} < \frac{M'(\sigma)}{A_{1p-2\nu_1}(M(\sigma))} < (\gamma + \varepsilon) \frac{(\log^{q-1} \sigma)^{\rho - 1}}{A_{1q-2\nu_1}(\sigma)}.
\]

Integrating the above inequalities between the suitable limits and dividing by \((\log^{q-1} \sigma)^p\) and then proceeding to limits, we at once have (3.2).

**Theorem 3.** Let \( f(s) \) be an entire function defined by a Dirichlet series of \((p, q)-order \rho(b < \rho < \infty)\) and \((p, q)-type \tau(0 < \tau < \infty)\). If limit in (3.1) exists, then \( \log^{q-1}(\log^{p-2} M(\sigma))^{p} \) is a proximate type of \( f(s) \).
Proof. Let
\[
\tau(\sigma) = \frac{\log \left( u^{-1} \log^{[p-1]} M(\sigma) \right)}{(\log^{[q-1]} \sigma)^\rho}.
\] (3.3)

Since \( \log^{[p-1]} M(\sigma) \) is a real, continuous and increasing function of \( \sigma \), which is differentiable in adjacent intervals, it follows that \( \tau(\sigma) \) satisfies (1.1).

The limit in (3.1) exists by assumption and hence (3.2) shows that \( f(s) \) is of perfectly regular \((p, q)\)-growth. Moreover \( \tau(\sigma) \to \tau \) as \( \sigma \to \infty \).

Further \( \tau(\sigma) \) is piecewise differentiable and from (3.3) we observed that
\[
\tau'(\sigma)(\log^{[q-1]} \sigma)^\rho + \frac{\rho \tau(\sigma)(\log^{[q-1]} \sigma)^{\rho-1}}{A_{[q-1]}(\sigma)} = \frac{M'(\sigma)}{A_{[p-2]}(M(\sigma))}.
\]

Hence,
\[
\lim_{\sigma \to \infty} A_{[q-1]}(\sigma) \tau'(\sigma) = \lim_{\sigma \to \infty} \left\{ \frac{M'(\sigma)A_{[q-2]}(\sigma)}{A_{[p-2]}(M(\sigma))(\log^{[q-1]} \sigma)^{\rho-1}} - \rho \tau(\sigma) \right\}.
\]

On using the hypothesis of Theorem 3 and the Lemma, we get \( \lim_{\sigma \to \infty} A_{[q-1]}(\sigma) \tau'(\sigma) = 0 \). Thus \( \tau(\sigma) \) satisfies (1.3) also.

Finally, (1.4) follows immediately from (3.3) and hence the theorem.

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