Nash-Type Equilibrium Theorems and Competitive Nash-Type Equilibrium Theorems

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Abstract—In this paper, we consider two types of equilibrium problems. We study the constrained Nash-type equilibrium problems with multivalued payoff functions. We also study the competitive Nash-type equilibrium problems with multivalued payoff functions. In these two equilibrium problems, we want to find a strategy combination such that each player wishes to find a minimal loss from his multivalued payoff function. We use a fixed-point theorem of Park to prove the existence results of these two types of equilibrium problems. © 2002 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Recently, many authors have studied the existence of solutions of Pareto equilibria of constrained (also for nonconstrained) multiobjective game with vector payoff functions, for example, see [1-4] and references therein. In all the previous results, the payoff functions are single-valued functions. But in some cases, for each strategy, the payoff may be one collection of things from many collections of things. Hence, the study of a vector equilibrium problem with multivalued payoff functions has some applications in the real world.

In this paper, we want to study two types of equilibrium problems with multivalued payoff functions.

First, we consider the constrained Nash-type equilibrium problem for multifunction of Type I, \( \Delta = (X_i, S_i, F_i)_{i \in I} \).

In this problem, let \( I \) be a finite or infinite index set and \( X_i \) be a topological space for each \( i \in I \), \( X = \prod_{i \in I} X_i \) and \( X^i = \prod_{j \in I, j \neq i} X_j \). For each \( x \in X \), let \( x_i \) and \( x^i \) denote the \( i \)-th coordinate of \( x \) and the projection of \( x \) on \( X^i \), respectively. In the sequel, we may write \( x = (x^i, x_i) \). For each \( i \in I \), let \( Z_i \) be a real topological vector space with proper closed convex solid cone \( C_i \), \( F_i : X^i \times X_i \to \partial Z_i \), \( S_i : X^i \to X_i \). One of our purposes is to study the problem of finding \( \bar{x} = (\bar{x})_{i \in I} \in X \) \( z_i \in F_i(\bar{x}^i, \bar{x}_i) \) such that for each \( i \in I \), \( \bar{x}_i \in S_i(\bar{x}^i) \), and

\[
    z_i - \bar{x}_i \notin -\text{int} C_i, \quad \text{for all} \ z_i \in F_i(\bar{x}^i, \bar{x}_i) \ \text{and all} \ u_i \in S_i(\bar{x}^i).
\]

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If for each \( i \in I, F_i \) is a single-valued function, \( Z_i = R \) and \( S_i(X^i) = X_i \), our problem is to find \( \bar{x} = (\bar{x}_i)_{i \in I} \in X \), such that for each \( i \in I, \bar{x}_i \in S_i(\bar{x}) \), and

\[
F_i(\bar{x}, \bar{x}_i) \leq F_i(\bar{x}, y_i), \quad \text{for all } y_i \in X_i.
\]

Then this problem reduces to the Nash equilibrium problem [5]. If for each \( i \in I, F_i \) is a single-valued function and \( Z_i = R \), then this problem reduces to the Debreu social equilibrium problem [6].

We also study the constrained competitive Nash-type equilibrium problems \( \Delta' = (X_i, Y_j, S_j, T_i, F_j, G_i)_{i \in I, j \in J} \).

In this problem, \( I \) and \( J \) denote the finite index sets, \( X_i \) and \( Y_j \) denote the strategy sets of the \( i^{th} \) and \( j^{th} \) players of these two families (say Families A and B) of players, respectively; \( S_j : X \rightarrow \omega Y_j \) is the constrained correspondence which restricts the strategy of the \( i^{th} \) player of Family B to the subset \( S_j(X) \) of \( Y_j \) when all the players of Family A have chosen their strategies \( x_i \in X_i \) \( i \in I \) and \( T_i : Y \rightarrow \omega X_i \) is the constrained correspondence which restricts the strategy of the \( i^{th} \) player of Family A to the subset \( T_i(Y) \) of \( X_i \) when all the players of Family B have chosen their strategies \( y_j \in Y_j \) \( j \in J \); and \( F_j : X \times Y_j \rightarrow \omega Z_j \) and \( G_i : X_i \times Y \rightarrow \omega Z_i \) are the payoff functions of the \( j^{th} \) player of Family A and the gain function of the \( i^{th} \) player of Family B, respectively, where \( Z_i \) and \( Z_j' \) are real topological vector spaces. In the constrained multiobjective game \( \Delta' \), we are interested in finding a strategies combination \( (\bar{x}, \bar{y}) \in X \times Y \) (called the constrained competitive Nash-type equilibria of the game) such that for each \( i \in I \) and \( j \in J \), \( \bar{y}_j \in S_j(\bar{x}) \) and \( \bar{x}_i \in T_i(\bar{y}) \) and there exist \( \tilde{z}_j \in F_j(\bar{x}, \bar{y}_j) \) and \( \tilde{a}_i \in G_i(\bar{x}_i, \bar{y}) \) satisfying the following properties:

\[
z_j - \tilde{z}_j \notin \text{int} C_j, \quad \text{for all } z_j \in F_j(\bar{x}, u_i), \ u_j \in S_j(\bar{x}), \ \text{and all } j \in J,
\]

and

\[
a_i - \tilde{a}_i \notin \text{int} C_i, \quad \text{for all } a_i \in G_i(v_i, \bar{y}), \ v_i \in T_i(\bar{y}), \ \text{and all } i \in I.
\]

This problem has applications in the real world.

We now demonstrate an example of this kind of equilibrium problem in real life. Let \( I \) denote the index set of factories in Company A and \( J \) denote the index set of factories in Company B. We assume that the products between the factories in the same company are different, while some collections of products are the same and some collections of products are different between different factories in different companies. Therefore, the strategy of the \( j^{th} \) factory in Company A depends on the strategies of all factories in Company B. The payoff function \( F_j \) of the \( j^{th} \) factory in Company A depends on its strategy and all the strategies of factories in Company B. Similarly, the strategy of the \( k^{th} \) factory in Company B depends on its strategy and all the strategies of factories in Company A. With this strategies combination, each factory can choose a collection of products which minimize the loss or gain of each factory.

If \( F_j \) and \( G_i \) are single real-valued functions for all \( i \in I \) and \( j \in J \), then the second problem reduces to the problem of finding \( \bar{x} = (\bar{x}_i)_{i \in I} \in \prod_{i \in I} X_i, \ \bar{y} = (\bar{y}_j)_{j \in J} \in \prod_{j \in J} Y_j \) such that

\[
F_j(\bar{x}, \bar{y}_j) \leq F_j(\bar{x}, y_j), \quad \text{for all } y_j \in S_j(\bar{x}) \ \text{and all } j \in J,
\]

and

\[
G_i(\bar{x}_i, \bar{y}) \geq G_i(x_i, \bar{y}), \quad \text{for all } x_i \in T_i(\bar{y}) \ \text{and all } i \in I.
\]

The payoff (or gain) functions of these two types of equilibrium problems are vector multimaps. The main tool in this paper is a fixed-point theorem of Park [7].
2. PRELIMINARIES

Let $X$, $Y$, and $Z$ be nonempty sets. A multimap (or map) $T : X \to Y$ is a function from $X$ into the power set of $Y$ with nonempty values. Let $x \in T^{-1}(y)$ if and only if $y \in T(x)$. Let $A \subset X$, and we define $T(A) = \bigcup \{ T(x) \mid x \in A \}$.

For topological spaces $X$ and $Y$, a multimap $T : X \to Y$ is said to be upper semicontinuous (in short, u.s.c.) if for every $x \in X$ and every open set $U$ in $Y$ with $T(x) \subset U$, there exists an open neighborhood $U(x)$ of $x$ such that $T(x') \subset U$ for all $x' \in U(x)$; lower semicontinuous (in short, l.s.c.) if for every $x \in X$ and every open set $V$ in $Y$ with $T(x) \cap V \neq \emptyset$, there exists an open neighborhood $V(x)$ of $x$ such that $T(x') \cap V \neq \emptyset$ for all $x' \in V(x)$; continuous if it is u.s.c. and l.s.c.; closed if its graph $\text{Gr}(T) = \{(x, y) \mid x \in X, y \in T(x)\}$ is closed in $X \times Y$; and compact if there exists a compact subset $K \subset Y$ such that $T(X) \subset K$. Moreover, let $A \subset X$, and we denote $\text{int} A$ and $\overline{A}$ to be the interior of $A$ and the closure of $A$, respectively.

A nonempty topological space is acyclic if all of its reduced Čech homology groups vanish. Note that any convex subset, starshaped set of a topological vector space are contractible and any contractible set is acyclic.

**Definition 1.** Let $Z$ be a real t.v.s. with a convex solid cone $C$ (i.e., $\text{int} C \neq \emptyset$), and $A$ be a nonempty subset of $Z$. Let $y_1, y_2 \in A$.

(i) We denote

$$y_1 \leq y_2, \quad \text{if } y_2 - y_1 \in C,$$

and

$$y_1 < y_2, \quad \text{if } y_2 - y_1 \in \text{int} C.$$

(ii) A point $\bar{y} \in A$ is called a vector minimal point (respectively, weakly vector minimal point) of $A$ if for any $y \in A$, $y - \bar{y} \notin -C \setminus \{0\}$ (respectively, $y - \bar{y} \notin -\text{int} C$). Moreover, the set of vector minimal points (respectively, weakly vector minimal points) of $A$ is denoted by $\text{Min}_C A$ (respectively, $\text{wMin}_C A$).

**Definition 2.** (See [8].) Let $X$ be a nonempty convex subset of a real t.v.s. $E$, $Z$ a real t.v.s. with a convex cone $C$ such that $C \neq Z$. Let $G : X \to Z$ be a multimap. Then $G$ is said to be $C$-quasi-convex if for any $z \in Z$, the set

$$\{ x \in X \mid \text{there is a } y \in G(x) \text{ such that } z - y \in C \}$$

is convex.

**Lemma 1.** (See [9].) Let $X$ and $Y$ be two nonempty subsets of topological spaces $E_1$ and $E_2$, respectively, $Z$ a real t.v.s. with a pointed closed convex solid cone $C$, $S : X \to Y$, and $F : X \times Y \to Z$. Let $m : X \to Z$ be defined by

$$m(x) = \text{wMin}_C F(x, S(x)), \quad \text{for } x \in X,$$

and $M : X \to Z$ be defined by

$$M(x) = \{ u \in S(x) \mid F(x, u) \cap m(x) \neq \emptyset \}.$$

If both $F$ and $S$ are compact continuous multimaps with closed values, then both $m$ and $M$ are closed compact u.s.c. multimaps.

**Lemma 2.** (See [10].) Let $X$ and $Y$ be two topological spaces and $T : X \to Y$ be a multimap.

(a) If $X$ is compact and $T$ is u.s.c. with compact values, then $T(X)$ is compact.

(b) If $Y$ is compact and $T$ is closed, then $T$ is u.s.c.

(c) If $T$ is a u.s.c. multimap with closed values, then $T$ is closed.
LEMMA 3. (See [7].) Let \( X \) be a nonempty compact convex subset of a locally convex t.v.s. \( E \), and \( F : X \to \mathcal{O}X \) be a u.s.c. multimap with nonempty closed acyclic values. Then \( F \) has at least one fixed point, i.e., there exists an \( \bar{x} \in X \) such that \( \bar{x} \in F(\bar{x}) \).

DEFINITION 3. (See [11].) A convex space \( X \) is a nonempty convex set in a vector space with any topology that induces the Euclidean topology on the convex hull of its finite subsets.

LEMMA 4. (See [12].) Let \( X \) be a paracompact space, \( Y \) a convex space, \( G, T : X \to \mathcal{O}Y \) two multimaps such that for each \( x \in X \), \( \co G(x) \subset T(x) \) and \( X = \bigcap_{y \in Y} \operatorname{int} G^{-1}(y) \). Then \( T \) has a continuous selection \( f : X \to Y \), that is, \( f : X \to Y \) is a continuous function such that \( f(x) \in T(x) \) for each \( x \in X \).

LEMMA 5. (See [13].) Let \( X \) be a convex subset of topological vector space and \( Z \) be a topological vector space with a closed convex cone \( C \) such that \( \operatorname{int} C \neq \emptyset \). Let \( F : X \to \mathcal{O}Z \) be a multimap.

\begin{enumerate}
\item For any fixed \( e \in \operatorname{int} C \) and any fixed \( a \in Z \), then the function \( \xi : Z \to \mathbb{R} \) defined by
\[ \xi(y) = \min \{ t \in \mathbb{R} : y \in a + te - C \} \]
for \( y \in Z \), is a continuous and strictly monotonically increasing function from \( Z \) to \( \mathbb{R} \), that is, \( \xi(a) > \xi(b) \) if \( a - b \in \operatorname{int} C \).
\item If \( F \) is \( C \)-quasiconvex, then the composition multimap \( \xi F : X \to \mathcal{O} \mathbb{R} \) is \( \mathbb{R}^+ \)-quasiconvex.
\end{enumerate}

LEMMA 6. (See [13].) Let \( \xi \) be a continuous function from a topological space \( Z \) to \( \mathbb{R} \) and \( F \) be a multimap from a topological space \( X \) to \( Z \).

\begin{enumerate}
\item If \( F \) is u.s.c., then \( \xi F : X \to \mathcal{O} \mathbb{R} \) is u.s.c.
\item If \( F \) is l.s.c., then \( \xi F : X \to \mathcal{O} \mathbb{R} \) is l.s.c.
\end{enumerate}

LEMMA 7. (See [14].) Let \( A \) be a nonempty compact subset of a real t.v.s., \( C \) a closed convex cone in \( Z \) such that \( C \neq \emptyset \), then \( \operatorname{Min} C A \neq \emptyset \).

Throughout this paper, all topological spaces are assumed to be Hausdorff.

3. NASH-TYPE EQUILIBRIUM THEOREMS

THEOREM 3.1. Let \( I \) be a finite index set and \( \{E_i\}_{i \in I} \) be a family of locally convex t.v.s. For each \( i \in I \), let \( X_i \) be a compact convex subset of \( E_i \), \( S_i : X_i \to \mathcal{O}X_i \) a continuous multimap with nonempty closed values, \( Z_i \) a real t.v.s. with a proper closed convex solid cone \( C_i \) which induces an order on \( Z_i \). Suppose that for each \( i \in I \), \( F_i : X_i \times X_i \to \mathcal{O}Z_i \) is a continuous multimap with compact values and for each \( x_i \in X_i \), \( M_i(x_i) \) is an acyclic set, where \( M_i : X_i \to \mathcal{O}X_i \) is defined by
\[ M_i(x_i) = \{ u_i \in S_i(x_i) \mid F_i(x_i, u_i) \cap \operatorname{wMin} C_i F_i(x_i) \neq \emptyset \} \]
for \( x_i \in X_i \). Then there exists an \( \bar{x} = (\bar{x}_i)_{i \in I} \in X \), \( \bar{z}_i \in F_i(\bar{x}_i, \bar{x}_i) \) such that for each \( i \in I \), \( \bar{x}_i \in S_i(\bar{x}_i) \), and
\[ z_i - \bar{z}_i \notin \operatorname{int} C_i, \quad \text{for all } z_i \in F_i(\bar{x}_i, u_i) \text{ and all } u_i \in S_i(\bar{x}_i) \].

PROOF. Since for each \( i \in I \), \( F_i \) and \( S_i \) are continuous multimaps with compact values and \( X_i \) is compact, it follows from Lemma 2 that \( F_i(X) \) and \( S_i(X_i) \) are compact and \( F_i \) and \( S_i \) are compact. Since for each \( i \in I \) and \( x^i \in X_i \), \( S_i(x^i) \) is compact and \( F_i \) is a u.s.c. multimap with compact values. Again, by Lemma 2, it follows that \( F_i(x^i, S_i(x^i)) \) is a compact set. It follows from Lemma 7, \( \emptyset \neq \operatorname{Min} C_i F_i(x^i, S_i(x^i)) \subset \operatorname{wMin} C_i F_i(x_i, S_i(x_i)) \). Therefore, \( M_i(x_i) \neq \emptyset \). By Lemma 1, for each \( i \in I \), \( M_i : X_i \to \mathcal{O}X_i \) is a closed compact u.s.c. multimap. By assumption, for each \( i \in I \), \( M_i \) is a u.s.c. multimap with compact acyclic values. Now we define \( M : X \to \mathcal{O}X \) by \( M(x) = \prod_{i \in I} M_i(x_i) \) for \( x \in X \). Then by Kunneth formula (see [15]) and Lemma 3 of [16], \( M \) is
also a u.s.c. multimap with compact acyclic values. Since $X$ is compact, it follows from Lemma 3 that there exists an $\bar{x} \in X$ such that $\bar{x} \in M(\bar{x})$. That is, for each $i \in I$, $\bar{x}_i \in M_i(\bar{x}_i)$. In other words, for each $i \in I$,

$$\bar{x}_i \in S_i(\bar{x}_i) \quad \text{and} \quad F_i(\bar{x}_i, \bar{x}_i) \cap \text{wMin}_{C_i} F_i(x_i, S_i(x_i)) \neq \emptyset.$$ 

This implies that, for each $i \in I$, there exists a $z_i \in F_i(\bar{x}_i, \bar{x}_i)$ such that

$$z_i - \bar{x}_i \notin \text{int} C_i, \quad \text{for all } z_i \in F_i(\bar{x}_i, u_i) \text{ and all } u_i \in S_i(\bar{x}_i).$$

**Remark.** In Theorem 3.1, if we assume that $I$ is any index set and for each $x_i \in X_i$, $M_i(x_i)$ is a convex set, then we have the same conclusion.

As a simple consequence of Theorem 3.1 and Lemma 4, we have the following theorem.

**Theorem 3.2.** Let $I$ be a finite index set and $\{E_i\}_{i \in I}$ be a family of locally convex t.v.s. and $\{V_i\}_{i \in I}$ be a family of t.v.s. For each $i \in I$, let $X_i$ be a compact convex subset of $E_i$, $Y_i$ a convex subset of $V_i$, $S_i : X_i \to \partial X_i$ a continuous multimap with nonempty closed values, $Z_i$ a real t.v.s. with a proper closed convex solid cone $C_i$ which induces an order on $Z_i$. For each $i \in I$, let $T_i : X_i \to \partial Y_i$ be a multimap with convex values and $X_i = \bigcup_{y_i \in Y_i} T_i^{-1}(y_i)$, $F_i : X_i \times Y_i \times X_i \to \partial Z_i$ a continuous multimap with compact values. Moreover, for each $i \in I$, and for each $(x_i, y_i) \in X_i \times Y_i$, $M_i(x_i, y_i)$ is an acyclic set, where $M_i : X_i \times Y_i \to \partial X_i$ is defined by

$$M_i(x_i, y_i) = \{u_i \in S_i(x_i) \mid F_i(x_i, y_i, u_i) \cap \text{wMin}_{C_i} F_i(x_i, y_i, S_i(x_i)) \neq \emptyset\},$$

for $(x_i, y_i) \in X_i \times Y_i$. Then there exist $\bar{x}_i = (\bar{x}_i)_{i \in I} \in X$ and $\bar{y}_i = (\bar{y}_i)_{i \in I} \in Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}_i)$, $\bar{y}_i \in T_i(\bar{x}_i)$, and there exists a $\bar{z}_i \in F_i(\bar{x}_i, \bar{y}_i, \bar{x}_i)$ satisfying

$$z_i - \bar{z}_i \notin \text{int} C_i, \quad \text{for all } z_i \in F_i(\bar{x}_i, \bar{y}_i, u_i) \text{ and all } u_i \in S_i(\bar{x}_i).$$

In particular, if for each $i \in I$, $F_i(x_i, y_i, x_i) \subset C_i$, for all $(x, y) \in X \times Y$, then $z_i \notin \text{int} C_i$, for all $z_i \in F_i(\bar{x}_i, \bar{y}_i, u_i)$ and all $u_i \in S_i(\bar{x}_i)$.

**Proof.** By Lemma 4, for each $i \in I$, $T_i$ has a continuous selection $f_i : X_i \to Y_i$. For each $i \in I$, we define two multimaps $G_i : X_i \times X_i \to \partial Z_i$ and $H_i : X_i \to \partial X_i$ as follows:

$$G_i(x_i, u_i) = F_i(x_i, f_i(x_i), u_i), \quad \text{for each } (x_i, u_i) \in X_i \times X_i,$$

$$H_i(x_i) = M_i(x_i, f_i(x_i)), \quad \text{for each } x_i \in X_i.$$

Then we have

$$H_i(x_i) = \{u_i \in S_i(x_i) \mid F_i(x_i, f_i(x_i), u_i) \cap \text{wMin}_{C_i} F_i(x_i, f_i(x_i), S_i(x_i)) \neq \emptyset\} = \{u_i \in S_i(x_i) \mid G_i(x_i, u_i) \cap \text{wMin}_{C_i} G_i(x_i, S_i(x_i)) \neq \emptyset\}. $$

For each $i \in I$, since $F_i$ is a continuous multimap with compact values, $G_i$ is a continuous multimap with compact values. Then by Theorem 3.1, there exists an $\bar{x} \in X$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}_i)$ and there exists a $\bar{z}_i \in G_i(\bar{x}_i, \bar{x}_i)$ satisfying

$$z_i - \bar{z}_i \notin \text{int} C_i, \quad \text{for all } z_i \in G_i(\bar{x}_i, u_i) \text{ and all } u_i \in S_i(\bar{x}_i).$$

For each $i \in I$, if we let $\bar{y}_i = f_i(\bar{x}_i)$, then $\bar{x}_i \in S_i(\bar{x}_i)$, $\bar{y}_i = f_i(\bar{x}_i) \in T_i(\bar{x}_i)$, and

$$z_i - \bar{z}_i \notin \text{int} C_i, \quad \text{for all } z_i \in F_i(\bar{x}_i, \bar{y}_i, u_i) \text{ and all } u_i \in S_i(\bar{x}_i).$$

This completes the proof of the theorem.
REMARK. In Theorem 3.1, if for each \( i \in I \), \( F_i \) is a single-valued function, \( Z_i = \mathbb{R} \) and 
\[ S_i(x_i) = x_i \]
and for each \( x_i \in X_i \), \( u_i = F_i(x_i, u_i) \) is convex, then Theorem 3.1 reduces to the Nash equilibrium theorem [5]. If for each \( i \in I \), \( F_i \) is a single-valued function, \( Z_i = \mathbb{R} \) and for each \( x_i \in X_i \), \( u_i = F_i(x_i, u_i) \) is convex, then Theorem 3.1 reduces to the Debreu social equilibrium theorem [6].

As a simple consequence of Theorem 3.1, we have the weight Nash-type equilibrium theorem.

**THEOREM 3.3.** Let \( I \) be a finite index set and for each \( i \in I \), \( X_i \) be a compact convex subset of a locally convex t.v.s. \( E_i \), \( S_i : X_i \rightarrow \mathbb{R}^i \) be a continuous multimap with closed values, and \( F_i : X \rightarrow \mathbb{R}^k \). Suppose that there exists a weight vector \( W = \sum_i w_i \) with \( w_i \in \mathbb{R}^k \setminus \{0\} \) such that for each \( i \in I \), we have the following.

(i) The function \( G_i : X_i \times X_i \rightarrow \mathbb{R} \) defined by \( G_i(x) = W_i \circ F_i(x) \) is a continuous multimap with compact values, where \( W_i \circ F_i(x) = \{ z_i \in F_i(x) \} \) and \( W_i \circ z_i \) denotes the inner product of \( W_i \) and \( z_i \).

(ii) For each \( x_i \in X_i \), the set
\[
\{ u_i \in S_i(x_i) \mid \min G_i(x_i, S_i(x_i)) \in G_i(x_i, u_i) \}
\]
is acyclic.

Then there exist \( \bar{x} = (\bar{x}_i)_{i \in I} \in X \) and \( \bar{z}_i \in F_i(\bar{x}_i, \bar{x}_i) \) such that for each \( i \in I \), \( \bar{z}_i \in S_i(\bar{x}_i) \), and
\[
W_i \circ z_i \geq W_i \circ \bar{z}_i, \quad \text{for all } z_i \in F_i(\bar{x}_i, u_i) \text{ and } u_i \in S_i(\bar{x}_i).
\]

**PROOF.** The conclusion of Theorem 3.3 follows from Theorem 3.1 by letting \( Z_i = \mathbb{R} \) and \( C_i = [0, 1] \) in Theorem 3.1 for all \( i \in I \).

**REMARK.** In Theorem 3.3, Condition (ii) can be replaced by (ii)', where

(ii)' for each fixed \( x_i \in X_i \), the function \( u_i \circ F_i(x_i, u_i) \) is \( \mathbb{R}^k \)-convex.

The following theorem gives a characterization of \( C \)-quasiconvexity of a multimap.

**THEOREM 3.4.** Let \( X \) be a nonempty convex subset of a real t.v.s. \( E \), \( Z \) be a real t.v.s. with a convex cone \( C \) such that \( C \neq Z \) and \( G : X \rightarrow Z \) be a multimap, then \( G \) is \( C \)-quasiconvex if and only if for any \( x_1, x_2 \in X \), \( \lambda \in [0, 1] \), \( z_1 \in G(x_1) \), and \( z_2 \in G(x_2) \), there exists a \( y \in G(\lambda x_1 + (1 - \lambda) x_2) \) such that \( y \in z - C \) for all \( z \in u(z_1, z_2) \), where \( u(z_1, z_2) \) is the set of all upper bound of \( z_1 \) and \( z_2 \), that is,
\[
u(z_1, z_2) = \{ z \in Z : z_1 \in z - C, z_2 \in z - C \}.
\]

**PROOF.** Suppose \( G \) is \( C \)-quasiconvex. Then for all \( z \in Z \), the set \( B(z) = \{ x \in X : \text{there is a } y \in G(x) \text{ such that } y \in z \} \) is convex. Let \( x_1, x_2 \in X \), \( \lambda \in [0, 1] \), \( z_1 \in G(x_1) \), and \( z_2 \in G(x_2) \). Let \( z \in u(z_1, z_2) \). Then \( z \geq z_1 \) and \( z \geq z_2 \). Hence, \( x_1, x_2 \in B(z) \). By assumption, \( B(z) \) is convex and \( \lambda x_1 + (1 - \lambda) x_2 \in B(z) \). Therefore, there exists a \( y \in G(\lambda x_1 + (1 - \lambda) x_2) \) such that \( y \in z \). This shows that \( y \in z - C \) for all \( z \in u(z_1, z_2) \).

Conversely, if for any \( x_1, x_2 \in X \), \( \lambda \in [0, 1] \), \( z_1 \in G(x_1) \), and \( z_2 \in G(x_2) \), there exists a \( y \in G(\lambda x_1 + (1 - \lambda) x_2) \) such that \( y \in z - C \) for all \( z \in u(z_1, z_2) \). Let \( z \in C \) and \( u_1, u_2 \in B(z) \), then there exist \( w_1 \in G(u_1) \) and \( w_2 \in G(u_2) \) such that \( w_1 \in z - C, w_2 \in z - C \). Then \( z \in u(w_1, w_2) \). By assumption, there exists a \( y \in G(\lambda u_1 + (1 - \lambda) u_2) \) such that \( y \leq z \). This shows that \( \lambda u_1 + (1 - \lambda) u_2 \in B(z) \) and \( B(z) \) is convex for all \( z \in Z \). Hence, \( G \) is \( C \)-quasiconvex.

As a simple consequence of Theorems 3.1 and 3.4, we have the following lemma.

**LEMMA 3.5.** Let \( I \) be an index set and \( \{ E_i \}_{i \in I} \) be a family of locally convex t.v.s. For each \( i \in I \), let \( X_i \) be a compact convex subset of \( E_i \) and \( S_i : X_i \rightarrow \mathbb{R}^i \) a continuous multimap with nonempty closed convex values. Suppose that for each \( i \in I \), \( F_i : X_i \times X_i \rightarrow \mathbb{R} \) is a continuous
multimap with compact values and for each \( x^i \in X^i, u_i - \circ F_i(x^i, u_i) \) is \( R_+ \)-quasiconvex. Then there exists \( \bar{x} = (\bar{x}_i)_{i \in I} \in X, \bar{z}_i \in F_i(\bar{x}^i, \bar{z}_i) \) such that \( \bar{z}_i \in S_i(\bar{x}^i) \) and
\[
z_i - \bar{z}_i \geq 0, \quad \text{for all } z_i \in F_i(\bar{x}^i, u_i), \ u_i \in S_i(\bar{x}^i), \ 	ext{and all } i \in I.
\]

**Proof.** It suffices to show that for each \( x^i \in X^i \), the set
\[
M_i(x^i) = \{ u_i \in S_i(x^i) \mid \text{Min}_{\bar{z}_i \in S_i(x^i)} F_i(x^i, u_i) \} \subset M_i(x_i)
\]
is convex. Indeed, let \( u_i, u'_i \in M_i(x^i), 0 \leq \lambda \leq 1 \). Then \( u_i, u'_i \in S_i(x^i) \), and there exist \( w_i \in F_i(x^i, u_i) \) and \( w'_i \in F_i(x^i, u'_i) \) such that \( u_i = w_i = \text{Min}_{\bar{z}_i \in S_i(x^i)} F_i(x^i, u_i) \). Since for each \( x^i \in X^i \), \( u_i - \circ F_i(x^i, u_i) \) is \( R_+ \)-quasiconvex, it follows from Theorem 3.4, there exists a \( v_i \in F_i(x^i, \lambda u_i + (1 - \lambda)u'_i) \) such that \( v_i \leq \text{Max}_{\bar{z}_i \in S_i(x^i)} F_i(x^i, u_i) \). Since \( \lambda u_i + (1 - \lambda)u'_i \in S_i(x^i) \) and \( v_i \in F_i(x^i, S_i(x^i)) \). Therefore, \( v_i = \text{Min}_{\bar{z}_i \in S_i(x^i)} F_i(x^i, u_i) \). This shows that \( \lambda u_i + (1 - \lambda)u'_i \in M_i(x^i) \) and \( M_i(x^i) \) is a convex set for each \( x^i \in X^i \). Then the conclusion of Lemma 3.5 follows from Theorem 3.1.

As a consequence of Lemma 3.5, we have the following theorem.

**Theorem 3.6.** Let \( I \) be an index set and \( \{E_i\}_{i \in I} \) be a family of locally convex t.v.s. For each \( i \in I, \) let \( X_i \) be a compact convex subset of \( E_i, \ S_i : X^i \to \circ X_i \) a continuous multimap with nonempty closed values, \( Z_i \) a real t.v.s. with a proper closed convex solid cone \( C_i \) which induces an order on \( Z_i \). Suppose that for each \( i \in I, \) let \( F_i : X^i \times X_i \to \circ Z_i \) is a continuous multimap with compact values and for each \( x^i \in X^i, u_i - \circ F_i(x^i, u_i) \) is \( C_i \)-quasiconvex. Then there exist an \( \bar{z} = (\bar{z}_i)_{i \in I} \in X, \bar{z}_i \in F_i(\bar{x}^i, \bar{z}_i) \) such that for each \( i \in I, \bar{z}_i \in S_i(\bar{x}^i), \) and
\[
z_i - \bar{z}_i \notin - \text{Int } C_i, \quad \text{for all } z_i \in F_i(\bar{x}^i, u_i) \text{ and all } u_i \in S_i(\bar{x}^i).
\]

**Proof.** For any fixed \( e_i \in \text{Int } C_i \) and any \( a_i \in Z_i, \) let
\[
\xi_i(y) = \min \{ t \in \mathbb{R} : y \in a_i + te_i - C_i \}, \quad \text{for } y \in Z_i.
\]
Then it is easy to see from Lemma 5 that for each \( i \in I, \xi_i F_i : X^i \times X_i - \circ \mathbb{R} \) is a continuous multimap with compact values and for each \( x^i \in X^i, u_i - \circ \xi_i F_i(x^i, u_i) \) is \( R_+ \)-quasiconvex. Then it follows from Lemma 3.5 that there exist an \( \bar{z} = (\bar{z}_i)_{i \in I} \in X \) and \( \bar{z}_i \in \xi_i F_i(\bar{x}^i, \bar{z}_i) \) such that \( \bar{z}_i \in S_i(\bar{x}^i) \) and
\[
\xi_i z_i - \xi_i \bar{z}_i \geq 0, \quad \text{for all } z_i \in F_i(\bar{x}^i, u_i), \ u_i \in S_i(\bar{x}^i), \ 	ext{and all } i \in I.
\]

It follows from Lemma 5 that \( \xi_i \) is a strictly monotonically increasing function from \( Z_i \) to \( \mathbb{R} \). Therefore,
\[
z_i - \bar{z}_i \notin - \text{Int } C_i, \quad \text{for all } z_i \in F_i(\bar{x}^i, u_i) \text{ and all } u_i \in S_i(\bar{x}^i), \ 	ext{and all } i \in I.
\]

### 4. VECTOR COMPETITIVE NASH-TYPE EQUILIBRIUM THEOREM

**Theorem 4.1.** Let \( I \) and \( J \) be finite index sets, \( \{E_i\}_{i \in I} \) and \( \{V_j\}_{j \in J} \) be families of locally convex t.v.s. For each \( i \in I \) and \( j \in J \), let \( X_i \) be a compact convex subset of \( E_i \) and \( Y_j \) be a compact convex subset of \( V_j \). \( Z_i \) and \( Z_j \) be real t.v.s. with proper closed convex solid cones \( C_i \) and \( C_j \), respectively. Let \( Z_i \) and \( Z'_j \) be ordered by \( C_i \) and \( C'_j \), respectively, \( S_i : = X = \prod_{i \in I} X_i - \circ Y_j \) a continuous multimap with nonempty closed values, \( T_i : Y = \prod_{i \in I} Y_j - \circ X_i \) a continuous multimap with nonempty closed values. Suppose that for each \( i \in I \) and \( j \in J \), \( F_j : X \times Y_j - \circ Z'_j \),
and \( G_i : X_i \times Y \rightarrow \sigma Z_i \) are continuous multimaps with compact values. Moreover, for each \( i \in I \) and \( j \in J \), and for each \( x \in X \) and \( y \in Y \), \( M_j(x) \) and \( H_i(y) \) are acyclic sets, where \( M_j : X \rightarrow oY \) and \( H_i : Y \rightarrow oX \) are defined by

\[
M_j(x) = \{ y_j \in S_j(x) \mid F_j(x, y_j) \cap \text{wMin}_{Y_j} F_j(x, S_j(x)) \neq \emptyset \}
\]

and

\[
H_i(y) = \{ x_i \in T_i(y) \mid G_i(x_i, y) \cap \text{wMax}_{X_i} G_i(T_i(y), y) \neq \emptyset \},
\]

for \( x \in X \) and \( y \in Y \).

Then there exist \( \bar{x} = (\bar{x}_i)_{i \in I} \in X \), \( \bar{y} = (\bar{y}_j)_{j \in J} \in Y \), \( \bar{x}_i \in T_i(\bar{y}) \), \( \bar{y}_j \in S_j(\bar{x}) \), \( \bar{z}_j \in F_j(\bar{x}, \bar{y}_j) \), and \( \bar{w}_i \in G_i(\bar{x}_i, \bar{y}) \) such that

\[
z_j - \bar{z}_j \notin - \text{int } C'_j, \quad \text{for all } z_j \in F_j(\bar{x}, \bar{y}_j), \ y_j \in S_j(\bar{x}), \ \text{and all } j \in J,
\]

and

\[
w_i - \bar{w}_i \notin \text{int } C_i, \quad \text{for all } w_i \in G_i(x_i, \bar{y}), \ x_i \in T_i(\bar{y}), \ \text{and all } i \in I.
\]

**Proof.** Since for each \( i \in I \) and \( j \in J \), \( F_j \) and \( G_i \) are continuous multimaps with compact values and \( X_j \) and \( Y_j \) are compact, it follows from Lemma 2 that for each \( i \in I \) and \( j \in J \), \( F_i(X) \) and \( G_i(Y) \) are compact. By assumptions and Lemma 1, for each \( i \in I \) and \( j \in J \), \( M_j : X \rightarrow oY \) and \( H_i : Y \rightarrow oX \) are u.s.c. multimaps with compact acyclic values.

Now we define \( A : X \times Y \rightarrow oX \times Y \) by

\[
A(x, y) = \left( \prod_{i \in I} H_i(y) \right) \times \left( \prod_{j \in J} M_j(x) \right),
\]

for \((x, y) \in X \times Y\). Then by Kunneth formula (see [15]) and Lemma 3 of [16], \( A \) is a u.s.c. multimap with compact acyclic values. By Lemma 5, there exists a \((\bar{x}, \bar{y}) \in X \times Y\) such that \((\bar{x}, \bar{y}) \in A(\bar{x}, \bar{y}) = (\prod_{i \in I} H_i(y)) \times (\prod_{j \in J} M_j(x))\). Therefore, \( \bar{x} \in \prod_{i \in I} H_i(y) \) and \( \bar{y} \in \prod_{j \in J} M_j(x) \). That is, for each \( i \in I \) and \( j \in J \), \( \bar{x}_i \in H_i(\bar{y}) \) and \( \bar{y}_j \in M_j(\bar{x}) \). In other words, for each \( i \in I \) and \( j \in J \), \( \bar{x}_i \in T_i(\bar{y}) \), \( \bar{y}_j \in S_j(\bar{x}) \), and there exist \( \bar{z}_j \in F_j(\bar{x}, \bar{y}_j) \) and \( \bar{w}_i \in G_i(\bar{x}_i, \bar{y}) \) such that

\[
z_j - \bar{z}_j \notin - \text{int } C'_j, \quad \text{for all } z_j \in F_j(\bar{x}, \bar{y}_j), \ y_j \in S_j(\bar{x}), \ \text{and all } j \in J,
\]

and

\[
w_i - \bar{w}_i \notin \text{int } C_i, \quad \text{for all } w_i \in G_i(x_i, \bar{y}), \ x_i \in T_i(\bar{y}), \ \text{and all } i \in I.
\]

**Remark.**

1. From Theorem 4.1, the equilibrium points \( \bar{x} \) and \( \bar{y} \) are the fixed points of \((\prod_{i \in I} T_i) \circ (\prod_{j \in J} S_j)\) and \((\prod_{j \in J} S_j) \circ (\prod_{i \in I} T_i)\), respectively. This gives a simple way of finding the equilibrium points of this problem.

2. In Theorem 4.1, we assume further that for each \( i \in I \) and \( j \in J \), \( S_j \) and \( T_i \) are multimaps with convex values and if the condition "for each \( x \in X \) and \( y \in Y \), \( M_j(x) \) and \( H_i(y) \) are acyclic sets" is replaced by the condition that "for each \( x \in X \), the map \( y_j \mapsto F_j(x, y_j) \) is \( C'_j \)-quasiconvex and for each \( y \in Y \), the map \( x_i \mapsto G_i(x_i, y) \) is \( C_i \)-quasiconcave, respectively", then we have the same conclusion.
Following the same arguments as Theorem 4.1, we have the following theorem.

**THEOREM 4.2.** In Theorem 4.1, if $H_i : Y \to X_i$ is defined by

$$H_i(y) = \{ x_i \in T_i(y) \mid G_i(x_i, y) \cap \text{wMin}_{C_i} G_i(T_i(y), y) \neq \emptyset \}.$$  

Then there exist $\bar{x} = (\bar{x}_i)_{i \in I}$, $\bar{y} = (\bar{y}_j)_{j \in J}$, $\bar{x}_i \in T_i(\bar{y})$, $\bar{y}_j \in S_j(\bar{x})$, $\bar{z}_j \in F_j(\bar{x}, \bar{y}_j)$, and $\bar{w}_i \in G_i(\bar{x}_i, \bar{y})$ such that

$$z_j - \bar{z}_j \notin \text{int} C_j', \quad \text{for all } z_j \in F_j(\bar{x}, \bar{y}_j), \ y_j \in S_j(\bar{x}), \text{and all } j \in J,$$

and

$$w_i - \bar{w}_i \notin \text{int} C_i, \quad \text{for all } w_i \in G_i(x_i, \bar{y}), \ x_i \in T_i(\bar{y}), \text{and all } i \in I.$$

**REMARK.** In Theorems 4.1 and 4.2, for each $i \in I$, $H_i(y)$ are different. In Theorem 4.1, we want to find $\bar{x} \in X$, $\bar{y} \in Y$, $\bar{z}_j \in F_j(\bar{x}, \bar{y}_j)$, and $\bar{w}_i \in G_i(\bar{x}_i, \bar{y})$ such that the $j^{th}$ player in Team A has weak minimum loss $\bar{z}_j$ and the $i^{th}$ player in Team B has weak maximum gain $\bar{w}_i$. In Theorem 4.2, $\bar{z}_j$ is the weak minimum loss of the $j^{th}$ player in Team A and $\bar{w}_i$ is also the weak minimum loss of the $i^{th}$ player in Team B.

Applying Theorem 4.1 and following the same argument as Theorem 3.2, we have the following theorem.

**THEOREM 4.3.** Let $I$ and $J$ be finite index sets, $\{E_i\}_{i \in I}$ and $\{V_j\}_{j \in J}$ be families of locally convex t.v.s. For each $i \in I$ and $j \in J$, let $X_i$ be a compact convex subset of $E_i$ and $Y_j$ be a compact convex subset of $V_j$, $S_j : X - \to Y_j$ and $T_i : Y - \to X_i$ be continuous multimaps with nonempty closed values, $Z_i$ and $Z'_j$ be real t.v.s. with proper closed pointed convex solid cones $C_i$ and $C'_j$, respectively. Let $Z_i$ and $Z'_j$ be ordered by $C_i$ and $C'_j$, respectively. Suppose that for each $i \in I$ and $j \in J$, $F_j : X \times Y_j \to Y_j$ and $G_i : X_i \times Y - \to Z_i$ are two continuous multimaps with compact values, $A_j : X - \to Y_j$ and $Q_j : Y - \to X_i$ are multimaps with convex values such that $X = \bigcup_{y_j \in Y_j} \text{int} A_j^{-1}(y_j)$ and $Y = \bigcup_{x_i \in X_i} \text{int} Q_i^{-1}(x_i)$. Moreover, for each $i \in I$ and $j \in J$ and for each $(x, y_j) \in X \times Y_j$ and each $(x_i, y) \in X_i \times Y$, $M_j(x, y_j)$ and $H_i(x_i, y)$ are acyclic sets, where $M_j : X \times Y_j - \to Y_j$ and $H_i : X_i \times Y - \to s \circ X_i$ are defined by

$$M_j(x, y_j) = \{ u_j \in S_j(x) \mid F_j(x, y_j, u_j) \cap \text{wMin}_{C'_j} F_j(x, y_j, S_j(x)) \neq \emptyset \}$$

and

$$H_i(x_i, y) = \{ u_i \in T_i(y) \mid G_i(x_i, y_i) \cap \text{wMin}_{C_i} G_i(T_i(y), x_i, y) \neq \emptyset \},$$

for $(x, y_j) \in X \times Y_j$ and $(x_i, y) \in X_i \times Y$. Then there exist $\bar{x} = (\bar{x}_i)_{i \in I} \in X$, $\bar{y} = (\bar{y}_j)_{j \in J} \in Y$, $\bar{z}_j \in F_j(\bar{x}, \bar{y}_j)$, and $\bar{w}_i \in G_i(\bar{u}_i, \bar{x}_i, \bar{y})$, $\bar{y}_j \in S_j(\bar{x})$, $\bar{x}_i \in T_i(\bar{y})$, $\bar{u}_i \in A_j(\bar{x})$, $\bar{u}_i \in Q_i(\bar{y})$ such that

$$z_j - \bar{z}_j \notin \text{int} C'_j, \quad \text{for all } z_j \in F_j(\bar{x}, \bar{v}_j, \bar{y}_j) \text{ and all } y_j \in S_j(\bar{x}), \ j \in J,$$

and

$$w_i - \bar{w}_i \notin \text{int} C_i, \quad \text{for all } w_i \in G_i(\bar{u}_i, x_i, \bar{y}) \text{ and all } x_i \in T_i(\bar{y}), \ i \in I.$$

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