On second minimal subgroups of Sylow subgroups of finite groups

A. Ballester-Bolinches\textsuperscript{a,}\textsuperscript{*}, R. Esteban-Romero\textsuperscript{b}, Yangming Li\textsuperscript{c,a}

\textsuperscript{a} Departament d'Àlgebra, Universitat de València, Dr. Moliner, 50, E-46100 Burjassot, València, Spain
\textsuperscript{b} Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València, Camí de Vera, s/n, E-46022 València, Spain
\textsuperscript{c} Department of Mathematics, Guangdong University of Education, Guangzhou, 510310, People's Republic of China

\textbf{Article info}

Article history:
Received 22 March 2011
Available online 12 July 2011
Communicated by E.I. Khukhro

MSC:
20D10
20D20

Keywords:
Finite group
Partial CAP-subgroup
Second minimal subgroup
Supersoluble group

\textbf{Abstract}

A subgroup \(H\) of a finite group \(G\) is a partial CAP-subgroup of \(G\) if there is a chief series of \(G\) such that \(H\) either covers or avoids its chief factors. Partial cover and avoidance property has turned out to be very useful to clear up the group structure. In this paper, finite groups in which the second minimal subgroups of their Sylow \(p\)-subgroups, \(p\) a fixed prime, are partial CAP-subgroups are completely classified.

© 2011 Elsevier Inc. All rights reserved.

1. Introduction

All groups considered in this paper are finite. Arguably, the study of subgroup embedding properties has been one of the most efficient methods to clear up the structure of the groups. In particular, the embedding properties of 2-maximal and 2-minimal subgroups tend to give additional information about the group \([4,17,23,27]\). During the past four decades, the subgroup property known as the cover-avoidance property has gained more and more currency, first in the context of soluble groups \([8\text{–}10,12,24,25]\) and \([2,\text{Chapter 4}]\), and more recently as a way of describing certain classes of soluble and supersoluble groups and their local versions \([3,11,13\text{–}16,20\text{–}22,26]\).

\textsuperscript{*} Corresponding author.
E-mail addresses: Adolfo.Ballester@uv.es (A. Ballester-Bolinches), resteban@mat.upv.es (R. Esteban-Romero), liyangming@gdei.edu.cn (Y. Li).

0021-8693/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2011.06.016
Let $A$ be a subgroup of a group $G$ and $H/K$ a section of $G$. We say that $A$ covers $H/K$ if $HA = KA$ and $A$ avoids $H/K$ if $A \cap H = A \cap K$. If $A$ either covers or avoids every chief factor of $G$, then we say that $A$ has the cover and avoidance property in $G$ or $A$ is a CAP-subgroup of $G$. Unfortunately the cover and avoidance property is not hereditary in intermediate subgroups, that is, if $A$ is a CAP-subgroup of $G$ and $A$ is contained in a subgroup $B$ of $G$, it does not follow in general that $A$ has the cover and avoidance property in $B$ (see [4, Example 3]). The failure of the cover and avoidance property to hold in intermediate subgroups leads to the following weaker property, which is persistent in subgroups and is also extremely useful in the structural study of the groups:

**Definition 1.1.** A subgroup $A$ of a group $G$ is called a partial CAP-subgroup of $G$ if there exists a chief series $\Gamma_A$ of $G$ such that $A$ either covers or avoids each factor of $\Gamma_A$ (see [11,21] for alternative terminologies).

Clearly, every CAP-subgroup is a partial CAP-subgroup, but the converse does not hold ([4, Example 3]). In [4], the authors considered the effect of imposing the partial cover and avoidance property to the second minimal subgroups of the Sylow $p$-subgroups, $p$ a fixed prime. In the present paper the emphasis is on second minimal subgroups, and we consider what might be considered an opposite extreme, where the second minimal subgroups (2-minimal subgroups for short) of the Sylow $p$-subgroups are partial CAP-subgroups. In one result, we characterise the groups with this property, identifying a remarkable analogy between the partial cover and avoidance property of the subgroups of index $p^2$ and the partial cover and avoidance property of the subgroups of order $p^2$.

**Main theorem.** Let $p$ be a prime number, let $G$ be a group, and let $G^+ = G/O_{p'}(G)$. Then every subgroup of $G$ of order $p^2$ is a partial CAP-subgroup of $G$ if and only if one of the following statements holds:

1. the order of the Sylow $p$-subgroups of $G$ is at most $p$;
2. $G$ is a $p$-supersoluble group;
3. $\Phi(G^+) = 1$ and, if $P$ is a Sylow $p$-subgroup of $G$, $P^+ = \text{Soc}(G^+) = V_1 \times \cdots \times V_r$, where $V_1, \ldots, V_r$ are minimal normal subgroups of $G^+$ which are $G^+$-isomorphic to a 2-dimensional irreducible $G^+$-module $V$ over the Galois field $GF(p)$. Furthermore, $V$ is not an absolutely irreducible $G^+$-module when $r > 1$.

It seems desirable now to give an example of a group satisfying condition 3 of our main theorem. Existence of such groups was already shown in [1]. For the sake of completeness, we reproduce here the example of that paper.

**Example 1.2.** Consider an elementary abelian group

$$H = \langle a, b \mid a^5 = b^5 = 1, \, ab = ba \rangle$$

of order 25 and let $\alpha$ be an automorphism of $H$ of order 3 satisfying that $a^\alpha = b$, $b^\alpha = a^{-1}b^{-1}$. Let $H_1 = H$, $H_2 = \langle a', b' \rangle$ be a copy of $H_1$ and $G = [H_1 \times H_2](\alpha)$. For any subgroup $A$ of $G$ of order 25, there exists a minimal normal subgroup $N$ such that $A \cap N = 1$. Then $A$ covers or avoids the factors of the chief series of $G$,

$$1 < N < AN < G.$$ 

In other words, $A$ is a partial CAP-subgroup of $G$. However, $G$ is not 5-supersoluble. Note that $H$ is not an absolutely irreducible $G$-module over the Galois field of 5 elements.

This example also shows that a group in which the second minimal subgroups of the Sylow subgroups are partial CAP-subgroups is not supersoluble in general. The best we are able to say is the following:
Corollary 1.3. A group in which the second minimal subgroups of the Sylow subgroups are partial CAP-subgroups is soluble.

2. Preliminaries

We begin with some preparatory lemmas before coming to the main result of the paper. The main basic properties of partial CAP-subgroups are listed in the following result appeared in [11]. They are particularly useful when induction arguments are applied.

Lemma 2.1. Let $S$ be a partial CAP-subgroup of a group $G$.

1. If $S \leq K \leq G$, then $S$ is a partial CAP-subgroup of $K$.
2. If $N \triangleleft S$ and $N \triangleleft G$, then $S/N$ is a partial CAP-subgroup of $G/N$.
3. If $N \triangleleft G$ and $(|S|, |N|) = 1$, then $SN/N$ is a partial CAP-subgroup of $G/N$.

The information given in the following lemma comes in extremely useful when studying the partial cover and avoidance property.

Lemma 2.2. (See [1, Lemma 2.2].) Let $H$ be a partial CAP-subgroup of a group $G$. Suppose that $Q$ is a normal subgroup of $G$ such that $H$ is contained in $Q$. Then there exists a chief series $\Omega_H$ of $G$ passing through $Q$ such that $H$ either covers or avoids each chief factor in $\Omega_H$.

Let $r$ be a positive integer and let $H$ be a subgroup of $G$. Then $H$ is called an $r$-minimal (respectively $r$-maximal) subgroup of $G$ if there exists a subgroup chain $1 = H_0 < H_1 < \cdots < H_r = H$ (respectively $H = H_0 < H_1 < \cdots < H_r = G$) such that $H_i$ is a maximal subgroup of $H_{i+1}$ for all $0 \leq i < r - 1$.

In the present paper we investigate the effect of imposing the partial cover and avoidance property on the $2$-minimal subgroups of the Sylow subgroups, and once more we get a sense of why the partial cover and avoidance property has such bearing in the study of soluble groups. In fact, we use a local approach and characterise the groups $G$ enjoying the following property:

$(\dagger)$ Every $2$-minimal subgroup of every Sylow $p$-subgroup of $G$ is a partial CAP-subgroup of $G$, $p$ a fixed prime.

In the following $p$ will be a fixed prime.

Since $2$-minimal subgroups of $p$-groups have order $p^2$, every group with Sylow $p$-subgroups of order $p$ satisfies property $(\dagger)$. All $p$-supersoluble groups, or $p$-soluble groups whose $p$-chief factors have order $p$, also satisfy $(\dagger)$. Therefore we must think about groups whose order is divisible by $p^2$ which are not $p$-supersoluble.

An interesting special case is when the Sylow $p$-subgroups of $G$ have order $p^2$. In this case, the structure of $G$ is quite restricted as the following lemma shows.

Lemma 2.3. Let $G$ be a group whose Sylow $p$-subgroups have order $p^2$. Suppose that $G$ satisfies property $(\dagger)$. Then $G$ is $p$-soluble and either $G$ is $p$-supersoluble or $\text{Soc}(G/O_p'(G)) = \text{PO}_{p'}(G)/O_p'(G)$ is an elementary abelian group of order $p^2$ for each Sylow $p$-subgroup $P$ of $G$.

Proof. We can assume without loss of generality that $O_{p'}(G) = 1$. Suppose that $G$ is not $p$-soluble. Then every minimal normal subgroup of $G$ is non-abelian and its order is divisible by $p$ by [4, Theorem 7]. It is clear then that a Sylow $p$-subgroup $P$ of $G$ neither covers nor avoids any minimal normal subgroup of $G$, a contradiction which shows that $G$ is $p$-soluble. In that case, $S = \text{Soc}(G)$ is a minimal normal subgroup of $G$ contained in $P$ by [4, Theorem 7]. Consequently, either $S$ is of order $p$ and $G$ is $p$-supersoluble or $S = P$ is the Sylow $p$-subgroup of $G$. □

The next lemmas will be applied to the consideration of groups satisfying property $(\dagger)$. 
Lemma 2.4. (See [5, Proposition 1].) Let $\mathcal{F}$ be a saturated formation. Assume that $G$ is a group such that $G$ does not belong to $\mathcal{F}$ and there exists a maximal subgroup $M$ of $G$ such that $M \in \mathcal{F}$ and $G = MF(G)$. Then $G^\mathcal{F}/(G^\mathcal{F})'$ is a chief factor of $G$, $G^\mathcal{F}$ is a $p$-group for some prime $p$, and $G^\mathcal{F}$ has exponent $p$ if $p > 2$ and exponent at most 4 if $p = 2$. Moreover, either $G^\mathcal{F}$ is elementary abelian or $(G^\mathcal{F})' = Z(G^\mathcal{F}) = \Phi(G^\mathcal{F})$.

As an important deduction we have:

Lemma 2.5. (See [5, Theorem 6].) Let $\mathcal{F}$ be a saturated formation and $G$ a group with a normal subgroup $K$ such that $G/K \in \mathcal{F}$. If for some prime $p$, every subgroup of order $p$ of $K$ is contained in the $\mathcal{F}$-hypercentre $Z_{\mathcal{F}}(G)$ of $G$, then $G/O_{p'}(K) \in \mathcal{F}$.

There are some places where we use a known criterion for a normal $p$-subgroup to be contained in the hypercentre. For convenience, this is stated here as:

Lemma 2.6. Suppose that $P$ is a normal $p$-subgroup of $G$. Then $P \leq Z_{\infty}(G)$ if and only if $O^p(G) \leq C_G(P)$.

3. Main results

In this section we analyse the structure of the groups satisfying property $(\dagger)$, and prepare the way for the proof of the main result. We begin with a theorem about the minimal normal subgroups of the groups satisfying property $(\dagger)$.

Theorem 3.1. Let $G$ be a group satisfying property $(\dagger)$ whose order is divisible by $p^2$. Then every minimal normal subgroup of $G$ is either a $p'$-group or a $p$-group. The minimal normal $p$-subgroups of $G$ are of the same order, and it is at most $p^2$.

Proof. Let $N$ be a minimal normal subgroup of $G$, and suppose that $N$ is not a $p'$-group. Let $1 \neq N_p$ be a Sylow $p$-subgroup of $N$, and let $Q$ be a subgroup of $G$ of order $p^2$ such that $Q \cap N_p \neq 1$. We consider a chief series of $G$,

$$(\Gamma): \quad 1 = G_0 < \cdots < G_i < \cdots < G_j < G_{j+1} < \cdots < G_m = G$$

such that $Q$ either covers or avoids each chief factor of $G$ in $(\Gamma)$. Then there exists an index $i \in \{1, \ldots, m\}$ such that $N \cap G_{i-1} = 1$ and $G_i = G_{i-1}N$. In that case, $G_i/G_{i-1}$ is $G$-isomorphic to $N$. Suppose that $Q$ avoids $G_{i+1}/G_i$, then $Q \cap N_p \leq Q \cap G_i \cap N = Q \cap G_{i-1} \cap N = 1$, against the choice of $Q$. Consequently, $Q$ covers $G_i/G_{i-1}$. Then $G_i/G_{i-1}$ is of order at most $p^2$ and $N$ is a $p$-group of order at most $p^2$.

Next we prove that all minimal normal $p$-subgroups of $G$ are of the same order. Suppose, arguing by contradiction, that $G$ has two minimal normal $p$-subgroups, $N_1$ and $N_2$ say, such that $|N_1| = p$ and $|N_2| = p^2$. Let $H$ be a subgroup of $N_2$ of order $p$. Then $N_1H$ is a subgroup of $G$ of order $p^2$ which is a partial CAP-subgroup of $G$. By Lemma 2.2, there exists a chief series of $G$,

$$(\Delta): \quad 1 \leq N_3 \leq N_1N_2 \leq \cdots \leq G,$$

passing through $N_1N_2$ such that $N_1H$ either covers or avoids each chief factor of $G$ in $(\Delta)$. In addition, the order of $N_3$ is $p$ or $p^2$. Assume that $N_3$ is of order $p$. If $N_1H \cap N_3 = 1$, then $N_1HN_2 = N_1N_2$ and $N_2 = H(N_2 \cap N_1N_3)$. It means that either $N_2 = H$ or $N_2 = HN_1N_3$. This contradiction shows that $N_3$ is a subgroup of $N_1H$. In particular, $N_1H$ cannot cover $N_1N_2/N_3$. Hence $N_1H = N_1N_2 \cap N_1H = N_1H \cap N_3 = N_3$, a contradiction which shows that $N_3$ must be of order $p^2$. Since $N_1N_2$ is of order $p^3$ and $N_1H$ is of order $p^2$, it follows that $N_1H$ covers $N_3$. Thus $N_1H = N_3$, which contradicts the fact that $N_1$ and $N_3$ are two different minimal normal subgroups of $G$. This proves the result. $\square$
We now touch the question of the $p$-length of $p$-soluble groups satisfying property ($\dagger$). We prove that these groups belong to the saturated formation $\mathfrak{F}$ of all $p$-soluble groups whose $p$-length is at most one.

**Theorem 3.2.** Let $G$ be a $p$-soluble group satisfying property ($\dagger$). Then the $p$-length of $G$ is at most 1.

**Proof.** We will obtain a contradiction by supposing that the result is false and choosing a counterexample $G$ of least order. For the ease of reading, we break the argument into separately-stated steps.

1. $O_p(G) = 1$.
   Assume that $O_p(G) \neq 1$. By Lemma 2.1 the hypothesis holds in the group $G/O_p(G)$. The minimal choice of $G$ implies that $G/O_p(G)$ belongs to $\mathfrak{F}$. Hence $G$ is an $\mathfrak{F}$-group, against the choice of $G$. Thus $O_p(G) = 1$.

2. Any proper subgroup of $G$ belongs to $\mathfrak{F}$.
   Let $H$ be a proper subgroup of $G$. If a Sylow $p$-subgroup $H_p$ of $H$ is of order at most $p$, we have that $H \in \mathfrak{F}$ by [3, Lemma 3.1]. Assume that $p^2$ divides $|H_p|$ and let $L$ be a subgroup of $H$ of order $p^2$. Then $L$ is a partial CAP-subgroup of $H$ by Lemma 2.1, and so $H$ satisfies property ($\dagger$). The minimality of $G$ yields $H \in \mathfrak{F}$. This confirms Step 2.

3. There exists a maximal subgroup $M$ of $G$ such that $M \in \mathfrak{F}$ and $G = MG^\mathfrak{F}$. Moreover, $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$ is a chief factor of $G$, and the exponent of $G^\mathfrak{F}$ is $p$ or at most 4 if $p = 2$.
   Since $G$ is not an $\mathfrak{F}$-group and $\mathfrak{F}$ is saturated, it follows that $G/\Phi(G)$ does not belong to $\mathfrak{F}$. Let $N/\Phi(G)$ be a non-trivial normal subgroup of $G/\Phi(G)$. Then $N/\Phi(G)$ is supplemented in $G/\Phi(G)$. By Step 2, $G/N$ belongs to $\mathfrak{F}$. Therefore, since $\mathfrak{F}$ is a formation, $G/\Phi(G)$ has a unique minimal normal subgroup, $T/\Phi(G)$ say. Moreover, $T/\Phi(G)$ is not a $p'$-group. Since $G$ is $p$-soluble, it follows that $T/\Phi(G)$ is an abelian $p$-chief factor of $G$ which is complemented in $G$ by a maximal subgroup $M$ of $G$. Then $G = MF(G)$ and $T = G^\mathfrak{F}/\Phi(G)$. Step 2 implies that $M \in \mathfrak{F}$ and so $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$ is a chief factor of $G$, and the exponent of $G^\mathfrak{F}$ is $p$ or at most 4 if $p = 2$ by Lemma 2.4. This proves our claim.

4. $\Phi(G^\mathfrak{F}) = 1$ and $|G^\mathfrak{F}| = p^2$.
   Suppose that $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$ has order $p$. Since $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$ is $G$-isomorphic to $\text{Soc}(G/M_G)$, it follows that $G/M_G$ is in $\mathfrak{F}$. This contradiction shows that $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$ has order greater than $p$. Let $H$ be a subgroup of $G^\mathfrak{F}$ of order $p^2$ such that $H \not\subseteq \Phi(G^\mathfrak{F})$. Then $H$ is a partial CAP-subgroup of $G$ and, by Lemma 2.2, there exists a chief series of $G$,

$$(\Gamma_1): \quad 1 = G_0 \leq G_1 \leq \cdots \leq K \leq G^\mathfrak{F} \leq \cdots \leq G_n = G,$$

passing through $G^\mathfrak{F}$ such that $H$ either covers or avoids each $G$-chief factor in $(\Gamma_1)$. Since $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$ is a chief factor of $G$ by Step 3, it follows that either $K/\Phi(G^\mathfrak{F}) = \Phi(G^\mathfrak{F})$ or $K/\Phi(G^\mathfrak{F}) = G^\mathfrak{F}$. If $K/\Phi(G^\mathfrak{F}) = G^\mathfrak{F}$, then $K = G^\mathfrak{F}$, contrary to assumption. Thus $K \not\subseteq \Phi(G^\mathfrak{F})$. Since $G^\mathfrak{F}/K$ is a chief factor of $G$, we have $K = \Phi(G^\mathfrak{F})$ and so $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$ is a chief factor of $G$ in $(\Gamma)$. It therefore follows that $H$ either covers or avoids $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$. If $H$ avoids $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$, then $H = H \cap G^\mathfrak{F} \leq \Phi(G^\mathfrak{F})$, contrary to the choice of $H$. Hence we have that $H$ covers $G^\mathfrak{F}/\Phi(G^\mathfrak{F})$. Consequently $G^\mathfrak{F} = H\Phi(G^\mathfrak{F}) = H$, $G^\mathfrak{F}$ is of order $p^2$ and $\Phi(G^\mathfrak{F}) = 1$.

5. $G$ is not a primitive group. In particular $M_G \neq 1$.
   Suppose, arguing by contradiction, that $G$ is primitive. Then $G^\mathfrak{F}$ is the unique minimal normal subgroup of $G$. Since $G$ has $p$-length greater than 1 and $G^\mathfrak{F}$ is a $p$-group, it follows that $p$ divides $|M|$. Then we can choose an element $a \in G^\mathfrak{F}$ and an element $b \in M$ such that $(a,b)$ is a subgroup of $G$ of order $p^2$. Obviously, $(a,b)$ neither covers nor avoids $G^\mathfrak{F}$, a contradiction which proves Step 5.

6. The final contradiction.
   By Step 5, $M_G \neq 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $M_G$. Then $N$ is a $p$-group by Step 1, and $N \cap G^\mathfrak{F} = 1$. Let us choose an element $a \in G^\mathfrak{F}$ and an element $b \in N$ such
that \( \langle a, b \rangle \) is a group of order \( p^2 \). By hypothesis, \( \langle a, b \rangle \) is a partial CAP-subgroup of \( G \). Applying Lemma 2.2, there exists a chief series of \( G \),

\[
(\Gamma_2): \quad 1 = G_0 \leq K \leq G^\delta N \leq \cdots \leq G_n = G,
\]

passing through \( G^\delta N \) such that \( \langle a, b \rangle \) either covers or avoids each chief factor of \( G \) in \( (\Gamma_2) \).

It is clear that \( \langle a, b \rangle \) neither covers nor avoids \( G^\delta \). Hence \( K \neq G^\delta \). Then \( K \) is an \( \delta \)-central chief factor of \( G \). Since \( M \) is an \( \delta \)-normaliser of \( G \) by [2, Theorem 4.2.17], we can apply [2, Theorem 4.2.4] to conclude that \( K \leq M \).

Assume that \( \langle a, b \rangle \) avoids \( K \). Then \( \langle a, b \rangle \) must cover \( G^\delta N/K \), and so \( G^\delta N \leq \langle a, b \rangle K \).

We therefore have that \( G^\delta = G^\delta \cap \langle a, b \rangle K = \langle a \rangle (G^\delta \cap \langle b \rangle K) = \langle a \rangle \), contrary to Step 4. Hence \( \langle a, b \rangle \) must cover \( K \).

Thus \( K = M \cap \langle a, b \rangle = \langle b \rangle \).

Now \( \langle a, b \rangle \) either covers or avoids \( G^\delta N/K \). If \( \langle a, b \rangle \) covers \( G^\delta N/K \), then \( G^\delta N \leq \langle a, b \rangle K = \langle a, b \rangle \).

Then \( G^\delta \) is of order \( p \), against Step 4. Thus \( \langle a, b \rangle \) avoids \( G^\delta N/K \). This gives the final contradiction \( \langle a, b \rangle = G^\delta N \cap \langle a, b \rangle = \langle a, b \rangle \cap K = K \). \( \Box \)

Our next result shows that a group satisfying property \((\dagger)\) whose order is divisible by \( p^2 \) must be \( p \)-soluble.

**Theorem 3.3.** Let \( G \) be a group satisfying property \((\dagger)\). Then either the Sylow \( p \)-subgroups of \( G \) are of order \( p \) or \( G \) is a \( p \)-soluble group.

**Proof.** Suppose the result false, and let the group \( G \) provide a counterexample of least possible order. Then \( p^2 \) divides the order of \( G \). According to Lemma 2.1, the property of \( G \) is inherited by \( G^2 = G/Op^2(G) \). Hence the minimality of \( G \) implies that \( Op^2(G) = 1 \). We reach a contradiction after the following steps.

1. **If \( K \) is a proper subgroup of \( G \) and \( p^2 \) divides \(|K|\), then \( K \) is \( p \)-soluble.** If, in addition, \( K \) is normal in \( G \), then a Sylow \( p \)-subgroup of \( K \) is also normal in \( G \).

   Assume that \( K \) is a proper subgroup of \( G \) such that \( p^2 \) divides \(|K|\), and let \( L \) be a subgroup of \( K \) of order \( p^2 \). Then \( L \) is a partial CAP-subgroup of \( G \) and so is in \( K \) by Lemma 2.1. Hence \( K \) satisfies property \((\dagger)\). The minimal choice of \( G \) implies that \( K \) is \( p \)-soluble. Suppose that \( K \) is normal in \( G \). Since \( Op^2(K) \leq Op^2(G) = 1 \), we conclude that \( Op^2(K) \) is a Sylow \( p \)-subgroup of \( K \) by virtue of Theorem 3.2.

2. **\( G \) has a unique maximal normal subgroup, \( M \) say. In particular, the chief factor \( G/M \) appears in every chief series of \( G \).**

   Suppose that \( G \) has two different maximal normal subgroups, \( M \) and \( N \) say. Then \( G = MN \). If the Sylow \( p \)-subgroups of \( M \) and \( N \) are normal in \( G \), then \( G \) has a normal Sylow \( p \)-subgroup and so \( G \) is \( p \)-soluble, contradicting the choice of \( G \). Hence, by Step 1, we may assume that the order of the Sylow \( p \)-subgroups of \( M \) is at most \( p \). If the order of the Sylow \( p \)-subgroups of \( N \) is also at most \( p \), then the order of the Sylow \( p \)-subgroups of \( G \) is at most \( p^2 \). Applying Lemma 2.3, it follows that either \( p^2 \) does not divide the order of \( G \) or \( G \) is \( p \)-soluble, contrary to assumption.

   Hence, we may assume that \( p^2 \) divides \(|N|\) and a Sylow \( p \)-subgroup \( N_p \) of \( N \) is normal in \( G \). In that case \( N_pM \) is a normal subgroup of \( G \) containing \( M \). Hence \( G = N_pM \). Then \( G/M \) is a cyclic group of order \( p \). This implies that the Sylow \( p \)-subgroups have order \( p^2 \). Applying Lemma 2.3, \( G \) is \( p \)-soluble. This contradiction proves our claim.

3. **A Sylow \( p \)-subgroup of \( M \) is a non-trivial normal subgroup of \( G \) and \( M \) is \( p \)-soluble.** Furthermore, \( G = Op^2(G) \).

   Assume that the order of a Sylow \( p \)-subgroup \( M_p \) of \( M \) is at most \( p \). Let \( H \) be a subgroup of \( G \) of order \( p^2 \) containing \( M_p \). Then \( H \) is a partial CAP-subgroup of \( G \) and so \( H \) either covers or avoids \( G/M \). If \( H \) avoids \( G/M \), then \( H = H \cap G = H \cap M = M_p \). This contradiction implies that \( H \) covers \( G/M \). Then \( G = HM \) and so the Sylow \( p \)-subgroups of \( G \) are of order \( p^2 \). By Lemma 2.3, \( G \) is \( p \)-soluble, contrary to supposition. It therefore follows that \( M_p \) is of order greater or equal than \( p^2 \). By Step 1, we conclude that \( M_p \) is normal in \( G \) and \( M \) is \( p \)-soluble. If \( Op^2(G) \neq G \), then
$O^p(G) \leq M$ by Step 2. Thus $O^p(G)$ is $p$-soluble. Since $G/O^p(G)$ is $p$-group, we have that $G$ is $p$-soluble, contradicting again our assumption. Hence $G = O^p(G)$.

4. Every subgroup of $G$ of order $p$ or $p^2$ is contained in $M$.

Let $1 \neq M_p$ be the Sylow $p$-subgroup of $M$ and let $G_p$ be a Sylow $p$-subgroup of $G$ containing $M_p$.

Let $T$ be a subgroup of order $p$ contained in $M_p \cap Z(G_p)$. Assume that $H$ is a subgroup of order $p$ which is not contained in $M$. Then $HT$ is a subgroup of $G$ of order $p^2$ which either covers or avoids $G/M$. If $G/M$ were avoided by $HT$, then we would have $HT = HT \cap G = HT \cap M = T$, and if $G/M$ were covered by $HT$, it would follow that $G = HTM = HM$. This would mean that $G/M$ had to be cyclic of order $p$ and then had to be $p$-soluble. In both cases, we get a contradiction. Hence every subgroup of order $p$ has to be contained in $M$. Assume now that $X$ is a subgroup of order $p^2$ which is not contained in $M$. Exactly similar reasoning shows that $X$ neither covers nor avoids $G/M$. This however contradicts the hypothesis that $X$ is a partial CAP-subgroup of $G$.

5. $M = \Phi(G) = O_p(G)$.

Since $O_p(G) = 1$, it follows that $\Phi(G)$ is a $p$-group. We want to show that $M$ is contained in $\Phi(G)$. Suppose to the contrary that $M$ is not contained in $\Phi(G)$, and therefore that there exists a proper subgroup $X$ of $G$ such that $G = MX$. Since every subgroup of order $p$ and $p^2$ of $G$ has to be contained in $M$ by Step 4, then $G/M$ would be a $p'$-group if $p^2$ did not divide the order of $X$, and so $G$ would be $p$-soluble, in contradiction to our assumption. Therefore $p^2$ divides the order of $X$ and $X$ is $p$-soluble by Step 1. Hence $G/M$ is $p$-soluble and so is $G$. This contradiction shows that $M \leq \Phi(G)$ and $M = \Phi(G) = O_p(G)$.

6. No chief factor of $G$ below $M$ has order $p^2$.

Let us denote $S = G/M$. Let $H/K$ be a chief factor of $G$ below $M$. Then $H/K$ is an elementary abelian $p$-group and $H/K$ has the structure of an irreducible and faithful $G/C_G(H/K)$-module over the Galois field $GF(p)$. Since $M = O_p(G) \leq C_G(H/K)$ by [6, A, 13.8] and $C_G(H/K) \neq G$, we have $M = C_G(H/K)$. Assume now that the order of $H/K$ is $p^2$. Then $S$ can be regarded as a subgroup of $GL_2(p)$. Since $S$ is a non-abelian simple group, it follows that $S \leq (GL_2(p))'$ = $SL_2(p)$, and $S \cap (Z(SL_2(p))) = 1$. Hence $S$ can be regarded as a subgroup of $PSL_2(p)$. According to the subgroup structure of $PSL_2(p)$ (see [18, II, 8.27]), either $S \cong A_5$, where $p = 5$ or $p^2 - 1 \equiv 0 \pmod{5}$ or $S \cong PSL_2(p)$. Suppose that $S \cong A_5$. Since $p$ is a prime divisor of $\Phi(G)$, $p \in \pi(G/\Phi(G)) = \pi(G/M) = \pi(A_5) = \{2, 3, 5\}$, and so we must have $p = 5$. This is contrary to the fact that the dimensions of the irreducible and faithful representations of $A_5$ over $GF(5)$ are 3 and 5 (see [19, VII, 3.10]). Hence $S$ must be isomorphic to $PSL_2(p)$. In that case, $p \geq 5$ and since the index of $S$ in $SL_2(p)$ is 2 and $SL_2(p)$ is perfect, it therefore follows that $SL_2(p) = (SL_2(p))' \leq S$. In this case we are also led to a contradiction, and therefore conclude that the result as stated is true.

7. Every chief factor of $G$ of order $p$ is central in $G$.

Suppose that $H/K$ is a chief factor of $G$ of order $p$. Then, by [6, A, 13.8], $M = O_p(G) \leq C_G(H/K) \leq G$, and consequently we have that either $C_G(H/K) = M$ or $C_G(H/K) = G$. If $C_G(H/K) = M$, then $G/M = G/C_G(H/K)$ is a $p'$-group and $G$ is $p$-soluble. This is in contradiction to the choice of $G$. Hence $C_G(H/K) = G$, that is, $H/K$ is central in $G$.

8. $M$ is contained in the hypercentre $Z_\infty(G)$ of $G$.

From Step 7, it suffices to prove that every chief factor of $G$ below $M$ has order $p$. Assume to the contrary that there exists a chief factor $A/B$ of $G$ below $M$ whose order is greater than $p$. We choose $A$ of minimal order. Then every chief factor $H/K$ of $G$ below $M$ with $|H| < |A|$ is of order $p$. Let $L$ be a subgroup of $A$ of order $p^2$. Then $L$ is a partial CAP-subgroup of $G$. Applying Lemma 2.2, there exists a chief series of $G$,

$$1 \leq \cdots \leq T < A < \cdots < G$$

passing through $A$ such that $L$ either covers or avoids each chief factor of this series. The choice of $A$ implies that every chief factor of $G$ below $T$ is of order $p$. Consequently, $T \leq Z_\infty(G)$ by Step 7. If the order of $A/T$ were $p$, then $A$ would be contained in $Z_\infty(G)$. This would imply that $|A/B| = p$, in contradiction to the hypothesis that $A/B$ has order greater than $p$. Therefore the order of $A/T$ is greater than $p^2$ by Step 6, and so $L$ cannot cover $A/T$. Hence $L$ avoids $A/T$. 


Then $L \leq T \leq Z_\infty(G)$. We therefore conclude that every subgroup of $A$ of order $p^2$ is contained in $Z_\infty(G)$. By Lemma 2.6, $G = O_p(G) \leq C_p(\Omega_2(A))$, that is, $\Omega_2(A) = Z(G)$. Therefore every $p'$-element of $G$ centralises every element of $A$ of order $p$ or $p^2$. Applying [18, IV, 5.12], every $p'$-element of $G$ centralises $A$. It therefore follows that $G = O_p(G) \leq C_G(A)$. Consequently $A/B$ must be of order $p$, contrary to the choice of $A/B$.


From Step 8 and Lemma 2.6, we have that $M \leq Z(G)$. Therefore, by Step 4, every element of $G$ of order $p$ or $p^2$ is contained in $Z(G)$. Applying [18, IV, 5.5], we have $G$ is $p$-nilpotent. This last contradiction establishes the theorem. □

**Theorem 3.4.** Let $G$ be a $p$-soluble group satisfying property $(\dagger)$. Then either $G$ is $p$-supersoluble or $G$ satisfies the following two conditions:

1. all non-cyclic $p$-chief factors of $G$ are $G$-isomorphic and have order $p^2$;
2. all complemented $p$-chief factors of $G$ are not cyclic.

**Proof.** We proceed by induction on $|G|$. We may assume without loss of generality that $O_p(G) = 1$. Then $F(G) = O_p(G)$ is a Sylow $p$-subgroup of $G$ by Theorem 3.2. According to [6, A, 10.6], $F(G)/\Phi(G) = N_1/\Phi(G) \times \cdots \times N_r/\Phi(G)$, where $N_i/\Phi(G)$ is a minimal normal subgroup of $G/\Phi(G)$ which is complemented in $G$ for all $i$.

Suppose first that $r = 1$, that is, $F(G)/\Phi(G)$ is a chief factor of $G$. If $F(G)/\Phi(G)$ has order $p$, it can be seen without difficulty that $G$ is $p$-supersoluble, as desired. Hence we can assume that $F(G)/\Phi(G)$ has order at least $p^2$. Let us denote $M = F(G)G_{p'}$. Then $G = F(G)M$, and $F(G) \cap M = \Phi(G)$. Let $A$ be a normal subgroup of $G$ such that $F(G)/A$ is a chief factor of $G$. If $A$ was not contained in $\Phi(G)$, it would follow that $F(G) = A\Phi(G)$. This would mean that $G = F(G)G_{p'} = A\Phi(G)G_{p'} = AG_{p'}$. Consequently, $F(G) = A$. This is in contradiction to the definition of $A$. Therefore $F(G)/\Phi(G)$ is a chief factor of $G$ which appears in every chief series of $G$ passing through $F(G)$.

Suppose we have an element $x$ of $F(G)$ of order $p$ which is not in $\Phi(G)$. Let $y$ be an element of $\Phi(G)$ of order $p$ such that $H = \langle x, y \rangle$ has order $p^2$. Then $H$ either covers or avoids $F(G)/\Phi(G)$ by Lemma 2.2. If $H$ covers $F(G)/\Phi(G)$, then $F(G) = H\Phi(G) = \langle x \rangle\Phi(G)$. Then $F(G)/\Phi(G)$ is of order $p$, and if $H$ avoids $F(G)/\Phi(G)$, it follows $H = H\cap F(G) = H \cap \Phi(G)$. In each case we are led to a contradiction, and therefore conclude that $F(G)$ contains every element of order $p$ of $F(G)$.

If $F(G) \leq Z_{Lk}(G)$, the $p$-supersoluble hypercentre of $G$, we can apply Lemma 2.5 to the saturated formation $\mathcal{L}_p$ of all $p$-supersoluble groups to conclude that $G$ is $p$-supersoluble. This contradicts that $F(G)/\Phi(G)$ is a chief factor of $G$ of order at least $p^2$.

Therefore $F(G) \not\leq Z_{Lk}(G)$. Hence we have a chief series of $G$ passing through $F(G)$:

\[ (*) : \quad 1 = L_0 \leq L_1 \leq \cdots \leq L_{k-1} \leq L_k \leq \cdots \leq L_n = F(G) \leq \cdots \leq G \]

such that all chief factors of $G$ below $L_{k-1}$ are of order $p$ and $L_k/L_{k-1}$ has order greater than $p$. Since $F(G)$ centralises each chief factor of $G$, it follows that $L_k/L_{k-1}$ is a non-cyclic chief factor of $L_kG_{p'}$. Hence $L_kG_{p'}$ is not $p$-supersoluble and $L_{k-1}$ is contained in the $p$-supersoluble hypercentre of $L_kG_{p'}$. By Lemma 2.5, there exists an element $x$ of $L_k$ of order $p$ such that $x$ does not belong to $L_{k-1}$. Then, as above, we can consider an element $y$ of $L_{k-1}$ such that $H = \langle x, y \rangle$ of order $p^2$. The hypothesis on $G$ implies that $H$ is a partial $\text{CAP}$-subgroup of $G$. Applying Lemma 2.2, we know there exists a chief series of $G$, 

\[ (**) : \quad 1 \leq \cdots \leq T \leq L_k \leq \cdots \leq G, \]

passing through $L_k$ such that $H$ either covers or avoids each chief factor of $G$ in $(**)$. If $T \leq L_{k-1}$, then $T = L_{k-1}$ and $H$ neither covers nor avoids $L_k/L_{k-1}$. Hence $T \not\leq L_{k-1}$. Then $L_k = TL_{k-1}$, and $L_k/T \cong L_{k-1}/L_{k-1} \cap T$. Since $L_{k-1} \leq Z_{Lk}(G)$, we have $L_k/T$ is of order $p$. Moreover $T/T \cap L_{k-1} \cong L_k/L_{k-1}$ is of order $p^2$. Therefore $(**)$ has a unique chief factor of order greater than $p$ below $T$. In particular, $TG_{p'}$.
is not $p$-supersoluble, $T \cap L_{k-1}$ is contained in the $p$-supersoluble hypercentre of $TG'p'$ and $|T| < |L_k|$. Repeating this argument, we finally get a chief series of $G$,

$$\Delta: \quad 1 \leq R \leq \cdots \leq G,$$

such that the minimal normal subgroup $R$ of $G$ is of order greater than $p$. By Theorem 3.1, we have every minimal normal subgroup of $G$ is of order $p^2$. However $L_1$ is a minimal normal subgroup of $G$ of order $p$. This contradiction yields $L_{k-1} = 1$. In particular, $Z_{1p}(G) = 1$.

Let $A/B$ be a chief factor of $G$ with $A \leq \Phi(G)$. Since the chief factors of $AG'p'$ are chief factors of $G$ and $AG'p'$ is not supersoluble because it contains a minimal normal subgroup of order greater than $p$, by minimality of $G$ the complemented chief factors of $AG'p'$ are non-cyclic and all non-cyclic chief factors of $AG'p'$ are $AG'p'$-isomorphic and have order $p^2$. Since $A/B$ is a complemented chief factor of $AG'p'$, it follows that $|A/B| = p^2$ and then, by taking $A = \Phi(G)$, all chief factors of $G$ below $\Phi(G)$ are $G$-isomorphic and have order $p^2$. Assume now that $\Phi(G) \neq 1$. Let $c$ be an element of order $p^2$ of $F(G)$ and write $S = \langle c \rangle$. Applying Lemma 2.2, there exists a chief series of $G$ passing through $F(G)$ such that every chief factor in this series is either covered or avoided by $S$. But no chief factor of $G$ below $F(G)$ can be cyclic, so that no chief factor of $G$ below $F(G)$ can be covered by $S$. It follows that all such chief factors must be avoided by $S$, contrary to the choice of $S$. Therefore the exponent of $F(G)$ is $p$. Since all elements of $F(G)$ of order $p$ must be contained in $\Phi(G)$, it follows $F(G) = \Phi(G)$. This contradiction shows that $\Phi(G) = 1$. Since $F(G)$ is a minimal normal subgroup of $G$, it has order $p^2$ by Theorem 3.1. This completes the proof for $r = 1$.

Suppose now that $r > 1$. Let $N/\Phi(G)$ be an arbitrary abelian minimal normal subgroup of $G/\Phi(G)$ and let $M$ be a maximal subgroup of $G$ such that $G = F(G)M$ and $N \cap M = \Phi(G)$. Consider a chief series of $G$ passing through $\Phi(G)$ and $N$:

$$\alpha: \quad 1 \leq \cdots \leq \Phi(G) \leq N \leq \cdots \leq G.$$
(\(\beta_1\)): \[ 1 < N_1 < N_1N_2 = F(G) < \cdots < G, \]

(\(\beta_2\)): \[ 1 < N < NN_2 = F(G) < \cdots < G. \]

By [2, 1.2.36], \(N_1\) is \(G\)-isomorphic to \(N\) as \(N_1\) is not \(G\)-isomorphic to \(NN_2/N\). Similarly, if we consider the following two chief series of \(G\):

(\(\beta_1'\)): \[ 1 < N_2 < N_1N_2 = F(G) < \cdots < G, \]

(\(\beta_2'\)): \[ 1 < N < NN_2 = F(G) < \cdots < G, \]

it follows that \(N\) is \(G\)-isomorphic to \(N_2\). Hence \(N_1\) is \(G\)-isomorphic to \(N_2\). This contradiction, together with [2, 1.2.36], allow us to conclude that all non-cyclic \(p\)-chief factors of \(G\) are \(G\)-isomorphic and of order \(p^2\), and they are all complemented in \(G\). The theorem holds in this case.

2. \(\Phi(G) \neq 1\).

Consider the following two chief series of \(G\):

(\(\gamma_1\)): \[ 1 \leq \cdots \leq \Phi(G) \leq N_1 \leq N_1N_2 \leq \cdots \leq G, \]

(\(\gamma_2\)): \[ 1 \leq \cdots \leq \Phi(G) \leq N_2 \leq N_1N_2 \leq \cdots \leq G. \]

Intersecting the series \((\gamma_1)\) term-by-term with \(M_i\), \(i = 1, 2\), and deleting repetitions, we get the chief series of \(M_1\) and \(M_2\), respectively:

(\(\gamma_1\) \(\cap M_1\)): \[ 1 \leq \cdots \leq \Phi(G) = N_1 \cap M_1 \leq N_2 = N_1N_2 \cap M_1 \leq \cdots \leq M_1, \]

(\(\gamma_2\) \(\cap M_2\)): \[ 1 \leq \cdots \leq \Phi(G) = N_2 \cap M_2 \leq N_1 = N_1N_2 \cap M_2 \leq \cdots \leq M_2. \]

Now we intersect these two chief series with \(X\) and delete repetitions:

(\(\gamma_1\) \(\cap X\) = \(\gamma_2\) \(\cap X\)): \[ 1 \leq \cdots \leq \Phi(G) = N_1 \cap X = N_2 \cap X \leq \cdots \leq X. \]

It is clear that \((\gamma_1) \(\cap M_1\) \in \(\gamma_2\) \(\cap M_1\) \in X\) is a chief series of \(X\) and the \(X\)-chief factors of this series below \(\Phi(G)\) are \(M_1\)-chief factors for all \(i = 1, 2\). We know that either \(M_i\) is \(p\)-supersoluble or all non-cyclic \(p\)-chief factors of \(M_i\) are \(M_1\)-isomorphic and have order \(p^2\), and every complemented \(M_1\)-chief factor of \(M_i\) is non-cyclic, \(i = 1, 2\). Hence the orders of \(N_1/\Phi(G)\) and \(N_2/\Phi(G)\) are either \(p\) or \(p^2\). Suppose that \(|N_1/\Phi(G)| = p^2\) and \(|N_2/\Phi(G)| = p\). Since \(N_2/\Phi(G)\) is a cyclic complemented \(p\)-chief factor of \(M_1\), it follows that \(M_1\) is \(p\)-supersoluble. Therefore every \(p\)-chief factor of \(M_1\) in \((\gamma_1) \(\cap M_1\) \in \(\gamma_2\) \(\cap M_1\) \in X\) is of order \(p\). Hence every \(G\)-chief factor below \(\Phi(G)\) in \((\gamma_1)\) is of order \(p\). On the other hand, \(N_1/\Phi(G)\) is a complemented chief factor of \(M_2\) of order \(p^2\). Hence \(M_2\) is not \(p\)-supersoluble and so every non-cyclic chief factor of \(M_2\) is of order \(p^2\). But every chief factor of \(M_2\) below \(\Phi(G)\) is of order \(p\). Consequently, \(N_1/\Phi(G)\) is the unique complemented chief factor of \(M_2\) in the chief series \((\gamma_2) \(\cap M_2\) \in X\). It follows that \(\Phi(G) \leq \Phi(M_2)\). Since \(\Phi(M_2)\) is a nilpotent group, we obtain by order considerations that \(\Phi(G) = \text{O}_p(\Phi(M_2))\). However, the same arguments of the proof for \(r = 1\) show now that \(\Phi(G) = 1\), contrary to supposition. Therefore \(N_1/\Phi(G)\) and \(N_2/\Phi(G)\) have the same order. If \(|N_1/\Phi(G)| = |N_2/\Phi(G)| = p\), then \(G\) is \(p\)-supersoluble, and the theorem holds.

Assume that \(|N_1/\Phi(G)| = |N_2/\Phi(G)| = p^2\). Since \(N_3-i/\Phi(G)\) is a chief factor of \(M_i\), \(M_i\) is not \(p\)-supersoluble, \(i = 1, 2\). Then all non-cyclic chief factors of \(M_i\) are \(M_1\)-isomorphic and have order \(p^2\), and every complemented chief factor of \(M_i\) is non-cyclic. Assume that \(X\) is \(p\)-supersoluble. Then every chief factor of \(X\) below \(\Phi(G)\) is cyclic. Certainly these chief factors are
also chief factors of $G$. Since $M_i$ has no cyclic complemented chief factors, exactly similar arguments to those used above show that $\Phi(G) = \Phi(M_i) = 1$. This contradiction shows that $X$ is not $p$-supersoluble and so $X$ satisfies the properties enunciated in the statement of the theorem. In particular, $N_i/\Phi(G)$ is $M_i$-isomorphic to an $X$-chief factor of the form $\Phi(G)/C$, which is also a $G$-chief factor. It implies that $N_i/\Phi(G)$ and $N_2/\Phi(G)$ are $G$-isomorphic, and $G$ satisfies properties 1 and 2 by [2, 1.2.36].

Suppose that $r \geqslant 3$. Denote $G = N_1M_1 = N_2M_2$, where $M_1$, $M_2$ are maximal subgroups of $G$ such that $N_1 \cap M_1 = N_2 \cap M_2 = \Phi(G)$. Consider the following two chief series of $G$:

\[ \begin{align*}
(\delta_1): & \quad 1 \leqslant \cdots \leqslant \Phi(G) \leqslant N_1 \leqslant N_1N_2 \leqslant N_1N_2N_3 \leqslant \cdots \leqslant G, \\
(\delta_2): & \quad 1 \leqslant \cdots \leqslant \Phi(G) \leqslant N_2 \leqslant N_1N_2 \leqslant N_1N_2N_3 \leqslant \cdots \leqslant G.
\end{align*} \]

Intersecting the series $(\delta_i)$ term-by-term with $M_i$, $i = 1, 2$, we get the series:

\[ \begin{align*}
(\delta_1) \cap M_1: & \quad 1 \leqslant \cdots \leqslant \Phi(G) \leqslant N_1 \cap M_1 \leqslant N_2 \cap M_1 \leqslant N_2N_3 \cap M_1 \leqslant \cdots \leqslant M_1, \\
(\delta_2) \cap M_2: & \quad 1 \leqslant \cdots \leqslant \Phi(G) \leqslant N_2 \cap M_2 \leqslant N_1 \cap M_2 \leqslant N_2N_3 \cap M_2 \leqslant \cdots \leqslant M_2.
\end{align*} \]

By induction, we have that $M_1$ is $p$-supersoluble or all non-cyclic chief factors of $M_1$ are $M_1$-isomorphic and have order $p^2$ and every complemented chief factor of $M_i$ is non-cyclic, $i = 1, 2$. We distinguish two possibilities:

1. $M_1$ or $M_2$ is $p$-supersoluble.
   Assume that $M_1$ is $p$-supersoluble. Then it follows that $N_2/\Phi(G)$ and $N_3/\Phi(G)$ have order $p$. If $M_2$ were not $p$-supersoluble, then it would follow that all complemented $p$-chief factors of $M_2$ are $M_2$-isomorphic and of order $p^2$. In that case, $N_3/\Phi(G)$ would have order $p^2$. This contradiction yields that $M_2$ is $p$-supersoluble. In that case, every chief factor $N_i/\Phi(G)$ has order $p$, $i = 1, \ldots, r$. In this case $G$ is $p$-supersoluble, and the result holds.

2. Neither $M_1$ nor $M_2$ is $p$-supersoluble.
   Then all non-cyclic $M_1$-chief factors are $M_1$-isomorphic and have order $p^2$ and every complemented $p$-chief factor of $M_i$ is non-cyclic. In particular, $N_2/\Phi(G)$, $N_3/\Phi(G)$, $\ldots$, $N_r/\Phi(G)$ are $M_1$-isomorphic (and so $G$-isomorphic), and have order $p^2$. Since $M_2$ is not $p$-supersoluble, it follows that $N_1/\Phi(G)$, $N_3/\Phi(G)$, $\ldots$, $N_r/\Phi(G)$ are $M_1$-isomorphic (and so $G$-isomorphic). Consequently, $N_1/\Phi(G)$, $N_2/\Phi(G)$, $\ldots$, $N_r/\Phi(G)$ are $G$-isomorphic. Applying [2, 1.2.36], $G$ satisfies the properties enunciated in the statement of the theorem.

Therefore we conclude that the result as stated is true. \( \square \)

The proof of Theorem 3.4 leads also to the following result:

**Corollary 3.5.** Let $G$ be a $p$-soluble group satisfying property (†) such that $O_p'(G) = 1$. If $G$ is not $p$-supersoluble and $F(G)/\Phi(G)$ is a chief factor of $G$, then $\Phi(G) = 1$.

As an interesting deduction we have the

**Corollary 3.6.** Let $G$ be a $p$-soluble group satisfying property (†). Assume that $O_p(G) = 1$. Then either $G$ is $p$-supersoluble or $\Phi(G) = 1$ and all complemented $p$-chief factors of $G$ are $G$-isomorphic and have order $p^2$.

**Proof.** Applying Theorem 3.3, it follows that $F(G) = O_p(G)$ is the unique Sylow $p$-subgroup of $G$. We shall proceed by induction on $|G|$. As usual, we write $F(G/\Phi(G)) = N_1/\Phi(G) \times \cdots \times N_r/\Phi(G)$, where
\[ N_i/\Phi(G) \] is a minimal normal subgroup of \( G/\Phi(G) \) for all \( i \). According to [2, 1.2.36], every complemented \( p \)-chief factor is \( G \)-isomorphic to \( N_i/\Phi(G) \) for some \( i \). Certainly, we can assume that \( r \geq 2 \) by Theorem 3.4 and Corollary 3.5. Suppose that \( G \) is not \( p \)-supersoluble. According to Theorem 3.4, \( N_i/\Phi(G) \) has order \( p^2 \) by Theorem 3.4. Let \( M_i \) be a maximal subgroup of \( G \) such that \( G = N_iM_i \) and \( N_i \cap M_i = \Phi(G) \). Consequently, \( N_i/\Phi(G) \) is a \( p \)-chief factor of \( M_i \) for all \( j \neq i \). It follows that \( M_i \) is not \( p \)-supersoluble. We observe that \( C_{M_i}(N_i/\Phi(G)) \) is the Sylow \( p \)-subgroup of \( M_i \) for all \( j \neq i \). Therefore \( O_{p'}(M_i) = 1 \). Consequently \( \Phi(M_i) = 1 \) by induction. It therefore follows that \( \Phi(N_i) = 1 \) for all \( i \) and \( N_i \) is centralised by \( N_j \) for all \( j \neq i \). Then \( N_i \) is elementary abelian and so it has the structure of G-module over the Galois field \( \text{GF}(p) \) in the natural way. Since \( N_i \) is centralised by \( F(G) \), Maschke’s theorem [6, A, 11.5] implies the complete reducibility of the representation space. In particular, there exists a minimal normal subgroup \( K_i \) of \( G \) of order \( p^2 \) such that \( N_i = \Phi(G) \times K_i \), \( i = 1, 2, \ldots, r \). Consequently, \( G = N_1 \cdots N_rG_{p'} = K_1 \cdots K_rG_{p'} \). \( F(G) = K_1 \cdots K_r \) and \( \Phi(G) = 1 \), as required. □

**Proof of the main theorem.** We prove the necessity of the condition by induction on the order of \( G \). Certainly, by Lemma 2.1, we may assume that \( O_{p'}(G) = 1 \). Let \( P \) be a Sylow \( p \)-subgroup of \( G \). It may be supposed that \( |P| \) is greater than \( p \). Then, applying Theorem 3.3, \( G \) is \( p \)-soluble and, by Theorem 3.2, the \( p \)-length of \( G \) is at most 1. Hence \( P = O_{p'}(G) = F(G) \) is the Sylow \( p \)-subgroup of \( G \). By Corollary 3.6, we have that either \( G \) is \( p \)-supersoluble or \( \Phi(G) = 1 \). Suppose that \( G \) is not \( p \)-supersoluble. Then \( \Phi(G) = 1 \), and \( F(G) \) is elementary abelian and it can be regarded as a completely reducible G-module over the Galois field \( \text{GF}(p) \) by [6, A, 11.5]. This means that \( P \) is expressible as a direct product of minimal normal subgroups of \( G \), say \( P = V_1 \times \cdots \times V_r \), where \( V_i \) is an irreducible \( G \)-module over \( \text{GF}(p) \) (\( i = 1, \ldots, r \)). By Corollary 3.6, each \( |V_i| = p^2 \), \( i = 1, \ldots, r \), and all of them are \( G \)-isomorphic. Now we consider the case that \( r > 1 \). Consider the submodule \( W = V_1 \times V_2 \). Write \( K = \text{GF}(p) \) and \( V = V_1 \). Suppose that \( V \) is an absolutely irreducible \( G \)-module. Then \( E = \text{End}_{kG}(V) = K \) and the \( G \)-endomorphisms of \( V \) are exactly those defined by \( \phi_t : V \to V \), given by \( v^\phi = v^t \), \( v \in V \), \( t \in K \). According to [6, B, 8.2], the irreducible submodules of \( W \) are \( V^t = \{ (v, v') : v \in V \} \), for \( t \in K \), and \( V_2 = \{ (1, v) : v \in V \} \). On the other hand, \( V \), as a vector \( K \)-space, has dimension 2. Let \( (a, b) \) be a basis of \( V \). With the obvious notation, consider the subgroup \( A = \langle (a_1 b_1, 1), (1, a_2 b_2) \rangle \) of \( W \). Then \( A \) has order \( p^2 \) and so \( A \) is a partial CAP-subgroup of \( G \). But \( A \cap V_2 = \langle (1, a_2 b_2) \rangle \) and \( A \cap V = \langle (a_1 b_1, (a_2 b_2)^t) \rangle \) for any \( t \in K \), that is, \( A \) neither covers nor avoids any irreducible submodule of \( W \), contrary to Lemma 2.2. Hence \( V \) is not an absolutely \( G \)-module and the necessity of the condition holds.

To prove the sufficiency, it may be assumed that \( G \) satisfies the condition 3 and \( O_{p'}(G) \neq 1 \). Hence \( F(G) = \text{Soc}(G) = N_1 \times \cdots \times N_r \), where \( N_i \subseteq V \) is a \( G \)-irreducible module over \( \text{GF}(p) \) of dimension 2, for every \( i = 1, \ldots, r \). Moreover, \( F(G) \) is a Sylow \( p \)-subgroup of \( G \). We may also assume that \( r \geq 2 \). Then \( V \) is not an absolutely irreducible \( G \)-module. Let \( Q = \langle a, b \rangle \) be a subgroup of \( G \) of order \( p^2 \). We prove that \( Q \) is a partial CAP-subgroup of \( G \) by induction on the order of \( G \). Obviously, we can suppose that \( Q \) is not a minimal normal subgroup of \( G \). Let \( N \) be a minimal normal subgroup of \( G \). Then \( Q \cap N \) is trivial or has order \( p \). Assume that \( Q \cap N \) is of order \( p \) for all minimal normal subgroups \( N \) of \( G \). Then two different minimal normal subgroups produce different subgroups of order \( p \) of \( Q \). Since the number of subgroups of order \( p \) of \( Q \) is exactly \( p + 1 \), it follows that \( G \) has exactly \( p + 1 \) minimal normal subgroups. However, according to [6, B, 8.2], \( G \) has at least \( p^k + 1 \) minimal normal subgroups, where \( p^k \) is the number of elements in \( \text{End}_{kG}(V) \), and \( k \geq 2 \) since \( V \) is not an absolutely irreducible \( G \)-module. This contradiction implies that there exists a minimal normal subgroup \( A \) of \( G \) such that \( Q \cap A = 1 \). Since the group \( G/A \) satisfies the conditions of the theorem, we have that \( QA/A \) is a partial CAP-subgroup of \( G/A \). It is clear then that \( Q \) is a partial CAP-subgroup of \( G \). We conclude that \( G \) satisfies property (\( \diamondsuit \)). □

**Acknowledgments**

The first and the second authors have been supported by the research grant MTM2010-19938-C03-01 from MICINN (Spain). Most of this research was carried out during a visit of the third author to the Departament d’Algebra, Universitat de València, Burjassot, València, Spain, during the academic year 2009–10.
References