Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Linear Algebra and its Applications 423 (2007) 230–245
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

Generalized Pascal functional matrix and its applications ☆

Yongzhi Yang *, Catherine Micek

Department of Mathematics, University of St. Thomas, 2115 Summit Ave., St. Paul, MN 55105-1079, USA

Received 8 May 2006; accepted 20 December 2006
Available online 19 January 2007
Submitted by R. Brualdi

Abstract

In this paper, we introduce the generalized Pascal functional matrix and show that the existing variations of Pascal matrices are special cases of this generalization. We study some algebraic properties of such generalized Pascal functional matrices. In addition, we demonstrate a direct application of these properties by deriving several novel combinatorial identities and a nontraditional approach for LU decompositions of some well-known matrices (such as symmetric Pascal matrices).

AMS classification: 15A15; 15A23

Keywords: Generalized Pascal functional matrix; Pascal matrices; Combinatorial identities; LU decomposition; Generalizing functions

1. Introduction

Over the past few decades, there has been an interest in Pascal matrices in mathematical literature (see [2,3,4,9,10]). The $(n+1)\times(n+1)$ Pascal matrix, denoted by $P_n[x]$, was defined in [4] as

$$(P_n[x])_{i,j} = \begin{cases} \binom{i}{j} x^{(i-j)} & \text{if } i \geqslant j, \\ 0, & \text{otherwise,} \end{cases} \qquad i,j = 0,1,\ldots,n. \tag{1}$$

☆ This research was supported by the University Scholar Grant of the University of St. Thomas.
* Corresponding author.
E-mail address: y9yang@stthomas.edu (Y. Yang).

0024-3795/$ - see front matter
doi:10.1016/j.laa.2006.12.014

In [4], Call and Velleman discussed the inverse of $P_n[x]$ and a few basic properties of the matrix. Some variations on this matrix and their factorizations were discussed by Brawer and Pirovino in [2], most notably the symmetric Pascal matrix, Q_n, defined as

$$(Q_n)_{i,j} = \binom{i+j}{j}, \quad i, j = 0, 1, \ldots, n. \tag{2}$$

They showed that Q_n can be decomposed as the product of a lower triangular Pascal matrix, $P_n[1]$, and an upper triangular Pascal matrix, $P_n[1]^{\mathrm{T}}$.

Zhang and Liu further elaborated on the results of [2] in [9] by introducing extended generalized rectangular Pascal matrix $\Psi_n[x, y]$ defined as

$$(\Psi_n[x, y])_{i,j} = x^{i-j} y^{i+j} \binom{i+j}{j}, \quad i, j = 0, 1, \ldots, n \tag{3}$$

and extended generalized lower triangular Pascal matrix $\Phi_n[x, y]$ defined as

$$(\Phi_n[x, y])_{i,j} = \begin{cases} x^{i-j} y^{i+j} \binom{i}{j} & \text{if } i \geqslant j, \\ 0, & \text{otherwise,} \end{cases} \quad i, j = 0, 1, \ldots, n. \tag{4}$$

They demonstrated that $\Psi_n[x, y]$ has the LU decomposition $\Psi_n[x, y] = \Phi_n[x, y] P_n^{\mathrm{T}}[y/x]$.

Another variation of Pascal functional matrix was introduced by Bayat and Teimoori in [3] and it is defined as

$$(H_{n,\lambda}[x])_{i,j} = \begin{cases} x^{(i-j)|\lambda} \binom{i}{j} & \text{if } i \geqslant j, \\ 0, & \text{otherwise,} \end{cases} \quad i, j = 0, 1, \ldots, n, \tag{5}$$

where $x^{n|\lambda}$ is the generalized upper factorial and is defined as following:

$$x^{n|\lambda} = \begin{cases} x(x+\lambda)(x+2\lambda)\cdots(x+(n-1)\lambda), & \text{if } n \geqslant 1, \\ 1, & \text{if } n = 0. \end{cases} \tag{6}$$

Their study of the algebraic properties of such a matrix yielded several interesting combinatorial identities.

In [10], Zhao and Wang extended the Pascal functional matrix defined by Eq. (5) to more general Pascal functional matrix, denoted by $G_n[x]$, and defined by

$$(G_n[x])_{i,j} = \begin{cases} g_{i-j}(x) \binom{i}{j} & \text{if } i \geqslant j, \\ 0, & \text{otherwise,} \end{cases} \quad i, j = 0, 1, \ldots, n. \tag{7}$$

where $\{g_n(x)\}$ is a sequence of binomial-type polynomials, i.e., for all of $\{g_n(x)\}$, $g_n(x+y) = \sum_{k=0}^{n} \binom{n}{k} g_k(x) g_{n-k}(y)$ for any x and y. The authors proved some algebraic properties of such a matrix and derived combinatorial identities from the properties.

In Section 2, we introduce a more general Pascal functional matrix and show that all the existing Pascal matrices are special cases of one generalized Pascal functional matrix. Also, we study algebraic properties of this generalized Pascal functional matrix. To show interesting applications of the new Pascal functional matrix and its algebraic properties, in Section 3, we develop new combinatorial identities and introduce a novel LU decomposition technique in Section 4. Finally, We conclude in Section 5 by discussing future work to be done in this area.

2. A more generalized Pascal functional matrix and its algebraic properties

The new generalized Pascal functional matrix is defined as follows. To avoid any unnecessary confusion, we use $f^{(k)}$ to stand for the kth order derivative of f and use f^k to represent the kth power of f in the entire paper. In addition, $f^{(0)} = f$ and $f^0 = 1$.

Definition 2.1. Let $f(t; x)$ be a function of t with a parameter x such that the nth-order derivatives with respect to t exist. The generalized Pascal functional matrix, denoted by $\mathscr{P}_n[f(t; x)]$, is an $(n+1) \times (n+1)$ matrix which is defined as

$$(\mathscr{P}_n[f(t;x)])_{ij} = \begin{cases} \binom{i}{j} f^{(i-j)}(t;x) & \text{if } i \geqslant j, \\ 0, & \text{otherwise,} \end{cases} \qquad i, j = 0, 1, 2, \ldots, n. \tag{8}$$

It can be shown that all well-known variations of Pascal matrices in [3,4,10] are special cases of this new generalization of Pascal functional matrix. Consider the following:

(1) Let $f(t; x) = \mathrm{e}^{xt}$ in Eq. (8). When $t = 0$,

$$\mathscr{P}_n[f(t;x)]|_{t=0} = \mathscr{P}_n[\mathrm{e}^{xt}]|_{t=0} = P_n[x], \tag{9}$$

which is the Pascal matrix $P_n[x]$ introduced by Call and Velleman in [4].

(2) Consider the truncated exponential generating function for the binomial-type polynomial sequence of $\{g_n(x)\}$

$$f(t;x) = \sum_{k=0}^{n} g_k(x) \frac{t^k}{k!}. \tag{10}$$

It is easy to see that

$$\mathscr{P}_n[f(t;x)]|_{t=0} = \mathscr{P}_n\left[\sum_{k=0}^{n} g_k(x)\frac{t^k}{k!}\right]\Bigg|_{t=0} = G_n[x], \tag{11}$$

which is the Pascal functional matrix introduced by Zhao and Wang in [10].

(3) Since $\{[x]^{k|\lambda}\}$ is a special binomial-type polynomial sequence (see the proof in [3]), then choosing $g_k(x) = [x]^{k|\lambda}$ in Eq. (11) leads

$$\mathscr{P}_n[f(x,t)]|_{t=0} = \mathscr{P}_n\left[\sum_{k=0}^{n} [x]^{k|\lambda}\frac{t^k}{k!}\right]\Bigg|_{t=0} = H_{n,\lambda}[x], \tag{12}$$

which is the Pascal functional matrix introduced by Bayat and Teimoori in [3].

Next, we explore some of the algebraic properties for the new generalized Pascal functional matrix. Using Definition 2.1 and the Leibniz rule of differentiation, we can obtain the following theorem.

Theorem 2.1. *Let $\mathscr{P}_n[f(t;x)]$ and $\mathscr{P}_n[g(t;x)]$ be any two $(n+1) \times (n+1)$ Pascal functional matrices defined in Eq.* (8). *Then*

$$\mathscr{P}_n[f(t;x)]\mathscr{P}_n[g(t;x)] = \mathscr{P}_n[f(t;x)g(t;x)] = \mathscr{P}_n[g(t;x)]\mathscr{P}_n[f(t;x)].$$

Proof. It is obvious that $(\mathscr{P}_n[f(t;x)]\mathscr{P}_n[g(t;x)])_{i,j} = 0$ for $i < j$ because $\mathscr{P}_n[f(t;x)]$ and $\mathscr{P}_n[g(t;x)]$ are lower triangular matrices. For $i \geqslant j$, we have, by the multiplication rule for matrices, the entry in the ith row and jth column is

$$
\begin{aligned}
(\mathscr{P}_n[f(t;x)]\mathscr{P}_n[g(t;x)])_{i,j} &= \sum_{k=0}^{n} \binom{i}{k}\binom{k}{j} f^{(i-k)}(t;x)g^{(k-j)}(t;x) \\
&= \sum_{k=j}^{i} \binom{i}{k}\binom{k}{j} f^{(i-k)}(t;x)g^{(k-j)}(t;x) \\
&= \binom{i}{j}\sum_{k=j}^{i} \frac{(i-j)!}{(i-k)!(k-j)!} f^{(i-k)}(t;x)g^{(k-j)}(t;x).
\end{aligned}
$$

Letting $m = k - j$ and using Leibniz rule for differentiation yields

$$
\begin{aligned}
(\mathscr{P}_n[f(t;x)]\mathscr{P}_n[g(t;x)])_{i,j} &= \binom{i}{j}\sum_{m=0}^{i-j} \binom{i-j}{m} f^{(i-j-m)}(t;x)g^{(m)}(t;x) \\
&= \binom{i}{j}(f(t;x)g(t;x))^{(i-j)} \\
&= (\mathscr{P}_n[f(t;x)g(t;x)])_{i,j}.
\end{aligned}
$$

This completes the proof. □

If $(f(t;x)^{-1})^{(k)} = \left(\frac{1}{f(t;x)}\right)^{(k)}$ exists for $k = 0, 1, \ldots, n$, then setting $g(t;x) = f(t;x)^{-1}$ in Theorem 2.1 leads the following corollary.

Corollary 2.1. *Let $\mathscr{P}_n[f(t;x)]$ be any $(n+1)\times(n+1)$ Pascal functional matrix. If $(f(t;x)^{-1})^{(k)} = \left(\frac{1}{f(t;x)}\right)^{(k)}$ exists for $k = 0, 1, \ldots, n$, then*

$$
\mathscr{P}_n^{-1}[f(t;x)] = \mathscr{P}_n[f(t;x)^{-1}] = \mathscr{P}_n\left[\frac{1}{f(t;x)}\right].
$$

Proof. Let $g(t;x) = f(t;x)^{-1}$ in Theorem 2.1. Then, we obtain

$$
\mathscr{P}_n[f(t;x)]\mathscr{P}_n[f(t;x)^{-1}] = \mathscr{P}_n[f(t;x)f(t;x)^{-1}] = \mathscr{P}_n[1] = I_{(n+1)},
$$

where $I_{(n+1)}$ is the corresponding $(n+1)\times(n+1)$ identity matrix. This implies that $\mathscr{P}_n[f(t;x)^{-1}] = \mathscr{P}_n^{-1}[f(t;x)]$. □

Consider Pascal functional matrix $G_n[x]$ defined by Zhao and Wang in [10] (see Eq. (7)) and let $f(t;x) = \sum_{k=0}^{n} g_k(x)\frac{t^k}{k!}$ and $h(t;y) = \sum_{k=0}^{n} g_k(y)\frac{t^k}{k!}$ in Theorem 2.1. Noting

$$
\begin{aligned}
f(t;x)h(t;y) &= \left(\sum_{k=0}^{n} g_k(x)\frac{t^k}{k!}\right)\left(\sum_{l=0}^{n} g_l(y)\frac{t^l}{l!}\right) \\
&= \sum_{j=0}^{n}\left(\sum_{m=0}^{j} \frac{g_{j-m}(x)}{(j-m)!}\frac{g_m(y)}{m!}\right)t^j + t^{n+1}\Omega(t;x,y)
\end{aligned}
$$

$$= \sum_{j=0}^{n}\left(\sum_{m=0}^{j}\frac{j!}{m!(j-m)!}g_{j-m}(x)g_m(y)\right)\frac{t^j}{j!}+t^{n+1}\Omega(t;x,y)$$
$$= \sum_{j=0}^{n}\left(\sum_{m=0}^{j}\binom{j}{m}g_{j-m}(x)g_m(y)\right)\frac{t^j}{j!}+t^{n+1}\Omega(t;x,y),$$

where $\Omega(t; x, y)$ is a $(n-1)$th degree polynomial of t, yields

$$\begin{aligned}G_n[x]G_n[y] &= \mathscr{P}_n[f(t;x)]|_{t=0}\mathscr{P}_n[h(t;y)]|_{t=0} = \mathscr{P}_n[f(t;x)h(t;y)]|_{t=0}\\ &= \mathscr{P}_n\left[\sum_{j=0}^{n}\left(\sum_{m=0}^{j}\binom{j}{m}g_{j-m}(x)g_m(y)\right)\frac{t^j}{j!}+t^{n+1}\Omega(t;x,y)\right]\Bigg|_{t=0}\\ &= \mathscr{P}_n\left[\sum_{j=0}^{n}\left(\sum_{m=0}^{j}\binom{j}{m}g_{j-m}(x)g_m(y)\right)\frac{t^j}{j!}\right]\Bigg|_{t=0}+\mathscr{P}_n[t^{n+1}\Omega(t;x,y)]|_{t=0}.\end{aligned}$$

Noting $g_n(x)$ is the binomial-type polynomial sequence and $\mathscr{P}_n[t^{n+1}\Omega(t;x,y)]\|_{t=0}$ vanishes leads

$$\begin{aligned}G_n[x]G_n[y] &= \mathscr{P}_n\left[\sum_{j=0}^{n}g_j(x+y)\frac{t^j}{j!}\right]\Bigg|_{t=0}\\ &= \mathscr{P}_n[f(t;x+y)]|_{t=0} = G_n[x+y].\end{aligned}$$

In light of this result, we redevelop Theorem 2.1 from [10] and summarize it as following.

Corollary 2.2. *For any real numbers x and y, we have*

$$G_n[x]G_n[y] = G_n[x+y].$$

By noting that $G_n[0]$ is an identity matrix, I_{n+1}, and letting $y = -x$ in Corollary 2.2, we obtain another corollary.

Corollary 2.3. *The inverse matrix of $G_n[x]$ is $G_n[-x]$, i.e., $G_n^{-1}[x] = G_n[-x]$.*

The immediate consequences of Corollary 2.3 are illustrated by the following examples. Note that they are consistent with the results in [4] and [3], respectively.

Example 2.1. Let $\{g_n(x)\} = \{x^n\}$ in Corollary 2.3. This yields $P_n^{-1}[x] = P_n[-x]$. For instance, when $n = 3$ we have

$$P_3[x]=\begin{bmatrix}1&0&0&0\\x&1&0&0\\x^2&2x&1&0\\x^3&3x^2&3x&1\end{bmatrix} \quad\text{and}\quad P_3^{-1}[x]=P_3[-x]=\begin{bmatrix}1&0&0&0\\-x&1&0&0\\x^2&-2x&1&0\\-x^3&3x^2&-3x&1\end{bmatrix}.$$

Example 2.2. Let $\{g_n(x)\} = \{[x]^{n|\lambda}\}$ in Corollary 2.3. This yields $H_{n,\lambda}^{-1}[x] = H_{n,\lambda}[-x]$. For instance, when $n = 3$ we have

$$H_{3,\lambda}[x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ x(x+\lambda) & 2x & 1 & 0 \\ x(x+\lambda)(x+2\lambda) & 3x(x+\lambda) & 3x & 1 \end{bmatrix} \quad \text{and}$$

$$H_{3,\lambda}^{-1}[x] = H_{3,\lambda}[-x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -x & 1 & 0 & 0 \\ x(x-\lambda) & -2x & 1 & 0 \\ -x(x-\lambda)(x-2\lambda) & 3x(x-\lambda) & -3x & 1 \end{bmatrix}.$$

3. Some combinatorial identities

3.1. A novel formula for $(f(t)^{-1})^{(k)}$

We next obtain some novel combinatorial identities by employing the algebraic properties of the Pascal functional matrix just developed.

Theorem 3.1. *If $f(t)$ has nth order derivatives and $(f(t;x)^{-1})^{(k)} = \left(\frac{1}{f(t;x)}\right)^{(k)}$ exists for $k = 0, 1, \ldots, n$, then*

$$\left(\frac{1}{f(t)}\right)^{(k)} = \frac{(-1)^n}{f^{n+1}(t)} \sum_{l=0}^{n-1} (-1)^l \binom{n+1}{l} (f^{n-l}(t))^{(k)} f^l(t), \tag{13}$$

where $k = 1, 2, \ldots, n$.

Proof. Since $\mathscr{P}_n[f(t)] - f(t)I_{n+1}$ is a lower triangular matrix with zeros along the diagonal, $(\mathscr{P}_n[f(t)] - f(t)I_{n+1})^{n+1} = 0$. Expanding $(\mathscr{P}_n[f(t)] - f(t)I_{n+1})^{n+1} = 0$ by binomial formula yields

$$\sum_{l=0}^{n+1} (-1)^l \binom{n+1}{l} f^l(t) \mathscr{P}_n^{n+1-l}[f(t)] = 0. \tag{14}$$

Moving the last term of Eq. (14) to its right side of the equation and then dividing $(-1)^n f^{n+1}(t)$ on both sides leads

$$\frac{\left(\mathscr{P}_n^n[f(t)] - \binom{n+1}{1} f(t)\mathscr{P}_n^{n-1}[f(t)] + \cdots + (-1)^n \binom{n+1}{n} f^n(t) I_{n+1}\right) \mathscr{P}_n[f(t)]}{(-1)^n f^{n+1}(t)} = I_{n+1}.$$

This suggests that

$$\mathscr{P}_n^{-1}[f(t)] = \frac{\left(\mathscr{P}_n^n[f(t)] - \binom{n+1}{1} f(t)\mathscr{P}_n^{n-1}[f(t)] + \cdots + (-1)^n \binom{n+1}{n} f^n(t) I_{n+1}\right)}{(-1)^n f^{n+1}(t)}. \tag{15}$$

By Corollary 2.1, we have

$$\mathscr{P}_n[f(t)^{-1}] = \frac{\mathscr{P}_n^n[f(t)] - \binom{n+1}{1} f(t)\mathscr{P}_n^{n-1}[f(t)] + \cdots + (-1)^n \binom{n+1}{n} f^n(t) I_{n+1}}{(-1)^n f^{n+1}(t)}. \tag{16}$$

Using Theorem 2.1 and comparing the entry in the ith row and jth column ($i > j$) on both sides of Eq. (16) yields

$$(f(t)^{-1})^{(i-j)} = \frac{(-1)^n}{f^{n+1}(t)}\left((f^n(t))^{(i-j)} - \binom{n+1}{1} f(t)(f^{n-1}(t))^{(i-j)} + \cdots + (-1)^{n-1}\binom{n+1}{n-1} f^{n-1}(t) f^{(i-j)}(t)\right). \tag{17}$$

Finally, letting $k = i - j$, $k = 1, 2, \ldots, n$, in Eq. (17) yields Eq. (13). □

Theorem 3.1 provides a formula that allows us to represent $(\frac{1}{f(t)})^{(k)}$ in terms of $(f^i(t))^{(k)}$, $i = 1, 2, \ldots, n$. Furthermore, by carefully choosing $f(t)$ in Theorem 3.1 we are not only able to develop many novel combinatorial identities but also redevelop some well-known identities. For the sake of brevity, we demonstrate only four of them in this section.

Corollary 3.1

$$\alpha^{\langle k\rangle} = \sum_{l=0}^{n-1}(-1)^{l-n}\binom{n+1}{l}((l-n)\alpha)^{\langle k\rangle}, \tag{18}$$

where $x^{\langle k\rangle} = x(x+1)\cdots(x+k-1)$ *is rising factorial and* $1 \leqslant k \leqslant n$.

Proof. Substituting $f(t) = t^\alpha$ in Theorem 3.1 yields

$$\left(\frac{1}{t^\alpha}\right)^{(k)} = (t^{-\alpha})^{(k)} = (-1)^n t^{-(n+1)\alpha}\sum_{l=0}^{n-1}(-1)^l\binom{n+1}{l}(t^{(n-l)\alpha})^{(k)} t^{l\alpha}. \tag{19}$$

Setting $t = 1$ in Eq. (19) leads that the left hand of Eq. (19) is

$$(-\alpha)(-\alpha-1)(-\alpha-2)\cdots(-\alpha-k+1) = (-1)^k\alpha^{\langle k\rangle} \tag{20}$$

and the right hand of Eq. (19) is

$$\sum_{l=0}^{n-1}(-1)^{l-n}\binom{n+1}{l}\{\alpha(n-l)\}\{\alpha(n-l)-1\}\{\alpha(n-l)-2\}\cdots\{\alpha(n-l)-k+1\} = \sum_{l=0}^{n-1}(-1)^{l-n+k}\binom{n+1}{l}((l-n)\alpha)^{\langle k\rangle}. \tag{21}$$

Equating Eqs. (20)–(21) and dividing $(-1)^k$ yields the corollary. □

If we substitute $f(t) = \mathrm{e}^t$ and set $t = 0$ in Theorem 3.1, we can yield the following combinatorial identity:

Corollary 3.2

$$\sum_{l=0}^{n-1}(-1)^{n-l}\binom{n+1}{l}(n-l)^k = (-1)^k, \quad k = 1, 2, \ldots, n. \tag{22}$$

Let us consider the Hermite polynomials $H_n^{(\nu)}(x)$ of variance ν and the Euler polynomials $E_n^{(\omega)}(x)$ of order ω as defined in [6]. It is well-known that the exponential generating functions for $H_n^{(\nu)}(x)$ and $E_n^{(\omega)}(x)$ are

$$\sum_{k=0}^{\infty} \frac{H_k^{(\nu)}(x)}{k!} t^k = \mathrm{e}^{xt-\nu t^2/2} \tag{23}$$

and

$$\sum_{k=0}^{\infty} \frac{E_k^{(\omega)}(x)}{k!} t^k = \left(\frac{2}{\mathrm{e}^t+1}\right)^{\omega} \mathrm{e}^{xt}, \tag{24}$$

respectively. Using Theorem 3.1 and Eqs. (23)–(24) we obtain the following two new identities.

Corollary 3.3

$$H_k^{(-\nu)}(-x) = \sum_{m=1}^{n} (-1)^m \binom{n+1}{m+1} H_k^{(m\nu)}(mx), \quad k = 1, 2, \ldots, n. \tag{25}$$

Proof. Substituting $f(t) = \mathrm{e}^{xt-\nu t^2/2}$ in Theorem 3.1 yields

$$\left(\frac{1}{\mathrm{e}^{xt-\nu t^2/2}}\right)^{(k)} = \frac{(-1)^n}{(\mathrm{e}^{xt-\nu t^2/2})^{n+1}} \sum_{l=0}^{n-1} (-1)^l \binom{n+1}{l} (\mathrm{e}^{[n-l]xt-[n-l]\nu t^2/2})^{(k)} \mathrm{e}^{lxt-l\nu t^2/2}.$$

Setting $t = 0$ in the above equation leads

$$H_k^{(-\nu)}(-x) = \sum_{l=0}^{n-1} (-1)^{n+l} \binom{n+1}{l} H_k^{([n-l]\nu)}([n-l]x).$$

Changing the dummy index variable l to $m = n - l$ yields the corollary. □

Corollary 3.4

$$E_k^{(-\omega)}(-x) = \sum_{m=1}^{n} (-1)^m \binom{n+1}{m+1} E_k^{(m\omega)}(mx), \quad k = 1, 2, \ldots, n. \tag{26}$$

Since the proof of Corollary 3.4 is similar to the proof of Corollary 3.3, we omit it.

3.2. General even–odd-subject identity

We next consider the well-known combinatorial identity

$$\sum_{l=1}^{n} (-1)^l \binom{n}{l} l^k = 0, \quad k = 1, 2, \ldots, n-1. \tag{27}$$

Specifically, for $k = 0$, we get the even–odd-subject identity, $\sum_{l=0}^{n} (-1)^l \binom{n}{l} = 0$. We shall now extend the Even–Odd-Subject identities to more general form.

Theorem 3.2. *If $f(t)$ has nth order derivatives, then*

$$\sum_{l=1}^{n}(-1)^l \binom{n}{l} (f^l(t))^{(k)} f^{n-l}(t) = 0, \quad k = 1, 2, \ldots, n-1. \tag{28}$$

Proof. Comparing the elements on the ith row and jth column, where $(i > j)$, on both sides of Eq. (14) yields

$$\sum_{l=0}^{n}(-1)^l \binom{n+1}{l} f^l(t)(f^{n+1-l}(t))^{(k)} = 0, \tag{29}$$

where $k = i - j$ and $k = 1, 2, \ldots, n$. Letting $m = n + 1 - l$ yields

$$\sum_{m=1}^{n+1}(-1)^m \binom{n+1}{m} f^{n+1-m}(t)(f^m(t))^{(k)} = 0. \tag{30}$$

Reindexing on n in Eq. (30) yields Eq. (28). □

Theorem 3.2 is a very rich identity. Using Theorem 3.2, we can derive several interesting combinatorial identities with the appropriately chosen function $f(t)$. For example, if $f(t) = \mathrm{e}^t$ and $t = 0$, we obtain Eq. (27). We demonstrate three more such identities below.

For $f(t) = t^\alpha$ in Theorem 3.2, we have the following identity.

Corollary 3.5. *For a positive integer k, $k < n$, $\sum_{l=1}^{n}(-1)^l \binom{n}{l} \langle \alpha l \rangle_k = 0$, where $\langle x \rangle_k = x(x-1) \cdots (x-k+1)$ is falling factorial.*

If $f(t) = \mathrm{e}^{xt-\nu t^2/2}$, Theorem 3.2 leads to a new identity for Hermite polynomials of variance ν.

Corollary 3.6. *For a positive integer k, $k < n$, $\sum_{l=1}^{n}(-1)^l \binom{n}{l} H_k^{(l\nu)}(lx) = 0$, where $H_k^{(\nu)}(x)$ the Hermite polynomials of variance ν.*

Proof. Substituting $f(t) = \mathrm{e}^{xt-\nu t^2/2}$ in Theorem 3.2 yields

$$\sum_{l=1}^{n}(-1)^l \binom{n}{l} (\mathrm{e}^{lxt-l\nu t^2/2})^{(k)} \mathrm{e}^{[n-l]xt-[n-l]\nu t^2/2} = 0.$$

Setting $t = 0$ in the above equation leads $\sum_{l=1}^{n}(-1)^l \binom{n}{l} H_k^{(l\nu)}(lx) = 0$. This comletes the proof of the corollary. □

Along the lines of the proof of Corollary 3.6, by letting $f(t) = (\frac{2}{\mathrm{e}^t+1})^\omega \mathrm{e}^{xt}$ in Theorem 3.2, we obtain the following novel identity for Euler polynomials with ω order:

Corollary 3.7. *For a positive integer k, $k < n$, $\sum_{l=1}^{n}(-1)^l \binom{n}{l} E_k^{(l\omega)}(lx) = 0$, where $E_k^{(\omega)}$ is the Euler polynomials of order ω.*

3.3. Extended version of general Tepper's identity

In addition to developing new identities, we can use our work done with the generalized Pascal functional matrix to derive a more general Tepper's identity. To this end, we need to introduce a series of lemmas.

Let e_i, $(0 \leqslant i \leqslant n)$, be the unit vector in $\mathbf{R}^{(n+1)\times 1}$ and let

$$e_f(t;y) = [f(t;y), f'(t;y), \ldots, f^{(n)}(t;y)]^{\mathrm{T}}.$$

Then we can obtain the following lemmas.

Lemma 3.1. *For any nonnegative integers k and i, $(0 \leqslant i \leqslant n)$, we have*

$$e_i^{\mathrm{T}} \mathscr{P}_n^k[f(t;x)] e_g(t;y) = [f^k(t;x) g(t;y)]^{(i)}. \tag{31}$$

Proof. Since Theorem 2.1,

$$\begin{aligned} \mathscr{P}_n^k[f(t;x)] e_g(t;y) &= \mathscr{P}_n[f^k(t;x)] e_g(t;y) \\ &= [(f^k(t;x)g(t;x)), (f^k(t;x)g(t;y))', \ldots, (f^k(t;x)g(t;y))^{(n)}]^{\mathrm{T}}. \end{aligned} \tag{32}$$

Then $\mathrm{e}_i^{\mathrm{T}} \mathscr{P}_n^k[f(t;x)] e_g(t;y) = [f^k(t;x)g(t;y)]^{(i)}$. $\square$

Lemma 3.2. *For any positive integer l and any function $f(t)$ with lth order derivative, we have*

$$[\mathscr{P}_l[f(t)] - f(t) I_{l+1}]^l = M_l, \tag{33}$$

where M_l is the $(l+1)\times(l+1)$ square matrix, in which all entries are zeros except $(M_l)_{l,0} = l!(f'(t))^l$.

Proof. Since $[\mathscr{P}_l[f(t)] - f(t)I_{l+1}]$ is a lower triangular matrix with zeros along the diagonal, $[\mathscr{P}_l[f(t)] - f(t)I_{l+1}]^l$ will be an $(l+1)\times(l+1)$ lower triangular matrix, in which all elements are zeros except $(M_l)_{l,0}$. To evaluate the entry $(M_l)_{l,0}$, we rewrite the matrix $[\mathscr{P}_l[f(t)] - f(t)I_{l+1}]$ as a sum of $Q[f(t)]$ and $R[f(t)]$, where

$$Q[f(t)] = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ f'(t) & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2f'(t) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & lf'(t) & 0 \end{bmatrix}_{(l+1)\times(l+1)}$$

and $R[f(t)] = [\mathscr{P}_l[f(t)] - f(t)I_{l+1}] - Q[f(t)]$. It is easy to show that $R[f(t)]Q[f(t)] = Q[f(t)]R[f(t)]$. Therefore,

$$[\mathscr{P}_l[f(t)] - f(t)I_{l+1}]^l = \sum_{k=0}^{l} \binom{l}{k} Q^k[f(t)] R^{l-k}[f(t)].$$

Noting $Q^k[f(t)]R^{l-k}[f(t)]$ vanishes for $k = 0, 1, \ldots, l-1$, we have $[\mathscr{P}_l[f(t)] - f(t)I_{l+1}]^l = Q^l[f(t)] = M_l$. In order to see the structure of M_l, we rewrite $Q[f(t)]$ as $f'(t)S$, where

$$
\begin{aligned}
S &= \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & l & 0 \end{bmatrix} \\
&= \begin{bmatrix} \binom{0}{0} z & 0 & 0 & \ldots & 0 & 0 \\ \binom{1}{0}(z)' & \binom{1}{1} z & 0 & \cdots & 0 & 0 \\ \binom{2}{0}(z)'' & \binom{2}{1}(z)' & \binom{2}{2} z & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \binom{l}{0}(z)^{(l)} & \binom{l}{1}(z)^{(l-1)} & \binom{l}{2}(z)^{(l-2)} & \cdots & \binom{l}{l-1}(z)' & \binom{l}{l} z \end{bmatrix}_{z=0} \\
&= \mathscr{P}_l[z]|_{z=0}.
\end{aligned}
$$

Thus, by Theorem 2.1, we have

$$Q^l[f(t)] = M_l = (f'(t))^l S^l = (f'(t))^l [\mathscr{P}_l[z]|_{z=0}]^l = (f'(t))^l \mathscr{P}_l[z^l]|_{z=0}.$$

Therefore,

$$
\begin{aligned}
M_l &= (f'(t))^l \begin{bmatrix} \binom{0}{0} z^l & 0 & 0 & \ldots & 0 & 0 \\ \binom{1}{0}(z^l)' & \binom{1}{1} z^l & 0 & \cdots & 0 & 0 \\ \binom{2}{0}(z^l)'' & \binom{2}{1}(z^l)' & \binom{2}{2} z^l & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \binom{l}{0}(z^l)^{(l)} & \binom{l}{1}(z^l)^{(l-1)} & \binom{l}{2}(z^l)^{(l-2)} & \cdots & \binom{l}{l-1}(z^l)' & \binom{l}{l} z^l \end{bmatrix}_{z=0} \\
&= (f'(t))^l \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ l! & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
\end{aligned}
$$

and consequently $(M_l)_{l,0} = l!(f'(t))^l$. This completes the proof. □

We can thus derive the extended version of general Tepper's identities found in [3,10].

Theorem 3.3. *For any positive integers l and k, $(k \leqslant l)$ and any functions $f(t)$ and $g(t)$ with lth order derivatives, we have*

$$\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}[f^m(t)g(t)]^{(k)}f^{l-m}(t)=l!(f'(t))^l g(t)\delta_{l,k}, \tag{34}$$

where $\delta_{l,k}$ *is Kronecker delta function.*

Proof. By Lemma 3.2, we have

$$\mathrm{e}_k^{\mathrm{T}}(\mathscr{P}_l[f(t)]-f(t)I_{l+1})^l e_g[t]=\mathrm{e}_k^{\mathrm{T}}M_l e_g[t]=l!(f'(t))^l g(t)\delta_{l,k}. \tag{35}$$

On the other hand, by Lemma 3.1, we obtain

$$\begin{aligned}
&\mathrm{e}_k^{\mathrm{T}}(\mathscr{P}_l[f(t)]-f(t)I_{l+1})^l e_g[t]\\
&\quad=\mathrm{e}_k^{\mathrm{T}}\left(\sum_{m=0}^{l}(-1)^{l-m}\binom{l}{m}\mathscr{P}_l[f^m(t)]f^{l-m}(t)\right)e_g[t]\\
&\quad=\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}[f^m(t)g(t)]^{(k)}f^{l-m}(t).
\end{aligned} \tag{36}$$

Equating Eq. (35) and Eq. (36) yields the theorem. □

An immediate consequence of Theorem 3.3 is the well-known Tepper's identity from [7].

Corollary 3.8 (Tepper's identity). *For any positive integer* l *and any real number* x,

$$\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(x+m)^l=l!. \tag{37}$$

Proof. Let $f(t)=\mathrm{e}^t$, $g(t)=\mathrm{e}^{xt}$, and $k=l$ in Theorem 3.3. Then

$$\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(x+m)^l\mathrm{e}^{(l+x)t}=l!\mathrm{e}^{lt}\mathrm{e}^{xt}. \tag{38}$$

Setting $t=0$ in Eq. (38) leads the identity $\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}(x+m)^l=l!$. □

In addition, by choosing particular $f(t)$ and $g(t)$ in Theorem 3.3, we can develop the following result.

Corollary 3.9. *Let* $\{g_k(x)\}$ *be the sequence of binomial-type polynomials. For any positive integers* l *and* k, $k\leqslant l$, *and any real numbers* x *and* y,

$$\sum_{m=0}^{l}\binom{l}{m}(-1)^{l-m}g_k(mx+y)=l!g_1^l(x)\delta_{l,k}. \tag{39}$$

Proof. Let $f(t)=\sum_{i=0}^{l}g_i(x)\frac{t^i}{i!}$ and $g(t)=\sum_{j=0}^{l}g_j(y)\frac{t^j}{j!}$ in Theorem 3.3. If we note that $\{g_k(x)\}$ is the sequence of binomial-type polynomials and then use induction, we can easily show that

$$f^m(t) = \sum_{i=0}^{l} g_i(mx)\frac{t^i}{i!} + t^{l+1}\Omega_1(t; x) \tag{40}$$

and

$$f^m(t)g(t) = \sum_{j=0}^{l} g_j(mx + y)\frac{t^j}{j!} + t^{l+1}\Omega_2(t; x, y), \tag{41}$$

where $\Omega_1(t; x)$ and $\Omega_2(t; x, y)$ are polynomials of t. An application of Theorem 3.3 yields

$$\sum_{m=0}^{l} \binom{l}{m} (-1)^{l-m} \left[\sum_{j=0}^{l} g_j(mx + y)\frac{t^j}{j!} + t^{l+1}\Omega_2(t; x, y)\right]^{(k)} \left(\sum_{i=0}^{l} g_i(x)\frac{t^i}{i!}\right)^{l-m} \Bigg|_{t=0}$$
$$= l!g_1^l(x)g_0(y)\delta_{l,k}. \tag{42}$$

Finally, noting $g_0(x) = g_0(y) = 1$ and $[t^{l+1}\Omega_2(t; x, y)]^{(k)}|_{t=0}$ vanishes for $k \leqslant l$, we have $\sum_{m=0}^{l} \binom{l}{m}(-1)^{l-m} g_k(mx + y) = l!g_1^l(x)\delta_{l,k}$. □

It is notable that, since $\{\varphi_n(x)\}$ defined in [10] and $\{[x]^{n|\lambda}\}$ in [3] are special sequences of binomial-type polynomials, Theorems 4.2 and 4.4 in [10] and Corollaries 4 and 5 in [3] are special cases of Corollary 3.9.

Another immediate consequence of Theorem 3.3 is Proposition 4.6 in [1], which is restated by the following corollary.

Corollary 3.10. *For any nth degree polynomial $p(x) = \sum_{i=0}^{n} p_i x^i$ and an integer q with $q \geqslant n$, we have*

$$\sum_{m=0}^{q} \binom{q}{m} (-1)^{q-m} p(m + x) = n!p_n\delta_{n,q}. \tag{43}$$

Proof. Let $f(t) = \mathrm{e}^t$, $g(t) = \mathrm{e}^{xt}$, and $t = 0$ in Theorem 3.3 yields

$$\sum_{m=0}^{l} (-1)^{l-m} \binom{l}{m} (x + m)^k = l!\delta_{l,k}, \qquad (1 \leqslant k \leqslant l).$$

Then,

$$\begin{aligned}\sum_{m=0}^{q} \binom{q}{m} (-1)^{q-m} p(m + x) &= \sum_{m=0}^{q} \binom{q}{m} (-1)^{q-m} \left(\sum_{i=0}^{n} p_i(m + x)^i\right)\\ &= \sum_{i=0}^{n} p_i \left(\sum_{m=0}^{q} \binom{q}{m} (-1)^{q-m}(m + x)^i\right)\\ &= \sum_{i=0}^{n} p_i q!\delta_{q,i}.\end{aligned}$$

Since $q \geqslant n$, $\sum_{m=0}^{q} \binom{q}{m} (-1)^{q-m} p(m + x) = n!p_n\delta_{q,n}$. This completes the proof. □

4. LU decompositions of some well-known matrices

The other interesting application, which we would like to discuss, of the generalized Pascal functional matrix and its properties is its use in the LU decomposition of a certain class of matrices, whose components can be represented by a Wronskian of some function set. Rather than directly obtaining the LU decomposition of a matrix, our method is to find the LU decomposition of its Wronskian matrix first. Let us start by defining the Wronskian matrix of functions $f_0(x), f_1, (x), \ldots, f_n(x)$.

Definition 4.1. Assume that $f_j(x)$, $j = 0, 1, \ldots, n$, has the mth order derivative. The Wronskian matrix of $\{f_0(x), f_1, (x), \ldots, f_n(x)\}$, denoted by $W_{m,n}[f_0, f_1, \ldots, f_n]$, is an $(m+1) \times (n+1)$ matrix and defined by

$$(W_{m,n}[f_0, f_1, \ldots, f_n])_{i,j} = f_j^{(i)}(x), \quad i = 0, 1, \ldots, m, \quad \text{and} \quad j = 0, 1, \ldots, n.$$

The following theorem is an immediate consequence of Theorem 2.1.

Theorem 4.1. *Let $h_k(t)$ be a kth degree polynomial, for $k = 0, 1, \ldots, n$, and $f(t)$ be any function with nth order derivative. Then $W_{n,n}[f(t)h_0(t), f(t)h_1(t), \ldots, f(t)h_n(t)]$ has LU decomposition form*

$$LU = \mathscr{P}_n[f(t)]W_{n,n}[h_0(t), h_1(t), \ldots, h_n(t)].$$

We can redevelop the LU decomposition of symmetric Pascal matrix Q_n defined in Eq. (2) (which is presented in [2,4]) by choosing special $f(t)$ and $\{h_k(t)\}$ in Theorem 4.1. The result is the following corollary:

Corollary 4.1. *An $(n+1) \times (n+1)$ symmetric Pascal matrix Q_n can be decomposed as the product of lower triangular and upper triangular Pascal matrices, i.e.,*

$$Q_n = P_n[1]P_n^{\mathrm{T}}[1] = P_L P_U$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{n} & \binom{n}{n-1} & \binom{n}{n-2} & \cdots & \binom{n}{0} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & \cdots & \binom{n}{n} \\ 0 & 1 & 2 & \cdots & \binom{n}{n-1} \\ 0 & 0 & 1 & \cdots & \binom{n}{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{0} \end{bmatrix}. \quad (44)$$

Proof. It is well known that $Q_n = W_{n,n}[\mathrm{e}^t h_0(t), \mathrm{e}^t h_1(t), \mathrm{e}^t h_2(t), \ldots, \mathrm{e}^t h_n(t)]|_{t=0}$, where $h_k(t) = \sum_{l=0}^{k} \binom{k}{l} \frac{t^l}{l!}$, and $\mathrm{e}^t h_k(t)$ is the exponential generating function of the kth column of Q_n.

By Theorem 4.1, we have

$$\begin{aligned} Q_n &= W_{n,n}[\mathrm{e}^t h_0(t), \mathrm{e}^t h_1(t), \mathrm{e}^t h_2(t), \ldots, \mathrm{e}^t h_n(t)]\,|_{t=0} \\ &= (\mathscr{P}_n[\mathrm{e}^t])|_{t=0}(W_{n,n}[h_0(t), h_1(t), h_2(t), \ldots, h_n(t)])|_{t=0} = P_n[1]P_n^{\mathrm{T}}[1] = P_L P_U. \quad \square \end{aligned}$$

We also can re-derive Theorem 5 in [9], the LU decomposition of generalized rectangular Pascal matrix $\Psi_n[x, y]$ defined in Eq. (3) by applying Theorem 4.1.

Corollary 4.2. $\Psi_n[x, y] = \Phi[x, y]P_n^{\mathrm{T}}[y/x]$, *where* $\Psi_n[x, y]$, $\Phi[x, y]$ *and* $P_n[x]$ *are defined in* [9] *and Eqs.* (3), (4), *and* (1), *respectively.*

Proof. It is not difficult to show that

$$\Psi_n[x, y] = W_{n,n}[\mathrm{e}^{xyt}h_0(t; x, y), \mathrm{e}^{xyt}h_1(t; x, y), \ldots, \mathrm{e}^{xyt}h_n(t; x, y)]|_{t=0},$$

where $h_k(t; x, y) = \sum_{l=0}^{k}\binom{k}{l}\frac{(y/x)^{k-l}t^l}{l!}y^{2l}$. By Theorem 4.1, we have that

$$\begin{aligned} \Psi_n[x, y] &= (\mathscr{P}_n[\mathrm{e}^{(xy)t}]W_{n,n}[h_0(t; x, y), h_1(t; x, y), \ldots, h_n(t; x, y)])|_{t=0} \\ &= (\mathscr{P}_n[\mathrm{e}^{(xy)t}] \cdot \mathrm{Diag}[1, y^2, \ldots, y^{2n}] \cdot W_{n,n}[a_0(t; x, y), a_1(t; x, y), \ldots, \\ &\quad a_n(t; x, y)])|_{t=0}, \end{aligned}$$

where $a_k(t; x, y) = \sum_{l=0}^{k}\binom{k}{l}(\frac{y}{x})^{k-l}\frac{t^l}{l!}$. Noting $\mathscr{P}_n[\mathrm{e}^{(xy)t}]|_{t=0} \cdot \mathrm{Diag}[1, y^2, \ldots, y^{2n}] = \Phi[x, y]$ and $W_{n,n}[a_0(t; x, y), a_1(t; x, y), \ldots, a_n(t; x, y)]|_{t=0} = P^{\mathrm{T}}[y/x]$ allows us to conclude the results of the corollary. $\square$

5. Future work

We have thus introduced a more generalized Pascal functional matrix and shown that its algebraic properties are very useful for deriving new combinatorial identities and finding LU decompositions of some special matrices. It is noteworthy that our LU decomposition technique is nontraditional. This novel approach sheds light upon the LU factorizations for classical Vandermonde matrix, three kinds of generalized Vandermonde matrices discussed in [8], and Striling matrices of the first and second kinds studied in [5]. Our future work will explore such factorizations.

Acknowledgments

The authors gratefully acknowledge the many helpful comments of the anonymous referee. Also, many thanks are due to the University of St. Thomas' Center for Applied Mathematics, which funded the researching and writing of this paper.

References

[1] T. Arponen, Matrix approach to polynomials 2, Linear Algebra Appl. 394 (2005) 257–276.
[2] R. Brawer, M. Pirovino, The linear algebra of the Pascal matrix, Linear Algebra Appl. 174 (1992) 13–23.
[3] M. Bayat, H. Teimoori, The linear algebra of the generalized Pascal functional matrix, Linear Algebra Appl. 295 (1999) 81–89.

[4] G. Call, D. Velleman, Pascal's matrices, Amer. Math. Monthly 100 (1993) 372–376.
[5] G.-S. Cheon, J.-S. Kim, Stirling matrix via Pascal matrix, Linear Algebra Appl. 329 (2001) 49–59.
[6] S. Roman, The Umbral Calculus, Academic Press, Orlando, FL, 1984.
[7] M. Tepper, A factorial conjecture, Math. Mag. 38 (1965) 303–304.
[8] Y. Yang, H. Holtti, The factorization of block matrices with generalized geometric progression rows, Linear Algebra Appl. 387 (2004) 51–67.
[9] Z. Zhang, M. Liu, An extension of the generalized Pascal matrix and its algebraic properties, Linear Algebra Appl. 271 (1998) 169–177.
[10] X. Zhao, T. Wang, The algebraic properties of the generalized Pascal functional matrices associated with the exponential families, Linear Algebra Appl. 318 (2000) 45–52.