Note

Pelikán's Conjecture and Cyclotomic Cosets

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The following conjecture was recently made by J. Pelikán. Let $a_0, \ldots, a_n$ be an $(n+1)$-tuple of 0's and 1's; let $A_k = \sum_{i=0}^{n-k} a_i a_{i+k}$ for $k = 0, \ldots, n$. Then if $n > 4$ some $A_k$ is even.

This paper shows that Pelikán's conjecture is false for infinitely many values of $n$. On the other hand it is also shown that the conjecture is true for most values of $n$, and a characterization is given of those values of $n$ for which it fails.

1. Introduction

J. Pelikán [3] recently made a conjecture, which we rephrase as follows: Let $a_0, \ldots, a_n \in GF(2)$; let

$$A_k = \sum_{i=0}^{n-k} a_i a_{i+k}$$

(the sum to be evaluated in $GF(2)$) for $k = 0, 1, \ldots, n$. Then if $n \geq 4$ some $A_k$ is zero.

In this paper we show that this conjecture is false, in fact we obtain the following results.

Theorem. A counterexample of length $n$ to Pelikán's conjecture exists if and only if $2n + 1 \in P$ where $P$ is a nonempty set of odd positive integers, with the following properties. (i) An integer $r$ belongs to $P$ if and only if all the prime factors of $r$ belong to $P$. (ii) If $p$ is a prime, then $p \in P$ if and only if $p$ divides $2^{2s+1} - 1$ for some $s$; in other words if and only if $\exp_p(2)$ is odd, where $\exp_p(a)$ is the smallest positive integer $m$ such that $a^m \equiv 1 \mod(p)$. This implies that if $p \equiv -1 \mod(8)$ then $p \in P$, and if $p = \pm 2 \mod(8)$, then $p \notin P$. 

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The last part of the theorem leaves open the question of the behavior of primes $p \equiv 1 \mod(8)$. It turns out that these are sometimes in $P$ and sometimes not; among primes $p \equiv 1 \mod(8)$, $p < 1000$ only 73, 89, 233, 337, 601, 881, and 937 belong to $P$. For further results, see [1, 2]. In particular, it is shown there that the Dirichlet density of the primes $p \in P$, $p \equiv 1 \mod(8)$, is $1/24$. Using stronger versions of the Chebotarev density theorem it can even be shown that if $\pi_p(x) = |\{p \in P; p \text{ prime}, p \leq x\}|$, then

$$\pi_p(x) \sim (7/6) \pi(x; 8, 1) \sim (7/24) \pi(x) \quad \text{as } x \to \infty.$$  

Since for every $r \in P$ all the prime factors $p$ of $r$ have to satisfy $p \equiv \pm 1 \mod(8)$, we conclude that the asymptotic density of $P$ is zero. Thus Pelikán's conjecture is almost always true.

It will be clear from the proof of the theorem that for those $n$ for which $2n + 1 \in P$, all the counterexamples can be constructed quite easily, and that their number is a power of 2.

2. Preliminaries

Recall that $a_0, a_1, ..., a_n \in GF(2)$, and

$$A_k = \sum_{i=0}^{n-k} a_i a_{i+k}.$$  

Set

$$f(x) = \sum_{i=0}^{n} a_i x^i \in GF(2)[x].$$  

Then

$$f(x) f(x^{-1}) = \left( \sum_{i=0}^{n} a_i x^i \right) \left( \sum_{j=0}^{n} a_j x^{-j} \right)$$

$$= \sum_{i,j} a_i a_j x^{i-j}$$

$$= \sum_{k=-n}^{n} x^k \sum_{i-j=k} a_i a_j$$

$$= A_0 + \sum_{k=1}^{n} A_k (x^k + x^{-k}).$$
Now suppose $A_i = 1$ for $0 \leq i \leq n$. Then

\[ f(x)f(x^{-1}) = 1 + \sum_{k=1}^{n} (x^k + x^{-k}) \]

\[ = x^{-n}(1 + x + \cdots + x^{2n}) \]

\[ = x^{-n}((x^{2n+1} - 1)/(x + 1)). \]

Set $\hat{f}(x) = x^n f(x^{-1})$, then

\[ f(x)\hat{f}(x) = ((x^{2n+1} + 1)/(x + 1)). \] (1)

The right side is a polynomial of degree $2n$, and its $2n$ zeros are precisely the $(2n + 1)$st roots of unity, excluding 1.

Now if $\alpha$ is a zero of $f(x)$, then $\alpha^{-1}$ is a zero of $\hat{f}(x)$, and conversely. Since $f(x)$ and $\hat{f}(x)$ are polynomials over $GF(2)$, this condition makes it possible to determine all solutions to (1).

3. FACTORIZATION OF $x^{2n+1} + 1$ OVER $GF(2)$

We first recall some standard terminology.

Let $\xi$ be a primitive $(2n + 1)$st root of unity. Suppose $g(x)$ is an irreducible factor of $x^{2n+1} + 1$ over $GF(2)$. If $\alpha$ is a zero of $g(x)$, then $\alpha = \xi^a$ for some positive integer $a$. The other zeros of $g(x)$ are then precisely $\xi^{2a}, \xi^{4a}, \ldots, \xi^{2k-1}$ where $k$ is the smallest positive integer such that $2^ka \equiv a \mod(2n + 1)$.

This result motivates the definition of cyclotomic cosets. The cyclotomic coset of $a \mod(2n + 1)$, which we call $C_a$, consists of the numbers

\[ a, 2a, 4a, \ldots, a2^{k-1} \]

(all reduced $\mod(2n + 1)$). Of course $C_a = C_{2a} = C_{4a} = \ldots$. The irreducible factors of $x^{2n+1} + 1$ over $GF(2)$ are the polynomials

\[ g_a(x) = \prod_{i \in C_a} (x + \xi^i). \]

Now factor $f(x)$ into irreducible factors

\[ f(x) = \prod_{a \in A} g_a(x) \]

where the cyclotomic cosets for $a \in A$ are distinct. Then it must be that

\[ \hat{f}(x) = \prod_{a \in A} g_{-a}(x). \]
Since (1) holds, any nontrivial \((2n + 1)\)st root of unity is a zero of exactly one of \(f(x), \frac{f(x)}{x}\). This implies that (1) can happen if and only if \(a\) and \(-a\) are never in the same cyclotomic coset \(\text{mod}(2n + 1)\), for \(1 \leq a \leq 2n\).

**Example.** For \(n = 11\), the cyclotomic cosets \(\text{mod} 23\) are

\[
C_0 = \{0\}, \\
C_1 = \{1, 2, 3, 4, 6, 8, 9, 12, 13, 16, 18\}, \\
C_5 = \{5, 7, 10, 11, 14, 15, 17, 19, 20, 21, 22\}.
\]

Thus in this case there is a solution to (1) and a counterexample to Pelikán’s conjecture. This is given by the coefficients of the polynomial

\[
f(x) = \prod_{i \in C_1} (x - \xi^i), \quad \text{i.e., } a_0, \ldots, a_{11} = 101011100011.
\]

Thus the question of when Pelikán’s conjecture fails is reduced to the question of finding the numbers \(n\) such that \(a\) and \(-a\) are in distinct cyclotomic cosets \(\text{mod}(2n + 1)\) for all \(1 < a < 2n\). Moreover for such \(n\), if the number of cyclotomic cosets \(\text{mod}(2n + 1)\) is \(2h + 1\) (including the trivial coset \(\{0\}\)), then there will be \(2^h\) counterexamples to the conjecture.

4. Structure of the Cyclotomic Cosets

Let \(P\) denote the set of odd integers \(r \geq 3\) for which the cyclotomic cosets \(\text{mod } r\) satisfy the condition that \(a\) and \(-a\) \(\text{mod}(r)\) are never in the same cyclotomic coset for \(1 \leq a \leq r - 1\). We study the conditions under which \(r \in P\).

Consider first the case where \(r = p\), a prime. Then \(C_a = aC_1 = \{a2^i \text{ (mod } p\}, 2^i \in C_1\}; thus \(-a \in C_a\) if and only if \(-1 \in C_1\), i.e., if and only if \(2^k \equiv -1 \text{ mod}(p)\) for some \(k\). Now suppose \(m = \exp_p(2)\) (the smallest positive integer such that \(2^m \equiv 1 \text{ mod}(p)\)). If \(m\) is even, say \(m = 2m'\), then

\[
2^{2m'} - 1 = (2^{m'} - 1)(2^{m'} + 1) = 0 \text{ mod}(p),
\]

and so \(2^{m'} + 1 \equiv 0 \text{ mod}(p)\) by minimality of \(m\). Thus \(p \notin P\) in this case. On the other hand, if \(2^k \equiv -1 \text{ mod}(p)\), then \(2^{2k} \equiv 1 \text{ mod}(p)\) and so \(m\) divides \(2k\). If \(m\) is odd this implies \(m\) divides \(k\), which is patently false. Thus the primes \(p \in P\) are precisely those for which \(\exp_p(2)\) is odd. They are the primes which divide \(2^{2s+1} - 1\) for some \(s\).
If \( p \equiv -1 \mod(8) \) then 2 is a quadratic residue, and since the number of quadratic residues is odd, \( \exp_p(2) \) is odd. If \( p \equiv \pm 3 \mod(8) \) then 2 is a nonresidue, i.e., if \( g \) is a primitive root of \( p \), \( 2 = g^{2s-1} \), and \( \exp_p(2) = (p - 1)/\gcd(p - 1, 2s - 1) \), which is even. This leaves open the case \( p \equiv 1 \mod(8) \), in which case \( \exp_p(2) \) can be odd or even, although the odd case is rare.

Now consider the general case. Suppose \( p \in P \). If \( p^i \notin P \) for some \( i > 1 \), then \( 2^k a \equiv -a \mod(p^i) \), i.e., \( (2^k + 1)a \equiv 0 \mod(p^i) \) for some \( a \). If \( 1 < a < p^i - 1 \). But \( p \in P \), so \( p \) does not divide \( 2^k + 1 \), i.e., \( p^i \) divides \( a \), which is impossible.

Now suppose \( r, s \in P \) and \( r \) and \( s \) are relatively prime. If \( rs \notin P \), then \( 2^k a \equiv -a \mod(rs) \) for some \( a \). If \( 1 < a < rs - 1 \). Then \( rs \mid (2^k + 1)a \). Since \( r \), \( s \in P \) we must have \( r \mid a \), \( s \mid a \), thus \( rs \mid a \), a contradiction. Hence \( rs \in P \). Thus if \( r = \prod p_i^{a_i} \), where \( p_i \in P \) for all \( i \), then \( p_i^{a_i} \in P \) for all \( i \), and since the \( p_i^{a_i} \) are relatively prime, we conclude \( r \in P \).

Finally suppose \( r \notin P \) and \( s \) is any positive integer. Since \( r \notin P \) there is an \( a \) such that \( 2^k a \equiv -a \mod(r) \). If \( 1 < a < r - 1 \). Then \( 2^k a s \equiv -as \mod(rs) \). If \( 1 < as < rs - 1 \). Thus \( rs \notin P \).

The preceding paragraph shows that if \( r \in P \) and \( d \) divides \( r \), then \( d \in P \). Thus if \( r \in P \), all the prime divisors of \( r \) are in \( P \), which completes the proof.

**REFERENCES**


2. H. Hasse, Über die Dichte der Primzahlen \( p \), für die eine vorgegebene ganzz rationale Zahl \( a \neq 0 \) von durch eine vorgegebene Primzahl \( \ell \neq 2 \) teilbarer bzw. unteilbarer Ordnung mod. \( p \) ist, *Math. Ann.* 162 (1965), 74–76.