Stochastic Processes and their Applications 19 (1985) 85-99
North-Holland Publishing Company

THE QUEUE GI/G/1: FINITE MOMENTS OF THE CYCLE VARIABLES AND UNIFORM RATES OF CONVERGENCE

Hermann THORISSON

Chalmers University of Technology and the University of Göteborg, S-412 96 Göteborg, Sweden

Received 9 September 1983
Revised 31 January 1984

We study the classical single server queue and establish finite geometric moments and φ moments of the cycle variables. Here $\varphi(x)=x^n\varphi_0(x)$ where n is integer and φ_0 is concave. More generally, we consider systems with different initial conditions and prove moment and stochastic domination results for the delay variables. This, together with the general results of [5], yields ergodic results for the time and customer dependent processes.

queue GI/G/1 * cycle and delay variables * time and customer dependent processes * regenerative processes * ergodicity

Introduction

In [5], uniform rates of convergence are established for general regenerative processes under moment conditions on delay and recurrence times. These results can be applied to the GI/G/1 queuing system because of its regenerative properties. For the time dependent process the delay is the time of the first arrival to an idle system and the recurrence times are the successive busy cycle times; for the customer dependent process the delay is the number (in order of arrival) of the first customer hitting the system idle and a recurrence time is the total number of customers arriving during a busy cycle. It is now natural to ask under what initial conditions and under what conditions on the arrival and service mechanisms these delays and recurrence times have moments of some order.

Our answer to this question is given in Theorem 1 and Corollary 1, and the resulting convergence results are stated in Theorem 2 (the time dependent process) and Theorem 3 (the customer dependent process). The proof of Theorem 1 is based on a result by Tweedie [8], an extension of the so-called Foster's criterion.

In Section 1 we establish notation and state results. In Section 2 and 3 we prepare for the proof of Theorem 1 and complete it in Section 4. In Section 5 we prove Theorem 2 and 3 and in Section 6 conclude with some remarks.

Supported by the Swedish Natural Science Research Council and by the Icelandic Science Foundation.

The results of the present paper are extended to the multi-server case in [6]. In [7], similar methods are used to study the regenerative and ergodic properties of the multi-server queue with nonstationary Poisson arrivals.

1. Statement of results

Consider a single server queuing system where customers arrive in the time-interval $[0, \infty)$ at times $t_0 \leq t_1 \leq \cdots$ and line up to be served under the 'first come, first served' discipline. The epoch of the first arrival is $u_0 = t_0$ and the *inter-arrival times* are $u_n = t_n - t_{n-1}$, $n \geq 1$. The arrivals are also described through the point process $n(\cdot)$ defined by $n(A)$ = the number of customers arriving in the time-set A. Let Q_0 be the *number of customers initially present* in the system and v_0 the *residual service time* of the customer being served at time 0. Let the $(Q_0 - 1)^+$ customers waiting for service at time 0 and the customers arriving in $[0, \infty)$ have *service times* $v_1, v_2, \ldots$. Assume that $(v_n)_1^\infty$, $(u_n)_1^\infty$ are independent sequences of i.i.d. random variables and independent of (Q_0, v_0, u_0); let $(\Omega, \mathcal{F}, P)$ be the underlying probability space and E denote expectation.

Call the Markov process $Z = (Z_t)_{[0,\infty)}$, where $Z_t = (Q_t, V_t, U_t)$,

Q_t = the number of customers present in the system at time t,

V_t = the residual service time of the customer being served at time t (=0 if the server is idle at that time),

U_t = the time from t until the next arrival in $[t, \infty)$,

(for convenience let $t \to Z_t$ be left-continuous), the *time dependent process* and the Markov chain $\tilde{Z} = (\tilde{Z}_n)_0^\infty$, where

$$\tilde{Z}_n = (Q_{t_n}, V_{t_n}, v_{(Q_0-1)^+ + n - (Q_{t_n}-1)^+ + 1}, \ldots, v_{(Q_0-1)^+ + n}),$$

the *customer dependent process*. Observe that the customer dependent variable W_n = the waiting time of the $(n+1)$th customer arriving in $[0, \infty)$, is determined by $\tilde{Z}_n$. Denote the transition function of Z by P_t and the n-step transition probabilities of $\tilde{Z}$ by $\tilde{P}_n$; both P_t and $\tilde{P}_n$ are determined by the distributions of v_1 and u_1. If λ is the distribution of $Z_0 = (Q_0, v_0, u_0)$ then λP_t is the distribution of Z_t. Let $\tilde{\lambda}$ be the initial distribution of $\tilde{Z}$ induced by λ.

Let S_n be the $(n+1)$th t such that $Z_t = (0, 0, 0)$. When the S_n's are finite, define the *delay variables* by

$$X_0 = S_0 = \textit{the delay} \quad (= \text{the delay of } Z),$$

$$N_0 = n[0, S_0) \quad (= \text{the delay of } \tilde{Z}),$$

$$T_0 = v_0 + \cdots + v_{Q_0 - 1 + N_0} = \textit{the busy delay},$$

$$I_0 = X_0 - T_0 = \textit{the idle delay},$$

and, for $n \geq 1$, the nth *cycle variables* by

$$X_n = S_n - S_{n-1} = \textit{the nth cycle} \quad (= \text{the } n\text{th recurrence time of } Z),$$

$$N_n = n[S_{n-1}, S_n) \quad (= \text{the } n\text{th recurrence time of } \tilde{Z}),$$

$$T_n = \inf\{t > 0: Q_{S_{n-1}+t} = 0\} = \textit{the nth busy period},$$

$$I_n = X_n - T_n = \textit{the nth idle period}.$$

The functions $\psi : [0, \infty] \to [0, \infty]$ considered below are measurable, bounded on bounded intervals and $\psi(\infty) = \infty$. Let $\bar{\psi}$ be defined by $\bar{\psi}(x) = \int_0^x \psi(y)\,\mathrm{d}y$. Two functions ψ and θ are *of the same order* if

$$\limsup_{t\to\infty} \frac{\psi(t)}{\theta(t)} < \infty \quad \text{and} \quad \limsup_{t\to\infty} \frac{\theta(t)}{\psi(t)} < \infty; \tag{1}$$

for any nonnegative random variable Y this implies: $E[\psi(Y)] < \infty \Leftrightarrow E[\theta(Y)] < \infty$. We call $E[\psi(Y)]$ the *ψ moment* of Y and say that Y has a *finite geometric moment* if there is a $\rho > 1$ such that $E[\rho^Y] < \infty$; we use analogous terminology for distributions.

Throughout the paper let φ be a function of the same order as $x \to x^n \varphi_0(x)$ where n is a nonnegative integer and φ_0 is concave, increasing and $\varphi_0(0) = 0$. If we define φ_n recursively by $\varphi_n = \bar{\varphi}_{n-1}$, $n \geq 1$, then φ is also of the same order as φ_n (see Lemma 1(b)).

For probability measures λ, μ on $[0, \infty)^k$ let $\lambda \leq^{\mathrm{D}} \mu$ mean that $\lambda(A) \geq \mu(A)$ for all sets of the form $A = [0, y_1] \times \cdots \times [0, y_k]$. Also, if Y_1 and Y_2 are two k-dimensional random variables with the distributions λ and μ respectively then $Y_1 \leq^{\mathrm{D}} Y_2$ means $\lambda \leq^{\mathrm{D}} \mu$. Subsequently, λ and μ are two initial distributions of Z. Let λ^0 denote the marginal distribution of (Q_0, v_0) when the distribution of (Q_0, v_0, u_0) is λ.

Theorem 1. *Let (Q_0, v_0, u_0) be distributed according to λ. The following statements hold when $E[v_1] < E[u_1]$:*

(a) *If Q_0, v_0 and v_1 have finite φ moments (geometric moments) then so have N_0 and T_0.*

(b) *If u_0, u_1 have finite φ moments (geometric moments) and Q_0, v_0 have finite first moments—φ moments if $\lim_{t\to\infty} \varphi(t)/t = 0$—then I_0 has finite φ moment (geometric moment).*

(c) *If Q_0, v_0, u_0 and v_1, u_1 have finite φ moments (geometric moments) then so has X_0.*

(d) *If $\lambda^0 \leq^{\mathrm{D}} \mu^0$ then $N_0 \leq^{\mathrm{D}} \bar{N}$ where $\bar{N}$ is a finite random variable with distribution independent of λ. If v_1 and the one-dimensional marginals of μ^0 have finite φ moments (geometric moments) then so has $\bar{N}$.*

(e) *If $\lambda \leq^{\mathrm{D}} \mu$ then $X_0 \leq^{\mathrm{D}} \bar{X}$ where $\bar{X}$ is a finite random variable with distribution independent of λ. If v_1, u_1 and the one-dimensional marginals of μ have finite φ moments (geometric moments) then so has $\bar{X}$.*

Corollary 1. *The following statements hold provided* $E[v_1] < E[u_1]$:

(a) *If* v_1 *has finite* φ *moment (geometric moment) then so have* N_1 *and* T_1.

(b) *If* u_1 *has finite* φ *moment (geometric moment) then so has* I_1.

(c) *If* v_1 *and* u_1 *have finite* φ *moments (geometric moments) then so has* X_1.

Proof. Put $Q_0 = 1$ and let v_0 and u_0 be independent with the same distributions as v_1 and u_1 respectively. Then the delay variables have the same distribution as the cycle variables and the corollary follows from Theorem 1(a), (b) and (c). □

The difference of two probability measures is a signed measure with total mass 0. For such signed measures ν the total variation norm satisfies $\|\nu\| = 2\sup_A \nu(A)$.

Theorem 2. *Suppose* $E[v_1] < E[u_1] < \infty$ *and there is an* n *such that* $u_1 + \cdots + u_n$ *has a nonsingular distribution. Then* Z *has an invariant distribution* π *and*:

(*uniform ergodicity*) $\sup_{\lambda \leq^D \mu} \|\lambda P_t - \pi\| \to 0 \quad \text{as } t \to \infty;$

(*uniform ergodicity of geometric order*) *if* v_1, u_1 *and the one-dimensional marginals of* μ *have finite geometric moments then there exists a* $\rho > 1$ *such that*

$$\rho^t \sup_{\lambda \leq^D \mu} \|\lambda P_t - \pi\| \to 0 \quad \text{as } t \to \infty;$$

(*uniform ergodicity of order* φ) *if* v_1, u_1 *have finite* $\bar{\varphi}$ *moments and the one-dimensional marginals of* μ *have finite* φ *moments then*

$$\varphi(t) \sup_{\lambda \leq^D \mu} \|\lambda P_t - \pi\| \to 0 \quad \text{as } t \to \infty;$$

(*weak ergodicity of order* φ) *if* v_1, u_1 *and the one-dimensional marginals of* λ *and* μ *have finite* φ *moments then* $\varphi(t)\|\lambda P_t - \mu P_t\| \to 0$ *as* $t \to \infty$.

Theorem 3. *Suppose* $E[v_1] < E[u_1]$. *Then* $\tilde{Z}$ *has an invariant distribution* $\tilde{\pi}$ *and*:

(*uniform ergodicity*) $\sup_{\lambda^0 \leq^D \mu^0} \|\tilde{\lambda} \tilde{P}_n - \tilde{\pi}\| \to 0 \quad \text{as } n \to \infty;$

(*uniform ergodicity of geometric order*) *if* v_1 *and the one-dimensional marginals of* μ^0 *have finite geometric moments then there exists a* $\rho > 1$ *such that*

$$\rho^n \sup_{\lambda^0 \leq^D \mu^0} \|\tilde{\lambda} \tilde{P}_n - \tilde{\pi}\| \to 0 \quad \text{as } n \to \infty;$$

(*uniform ergodicity of order* φ) *if* v_1 *has finite* $\bar{\varphi}$ *moment and the one-dimensional marginals of* μ^0 *have finite* φ *moments then*

$$\varphi(n) \sup_{\lambda^0 \leq^D \mu^0} \|\tilde{\lambda} \tilde{P}_n - \tilde{\pi}\| \to 0 \quad \text{as } n \to \infty;$$

(*weak ergodicity of order* φ) *if* v_1 *and the one-dimensional marginals of* λ^0 *and* μ^0 *have finite* φ *moments then* $\varphi(n)\|\tilde{\lambda} \tilde{P}_n - \tilde{\mu} \tilde{P}_n\| \to 0$ *as* $n \to \infty$.

2. Lemmata

Let Ψ_0 be the class of all concave nondecreasing ψ with $\psi(0)=0$; Φ_0 the class of all convex ψ satisfying $\psi(2x)\leq a\psi(x)$ for some $a<\infty$ and $\psi=\bar{\theta}$ where $\theta(0)=0$ and $\theta(x)\uparrow\infty$ as $x\to\infty$; and Λ_0 the class of increasing ψ satisfying $\psi\geq 2$ and $\log\psi(x)/x\downarrow 0$ as $x\to\infty$.

Lemma 1. (a) *If* $\lim_{x\to\infty}\varphi_0(x)=\infty$ *then* $\varphi_n\in\Phi_0$ *for* $n\geq 1$.

(b) φ_n *and* $x\to x^{n-k}\varphi_k(x)$, $k=0,\ldots,n$, *are of the same order.*

(c) *If* $\psi\in\Psi_0\cup\Phi_0$ *then there is a c such that* $x\to\max\{c,\psi(x)\}$ *is a member of* Λ_0.

(d) *If* $\psi\in\Lambda_0$ *then* $\psi(x+y)\leq\psi(x)\psi(y)$ *for all* $x,y\in[0,\infty)$.

(e) *If* $\psi\in\Lambda_0$ *then for each* $a\in(0,\infty)$, $\psi(x+a)/\psi(x)\to 1$ *as* $x\to\infty$.

Proof. (a) For $n\geq 1$, $\varphi_n=\bar{\varphi}_{n-1}$ where $\varphi_{n-1}(0)=0$ and $\varphi_{n-1}(x)\uparrow\infty$ as $x\to\infty$. Further, observe that $\varphi_0\in\Psi_0$ implies $\varphi_0(2x)\leq 2\varphi_0(x)$ and the induction assumption $\varphi_{n-1}(2x)\leq 2^n\varphi_{n-1}(x)$ yields

$$\varphi_n(2x)=\int_0^{2x}\varphi_{n-1}(y)\,\mathrm{d}y=\int_0^x 2\varphi_{n-1}(2y)\,\mathrm{d}y\leq 2\int_0^x 2^n\varphi_{n-1}(y)\,\mathrm{d}y$$
$$=2^{n+1}\varphi_n(x). \tag{2}$$

(b) The φ_n's are nondecreasing and thus

$$\varphi_n(x)\leq x\varphi_{n-1}(x)\leq\cdots\leq x^n\varphi_0(x).$$

From (2) we obtain the third inequality in:

$$\varphi_n(x)\geq\int_{x/2}^x\varphi_{n-1}(y)\,\mathrm{d}y\geq\frac{x}{2}\varphi_{n-1}\left(\frac{x}{2}\right)\geq(\tfrac{1}{2})^{n+1}x\varphi_{n-1}(x)\geq\cdots\geq c_n x^n\varphi_0(x),$$

for some $c_n>0$. This yields (b).

(c) If $\psi\in\Phi_0$ then (see e.g. [5, Reference [5]]) there exists a finite constant c such that $x\theta(x)/\psi(x)<c$, $x\in[0,\infty)$; the same holds for $\psi\in\Psi_0$ since then $\psi=\bar{\theta}$ where θ is non-increasing and thus $\psi(x)\geq x\theta(x)$. Take x so large that $\log\psi(x)\geq c$. Then $x\theta(x)/\psi(x)<\log\psi(x)$ which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\log\psi(x)}{x}\right)=\frac{\theta(x)}{x\psi(x)}-\frac{\log\psi(x)}{x^2}<0.$$

Hence $\log(\max\{c,\psi(x)\})/x$ decreases as $x\to\infty$ and the limit must be 0 since $\log\psi(2^n)/2^n\leq\log a^n\psi(1)/2^n\to 0$ as $n\to\infty$. Choose $c\geq 2$ to obtain the desired result.

(d) and (e) Take $x>0$ ((d) holds for $x=0$ since $\psi \geq 2$). Then $\log \psi(t)/t$ nonincreasing in t renders

$$\begin{aligned}
&\log \psi(x+y)-\log \psi(x)\\
&=\left(\frac{\log \psi(x+y)}{x+y}-\frac{\log \psi(x)}{x}\right)x+\frac{\log \psi(x+y)}{x+y}y\\
&\leq \frac{\log \psi(x+y)}{x+y}y\\
&\begin{cases} \leq \log \psi(y), & \text{implying (d)},\\ \downarrow 0 \quad \text{as } x\to\infty, & \text{implying (e)}. \end{cases} \qquad \square
\end{aligned}$$

Lemma 2. *Let M, Y_1, Y_2, ... be independent nonnegative random variables, Y_1, Y_2, ... i.i.d. and M integer valued.*

(a) *If M and Y_1 have finite φ_n moments where $n \geq 1$ (geometric moments) then so has $\sum_{i=1}^M Y_i$.*

(b) *If $\varphi_0(x)=\psi(\theta(x))$ where $\psi, \theta \in \Psi_0$ and $E[\psi(M)]<\infty$, $E[\theta(Y_1)]<\infty$ then $E[\varphi_0(\sum_{i=1}^M Y_i)]<\infty$.*

Proof. (a) It is no restriction to consider $\varphi(x)=x^n\varphi_0(x)$ instead of φ_n, due to Lemma 1(b). Suppose M and Y_1 have finite φ moments. Then $E[Y_1^\alpha]$ and $E[Y_1^\alpha\varphi_0(Y_1)]\leq$ some finite a for all $\alpha=0,\dots,n$ and Minkowski's inequality gives $E[(\sum_{i=1}^k Y_i)^\alpha]\leq ak^\alpha$. This together with formula (4.3) in [1] yields, for a large enough,

$$E\left[\varphi\left(\sum_{i=1}^k Y_i\right)\right]\leq k^n a\varphi_0(ak)+nak^{n-1}aka\leq \varphi(k)(a^2+na^3/\varphi_0(1)).$$

Hence

$$\begin{aligned}
E\left[\varphi\left(\sum_{i=1}^M Y_i\right)\right]&=\sum_{k=0}^\infty E\left[\varphi\left(\sum_{i=1}^k Y_i\right)\right]P(M=k)\\
&\leq (a^2+na^3/\varphi_0(1))E[\varphi(M)]<\infty.
\end{aligned}$$

Let $\rho_0, \rho_1>1$ and suppose $E[\rho_0^M]<\infty$, $E[\rho_1^{Y_1}]<\infty$. Put $\psi(x)=\rho_1^x$ and take ρ_1 sufficiently close to 1 for $E[\psi(Y_1)]\leq \rho_0$ to hold. Then

$$E\left[\psi\left(\sum_{i=1}^k Y_i\right)\right]=E[\psi(Y_1)]^k\leq \rho_0^k$$

and thus

$$E\left[\psi\left(\sum_{i=1}^M Y_i\right)\right]\leq E[\rho_0^M]<\infty.$$

(b) Suppose $E[\psi(M)]<\infty$ and $E[\theta(Y_1)]<\infty$. If $\psi, \theta \in \Psi_0$ then

$$E\left[\varphi_0\left(\sum_{i=1}^{k} Y_i\right)\right] \leqslant \psi\left(E\left[\sum_{i=1}^{k} \theta(Y_i)\right]\right) = \psi(kE[\theta(Y_1)])$$

$$\leqslant \psi(k)(1+E[\theta(Y_1)]).$$

Replacing k by M and taking expectations yields the desired result. □

Lemma 3. *Let Y_0 and Y_1 be nonnegative random variables. If $\psi \in \Psi_0 \cup \Phi_0$ and Y_0, Y_1 have finite ψ moments (geometric moments) then so has $Y_0 + Y_1$.*

Proof. Suppose Y_0, Y_1 have finite ψ moments. If $\psi \in \Psi_0$ then $E[\psi(Y_0+Y_1)] \leqslant E[\psi(Y_0)] + E[\psi(Y_1)] < \infty$. If $\psi \in \Phi_0$ then the Orlicz norm

$$\|Y_0\|_\psi = \inf\left\{a\colon E\left[\psi\left(\frac{1}{a} Y_0\right)\right] \leqslant 1\right\}$$

(an extension of the L_α-norm, see the appendix of [3]) is finite if and only if $E[\psi(Y_0)]<\infty$. Thus $E[\psi(Y_0+Y_1)]<\infty$ because

$$\|Y_0+Y_1\|_\psi \leqslant \|Y_0\|_\psi + \|Y_1\|_\psi < \infty.$$

For the geometric moment result, apply Hölder's inequality to get

$$E[\sqrt{\rho}^{\,Y_0+Y_1}] \leqslant \sqrt{E[\rho^{Y_0}]}\sqrt{E[\rho^{Y_1}]}. \qquad \square$$

3. Random walk on $(-\infty, \infty)$ with negative drift

Let $(R_n)_0^\infty$ be a Markov chain on a state space $(E, \mathscr{E})$. Let E_x denote expectation when $R_0 = x$. For $A \in \mathscr{E}$ define τ_A by

$$\tau_A = \inf\{n\colon R_n \in A\}.$$

Let g be a nonnegative measurable function on E. In [8, Section 3], we find the following powerful result (the (a)-part is the so called Foster's criterion in the more general setting).

Theorem 4. (a) *If, for some $\varepsilon > 0$,*

$$E_x[g(R_1)] \leqslant g(x) - \varepsilon, \quad x \in A^c,$$

then

$$E_x[\tau_A] \leqslant g(x)/\varepsilon, \quad x \in A^c.$$

(b) *If $g(x) \geqslant 1$ for $x \in A$ and, for some $\varepsilon > 0$,*

$$E_x[g(R_1)] \leqslant (1-\varepsilon) g(x), \quad x \in A^c,$$

then

$$E_x[\rho^{\tau_A}] \leq g(x)/(1-\rho(1-\varepsilon)), \quad x \in A^c,$$

for any $\rho < (1-\varepsilon)^{-1}$.

(c) *Let* $\psi: [0,\infty) \to [0,\infty)$ *be increasing. If*

$$E_x[g(R_1)] \leq g(x) - E[\psi(\tau_A)], \quad x \in A^c,$$

then

$$E_x[\bar{\psi}(\tau_A)] \leq g(x), \quad x \in A^c.$$

Now let $(R_n)_0^\infty$ be a random walk on $(-\infty,\infty)$ with negative drift, i.e. $R_n = \sum_{i=0}^n Y_i$ where $Y_1, Y_2, \ldots$ are i.i.d. random variables, independent of Y_0 and $E[Y_1^+] < E[Y_1^-]$. The following lemma slightly improves Proposition 1 in [8] by removing the condition that $E[Y_1^2] < \infty$.

Lemma 4. *Suppose* $\psi \in \Lambda_0$, $E[Y_1^+\psi(Y_1^+)] < \infty$ *and*

$$\exists c < \infty\colon E_x[\psi(\tau_{(-\infty,0)})] \leq c\psi(x) \quad \text{for } x \text{ large.} \tag{3}$$

Then (3) *holds with* ψ *replaced by* $\bar{\psi}$.

Proof. By Lemma 1(e), we can for each $a, \varepsilon > 0$ take b sufficiently large for

$$\psi(x+a) \leq (1+\varepsilon)\psi(x), \quad \psi(x-a) \geq (1-\varepsilon)\psi(x), \qquad x \geq b.$$

Since ψ is increasing we have

$$\bar{\psi}(x+y) \leq \bar{\psi}(x) + \psi(x+y)y, \quad x \geq a, y \geq -a.$$

Thus, with $Y_0 = x \geq b \geq a$ and $Y = \max\{Y_1, -a\}$, we obtain

$$\begin{aligned}
\bar{\psi}(R_1^+) &\leq \bar{\psi}(x+Y) \leq \bar{\psi}(x) + \psi(x+Y)Y \\
&\leq \bar{\psi}(x) + \psi(x+Y_1)Y_1 1_{\{Y_1>a\}} + \psi(x+a)Y_1 1_{\{0\leq Y_1\leq a\}} \\
&\quad + \psi(x-a)Y_1 1_{\{-a\leq Y_1<0\}} \\
&\leq \bar{\psi}(x) + \psi(x)\psi(Y_1)Y_1 1_{\{Y_1>a\}} + \psi(x)(1+\varepsilon)Y_1 1_{\{0\leq Y_1\leq a\}} \\
&\quad + \psi(x)(1-\varepsilon)Y_1 1_{\{-a\leq Y_1<0\}}
\end{aligned} \tag{4}$$

where $1_A(y) = 1$ or 0 according as $y \in A$ or $y \notin A$. Take $\delta > 0$ and let δ and ε be close enough to 0 and a sufficiently large for

$$E[(1+\varepsilon)Y_1 1_{\{0\leq Y_1\leq a\}} + (1-\varepsilon)Y_1 1_{\{-a\leq Y_1<0\}}] \leq -2\delta,$$

$$E[Y_1^+\psi(Y_1^+)1_{\{Y_1>a\}}] \leq \delta,$$

to hold. Then take expectations in (4) and apply (3) to obtain

$$E_x[\bar{\psi}(R_1^+)] \leq \bar{\psi}(x) - \delta\psi(x) \leq \bar{\psi}(x) - \frac{\delta}{c} E_x[\psi(\tau_{(-\infty,0)})], \quad x \geq b,$$

for b large enough. An application of Theorem 4(c) now yields the second step in

$$E_x[\bar{\psi}(\tau_{(-\infty,0)})] = E_{x+b}[\bar{\psi}(\tau_{(-\infty,b)})] \leqslant \frac{c}{\delta}\,\bar{\psi}(x+b), \quad x \geqslant 0,$$

the first being obvious. That (3) holds with ψ replaced by $\bar{\psi}$ follows from this and

$$\begin{aligned}\bar{\psi}(x+b) &= \bar{\psi}(b) + \int_0^x \psi(y+b)\,\mathrm{d}y \leqslant \bar{\psi}(b) + \psi(b)\bar{\psi}(x)\\ &\leqslant (1+\psi(b))\bar{\psi}(x), \quad x \geqslant b,\end{aligned}$$

where the second step is due to Lemma 1(d). □

Theorem 5. *If Y_0^+ and Y_1^+ have finite φ moments (geometric moments) then so has $\tau_{(-\infty,0)}$.*

Proof. It is no restriction to take $\varphi = \varphi_n$. Since $E_x[\varphi_n(\tau_{(-\infty,0)})]$ is increasing in x we obtain the φ moment result if we can establish that (3) holds with $\psi = \varphi_n$. We prove this by induction.

Take $a, \varepsilon > 0$ such that $E[Y] = -\varepsilon$ where $Y = \max\{Y_1, -a\}$. Then

$$E_x[R_1^+] \leqslant E[x+Y] = x - \varepsilon, \quad x \geqslant a,$$

and Theorem 4(a) yields the inequality in

$$E_x[\tau_{(-\infty,0)}] = E_{x+a}[\tau_{(-\infty,a)}] \leqslant \frac{x+a}{\varepsilon}, \quad x \geqslant 0. \tag{5}$$

An application of Jensen's inequality yields

$$E_x[\varphi_0(\tau_{(-\infty,0)})] \leqslant \varphi_0\left(\frac{x+a}{\varepsilon}\right) \leqslant \left(1 + \frac{2}{\varepsilon}\right)\varphi_0(x), \quad x \geqslant a,$$

and thus (3) holds with $\psi = \varphi_0$.

Now suppose $E[\varphi_n(Y_1^+)] < \infty$ where $n \geqslant 1$. Then $E[Y_1^+\varphi_k(Y_1^+)] < \infty$ for $k = 0, \ldots, n-1$, due to Lemma 1(b). Thus, by Lemma 4 and Lemma 1(a) and (c), (3) holds with $\psi = \varphi_{k+1}$ if it holds for $\psi = \varphi_k$, and the induction is completed.

In order to prove the geometric moment result, take $\rho > 1$ close enough to 1 for $E[Y_1^+\rho^{Y_1^+}] < \infty$, take $b \in (0, \log\rho]$ and put $Y = \max\{Y_1, -a\}$ where a is large enough for $E[Y] < 0$. Since $\lim_{b\downarrow 0} Y\,\mathrm{e}^{bY} = Y$ and $Y\,\mathrm{e}^{bY} \leqslant Y_1^+\rho^{Y_1^+}$, we can by dominated convergence take b sufficiently close to 0 for $E[Y\,\mathrm{e}^{bY}] = -\varepsilon/b$ where $\varepsilon > 0$. Hence, for $x \geqslant a$,

$$E_x[\mathrm{e}^{bR_1^+}] \leqslant E[\mathrm{e}^{b(x+Y)}] \leqslant \mathrm{e}^{bx}(1 + bE[Y\,\mathrm{e}^{bY}]) = \mathrm{e}^{bx}(1-\varepsilon)$$

and Theorem 4(b) yields the existence of a $\rho_0 > 1$ and a $c < \infty$ such that

$$E_x[\rho_0^{\tau_{(-\infty,0)}}] = E_{x+a}[\rho_0^{\tau_{(-\infty,a)}}] \leqslant c\,\mathrm{e}^{b(x+a)} = c\,\mathrm{e}^{ba}\,\mathrm{e}^{bx}, \quad x \geqslant 0.$$

Taking b sufficiently close to 0 for $E[\mathrm{e}^{bY_0}] < \infty$ completes the proof. □

4. Proof of Theorem 1

Proposition 1. *There exists a* D/G/1 *system (i.e. a single server queuing system with deterministic arrival times* $\hat{t}_n = nd$; *denote the service times* $\hat{v}_n$ *and the number of customers initially present* $\hat{Q}_0$) *with busy delay* $\hat{T}_0$ *such that* $T_0 \leqslant \hat{T}_0$. *Further, suppose* $E[v_1] < E[u_1]$. *Then* $E[\hat{v}_1] < d$ *and* $\hat{Q}_0$, $\hat{v}_0$, $\hat{v}_1$ *have* φ *moments (geometric moments) provided* Q_0, v_0, v_1 *have such moments.*

Proof. We use the following domination technique: Let $(t'_n)_0^\infty$ be increasing, $n'(\cdot)$ the associated counting process and T'_0 the busy delay obtained by replacing $(t_n)_0^\infty$ by (t'_n); if $n'(\cdot)$ dominates $n(\cdot)$, i.e. $n'[0, t] \geqslant n[0, t]$ for all $t \geqslant 0$, then it is easily seen that $T_0 \leqslant T'_0$.

For a measure μ on $[0, \infty)$ put $\theta_t\mu(A) = \mu(t + A)$, $A \in \mathscr{B}[0, \infty)$. Fix a $d > 0$. Dominate $n(\cdot)$ by the zero-delayed process $n_0(\cdot) = \theta_{t_0} n(\cdot)$ and then dominate $n_0(\cdot)$ by clumping together the arrivals in $[0, d)$ to one arrival epoch, 0. Then we have a group of $M_0 = n_0[0, d)$ customers arriving at time 0 while the arrivals in $[d, \infty)$ are described by $n_0([d, \infty) \cap \cdot)$. Dominate $n_0([d, \infty) \cap \cdot)$ by the d-delayed process $n_1(\cdot) = \theta_{t_{M_0} - d} n_0([d, \infty) \cap \cdot)$ and observe that $\theta_d n_1(\cdot)$ is independent of M_0 and has the same distribution as $n_0(\cdot)$. Dominate $n_1(\cdot)$ by clumping together the arrivals in $[d, 2d)$ to one arrival epoch, d. Then we have groups of M_0 and $M_1 = \theta_d n_1[0, d)$ customers arriving at 0 and d, respectively, while the arrivals in $[2d, \infty)$ are described by $n_1([2d, \infty) \cap \cdot)$. Dominate $n_1([2d, \infty) \cap \cdot)$ by the $2d$-delayed process $n_2(\cdot) = \theta_{t_{M_0 + M_1} - 2d} n_1([2d, \infty) \cap \cdot)$ etc. Proceed in this way to obtain an arrival process with groups containing $M_0, M_1, M_2, \ldots$ customers arriving at $0, d, 2d, \ldots$.

The M_n's are i.i.d. with the same distribution as $\theta_{t_0} n[0, d)$. This arrival process dominates $n(\cdot)$ and thus $T_0 \leqslant T'_0$.

Define the D/G/1 system as follows: put $\hat{Q}_0 = 1$, $\hat{v}_0 = \sum_{i=0}^{Q_0 - 1} v_i$ and regard the group arriving at nd as one individual with service time

$$\hat{v}_{n+1} = \sum_{i=1}^{M_n} v_{Q_0 - 1 + M_0 + \cdots + M_{n-1} + i}, \quad n \geqslant 0.$$

This does not affect the busy delay, i.e. $\hat{T}_0 = T'_0$, and thus $T_0 \leqslant \hat{T}_0$.

Suppose $E[v_1] < E[u_1]$. Then the elementary renewal theorem allows us to take d large enough for $(1/d) E[\theta_{t_0} n[0, d]] < 1/E[v_1]$ to hold. Hence $E[\hat{v}_1] = E[v_1] E[M_0] < d$. Also, $M_0 = \theta_{t_0} n[0, d)$ has finite geometric moment (and thus φ moment) and an application of Lemma 2(a) renders the moment results for $\hat{v}_1$. Finally, $\hat{v}_0$ has the desired moment properties, since $\sum_{i=1}^{Q_0 - 1} v_1$ has these properties, due to Lemma 2 (when applying Lemma 2(b) put $\psi = \varphi_0$ and $\theta(x) = x$, and observe that $E[v_1] < E[u_1]$ implies $E[v_1] < \infty$), and since $\hat{v}_0 = v_0 + \sum_{i=1}^{Q_0 - 1} v_i$ (apply Lemma 3). □

Proof of Theorem 1. Let $v'_1, v'_2, \ldots$ be i.i.d., distributed as v_1 and independent of (Q_0, v_0, u_0), $(v_n)_1^\infty$, $(u_n)_1^\infty$. For convenience, we let the Q_0 customers initially present have the service times $v_0, v'_1, \ldots, v'_{Q_0-1}$ and those arriving in $[0, \infty)$ the service times $v_1, v_2, \ldots$; this does not affect the distribution of the delay variables. We only prove the φ moment results since those for the geometric moments are established in the same way. Take $\varphi = \varphi_n$.

(a) Apply the results of the preceding section. Put

$$Y_k = \begin{cases} v_0 + v'_1 + \cdots + v'_{Q_0-1} - u_0, & k = 0, \\ v_k - u_k, & k \geq 1, \end{cases}$$

and observe that $E[Y_1] = E[v_1] - E[u_1] < 0$, $E[\varphi(Y_1^+)] \leq E[\varphi(v_1)] < \infty$, $E[\varphi(Y_0^+)] \leq E[\varphi(v_0 + v'_1 + \cdots + v'_{Q_0-1})] < \infty$ (see the end of the proof of Proposition 1) and $N_0 = \tau_{(-\infty,0)}$. Thus Theorem 5 gives $E[\varphi(N_0)] < \infty$, and Proposition 1 yields $E[\varphi(T_0)] < \infty$ if we can prove that this holds when $t_n = nd$, $n \geq 0$. But then $T_0 \leq X_0 = dN_0$, and (a) is established.

(b) Clearly $I_0 \leq u_0 + \sup_{1 \leq i \leq N_0} u_i$ and (b) is established if we can prove $E[\varphi(\sup_{1 \leq i \leq N_0} u_i)] < \infty$ (apply Lemma 3). When $\varphi = \varphi_0$ we have

$$\begin{aligned} E\left[\varphi_0\left(\sup_{1 \leq i \leq N_0} u_i\right) \middle| Y_0\right] &\leq \varphi_0\left(E\left[\sup_{1 \leq i \leq N_0} u_i \middle| Y_0\right]\right) \\ &\leq \varphi_0\left(E\left[\sum_{i=1}^{\infty} u_i 1_{\{N_0 \geq i\}} \middle| Y_0\right]\right) = \varphi_0\left(\sum_{i=1}^{\infty} E[u_i] P(N_0 \geq i \mid Y_0)\right) \\ &= \varphi_0(E[u_1] E[N_0 \mid Y_0]) \leq \varphi_0\left(E[u_1] \frac{Y_0 + a}{\varepsilon}\right); \end{aligned}$$

the first inequality is due to Jensen, the first equality follows from the independence of u_i and $(Y_0, 1_{\{N_0 \geq i\}})$, and the final inequality is due to (5). Take expectations to obtain $E[\varphi_0(I_0)] < \infty$. When $\varphi = \varphi_n$, where $n \geq 1$, we have

$$\begin{aligned} E\left[\varphi\left(\sup_{1 \leq i \leq N_0} u_i\right)\right] &= E\left[\sup_{1 \leq i \leq N_0} \varphi(u_i)\right] \leq E\left[\sum_{i=1}^{\infty} \varphi(u_i) 1_{\{N_0 \geq i\}}\right] \\ &= \sum_{i=1}^{\infty} E[\varphi(u_i)] P(N_0 \geq i) = E[\varphi(u_1)] E[N_0] < \infty, \end{aligned}$$

where $E[N_0] < \infty$ due to (a) and the condition that Q_0, v_0 have finite first moments.

(c) By (a) we have that $E[\varphi(T_0)] < \infty$ and by (b) that $E[\varphi(I_0)] < \infty$. Hence (c) follows from $X_0 = T_0 + I_0$ and Lemma 3.

(d) Let $(\bar{Q}, \bar{v})$ be governed by μ^0. Since $\lambda^0 \leq^D \mu^0$ implies $\lambda^0([0, x]^2) \geq \mu^0([0, x]^2)$ we obtain $\max\{Q_0, v_0\} \leq^D \max\{\bar{Q}, \bar{v}\} = \bar{Y}$ (say). Thus we may assume that $\max\{Q_0, v_0\}$ and $\bar{Y}$ are defined on the same probability space in such a way that $\max\{Q_0, v_0\} \leq \bar{Y}$, cf. [4, Satz 1.2.1]. Further, Construction 1.1 in [5] allows us to assume that Q_0, v_0, u_0 and $\bar{Y}$ are defined on the same probability space in such a way that $Q_0 \leq \bar{Y}$ and $v_0 \leq \bar{Y}$. Finally, we may take $(Q_0, v_0, u_0, \bar{Y})$ independent of $(v'_n)_1^\infty$, $(v_n)_1^\infty$, $(u_n)_1^\infty$.

Now replace (Q_0, v_0, u_0) by $([\bar{Y}], \bar{Y}, 0)$ and let $\bar{N}$ be the new N_0. Since $\bar{N} = \bar{\tau}_{(-\infty,0)} = \inf\{n \geq 0: \bar{R}_n < 0\}$, where $\bar{R}_n = \bar{Y} + v'_1 + \cdots + v'_{[\bar{Y}]-1} + \sum_{i=1}^{n} Y_i$, and since $\bar{R}_n \geq R_n$, we get $\bar{N} = \bar{\tau}_{(-\infty,0)} \geq \tau_{(-\infty,0)} = N_0$. The distribution of $\bar{N}$ is determined by μ^0 and the distributions of v_1, u_1 and thus does not depend on λ. If $\bar{Q}$, $\bar{v}$ have finite φ moments then $E[\varphi(\bar{Y})] \leq E[\varphi(\bar{Q})] + E[\varphi(\bar{v})] < \infty$ and (a) yields $E[\varphi(\bar{N})] < \infty$ provided $E[\varphi(v_1)] < \infty$.

(e) Let $(\bar{Q}, \bar{v}, \bar{u})$ be governed by μ. Proceed as in the proof of (d) but now put $\bar{Y} = \max\{\bar{Q}, \bar{v}, \bar{u}\}$ and use $\lambda \leq^{D} \mu$ to obtain $u_0 \leq \bar{Y}$ in addition to $Q_0 \leq \bar{Y}$ and $v_0 \leq \bar{Y}$. Put $\bar{X} = \bar{T} + \bar{I}$ where

$$\bar{T} = \bar{Y} + v'_1 + \cdots + v'_{[\bar{Y}]-1} + \sum_{i=1}^{\bar{N}} v_i \geq v_0 + v'_1 + \cdots + v'_{Q_0-1} + \sum_{i=1}^{N_0} v_i = T_0$$

and

$$\bar{I} = \bar{Y} + \sup_{1 \leq i \leq \bar{N}} u_i \geq u_0 + \sup_{1 \leq i \leq N_0} u_i \geq I_0.$$

Then $X_0 = T_0 + I_0 \leq \bar{T} + \bar{I} = \bar{X}$. The distribution of $(\bar{T}, \bar{I})$, and thus that of $\bar{X}$, does not depend on λ. If $\bar{Q}$, $\bar{v}$, $\bar{u}$ have finite φ moments then $E[\varphi(\bar{Y})] \leq E[\varphi(\bar{Q})] + E[\varphi(\bar{v})] + E[\varphi(\bar{u})] < \infty$. Thus (a) implies that $E[\varphi(\bar{T})] < \infty$ (provided $E[\varphi(v_1)] < \infty$) and computations similar to those in the proof of (b) yield $E[\varphi(\bar{I})] < \infty$ (provided $E[\varphi(u_1)] < \infty$). An application of Lemma 3 completes the proof. □

5. Proof of Theorem 2 and 3

Proof of Theorem 3. The condition $E[v_1] < E[u_1]$ implies $P(N_1 = 1) = P(u_1 > v_1) > 0$; thus the recurrence distribution of $\tilde{Z}$ is aperiodic. Combining Theorem 1(d), Corollary 1(a) and [5, Corollary 1.2(a′), (b′) and (c′)] yields the results on uniform ergodicity, and combining Theorem 1(a), Corollary 1(a), and [5, Theorem 1.4(c)] yields the result on weak ergodicity. □

Proposition 2. *Consider the zero-delayed system (i.e. $Q_0 = v_0 = u_0 = 0$) and suppose there is an n such that $t_n = u_1 + \cdots + u_n$ has a nonsingular distribution. Then so has X_1 provided $P(u_1 < v_1) > 0$ and $P(u_1 > v_1) > 0$. If $P(u_1 < v_1) = 0$ then $X_1 + \cdots + X_n$ has a nonsingular distribution. When $P(u_1 > v_1) = 0$ we have $P(X_1 = \infty) = 1$.*

Proof. Suppose $P(u_1 < v_1) > 0$ and $P(u_1 > v_1) > 0$. Then there exist x_0, x_1, y_0, y_1 satisfying $x_0 < y_0$, $x_1 > y_1$ and such that the sub-probability measures μ_0, μ_1, ν_0, ν_1 defined by

$$\mu_i(B) = P(u_1 \in B, x_i \leq u_1 \leq x_i + \varepsilon) \quad \text{and}$$

$$\nu_i(B) = P(v_1 \in B, y_i \leq v_1 \leq y_i + \varepsilon), \qquad i = 0, 1,\ B \in \mathscr{B}[0, \infty),$$

have a strictly positive mass for each $\varepsilon > 0$. Let μ be defined by

$$\mu(B) = P(t_n \in B, x \leq t_n \leq x + \varepsilon), \quad B \in \mathscr{B}[0, \infty),$$

where x is chosen so that μ has a nontrivial absolutely continuous part for each $\varepsilon > 0$. Take an integer m such that

$$mx_0 + x < my_0$$

and let k be the smallest integer satisfying

$$mx_0 + x + kx_1 > my_0 + (n+k)y_1.$$

Then choosing ε sufficiently close to 0 we have

$$X_1 = u_1 + \cdots + u_{m+n+k} = t_{m+n+k}$$

on

$$\begin{aligned} A = &\{x_0 \leq u_i \leq x_0 + \varepsilon\,;\ i = 1, \ldots, m\} \cap \{y_0 \leq v_i \leq y_0 + \varepsilon\,;\ i = 1, \ldots, m\} \\ &\cap \{x \leq u_{m+1} + \cdots + u_{m+n} \leq x + \varepsilon\} \\ &\cap \{x_1 \leq u_{m+n+i} \leq x_1 + \varepsilon\,;\ i = 1, \ldots, k\} \\ &\cap \{y_1 \leq v_{m+i} \leq y_1 + \varepsilon\,;\ i = 1, \ldots, n+k\}. \end{aligned}$$

Hence, for each $B \in \mathscr{B}[0, \infty)$,

$$\begin{aligned} P(X_1 \in B) &\geq P(\{X_1 \in B\} \cap A) \\ &= (\nu_0[y_0, y_0 + \varepsilon])^m (\nu_1[y_1, y_1 + \varepsilon])^{n+k} (\mu_0^{*m} * \mu * \mu_1^{*k})(B), \end{aligned}$$

so $P(X_1 \in \cdot)$ has an absolutely continuous component because μ has such.

When $P(u_1 < v_1) = 0$ we have $X_1 + \cdots + X_n = u_1 + \cdots + u_n$ on $\{u_i > v_i\,;\ i = 1, \ldots, n\}$ and the desired result follows easily. The final statement is obvious. □

Proof of Theorem 2. The condition $E[v_1] < E[u_1]$ implies $P(u_1 > v_1) > 0$ and combining Proposition 2 (when $P(u_1 < v_1) = 0$ consider $(S_{nk})_{k=0}^{\infty}$ instead of $(S_k)_0^{\infty}$), Theorem 1(e), Corollary 1(c), and [5, Corollary 1.1(a'), (b') and (c')] yields the results on uniform ergodicity while combining Proposition 2, Theorem 1(c), Corollary 1(c), and [5, Theorem 1.3(c)] yields the result on weak ergodicity. □

6. Remarks

Remark 1 (Random measures). Theorems 2 and 3 are not the only consequences of Theorem 1. For example, let η be the point process defined by $\eta(A) =$ the number of customers leaving the system in the time-set A, see [5, Example 1.3]. Then [5, Section 1.6] yields

$$\varphi_n(t) \sup_{\lambda \leq^D \mu} \left\| E_\lambda[\eta] - \frac{1}{E[u_1]}\, l \right\|_{[t,\infty)} \to 0 \quad \text{as } t \to \infty$$

and

$$\exists \rho > 1: \rho^t \sup_{\lambda \leq^D \mu} \left\| E_\lambda[\eta] - \frac{1}{E[u_1]} l \right\|_{[t,\infty)} \to 0 \quad \text{as } t \to \infty$$

provided $u_1 + \cdots + u_n$ has a nonsingular distribution for some n, $E[v_1] < E[u_1] < \infty$, v_1 and u_1 have finite φ_{n+2} moments (geometric moments) and the one-dimensional marginals of μ have finite φ_{n+1} moments (geometric moments). Here $E_\lambda[\eta]$ is the intensity measure of η when (Q_0, v_0, u_0) is governed by λ, l = Lebesgue measure and $\|\cdot\|_{[t,\infty)}$ = total variation norm for signed measures on $[t, \infty)$. For uniform convergence of means (such as $E[Q_t]$), see [5, Remark 1.6].

Remark 2 (Weak ergodicity). Suppose $P(S_n < \infty) = 1$ or, equivalently, $P(N_n < \infty) = 1$, for $n \geq 0$. Then

$$\|\tilde{\lambda} \tilde{P}_n - \tilde{\mu} \tilde{P}_n\| \to 0, \quad n \to \infty,$$

and, provided there exists an n such that $u_1 + \cdots + u_n$ has a nonsingular distribution,

$$\|\lambda P_t - \mu P_t\| \to 0, \quad t \to \infty,$$

see [5, Remark 1.4]. The condition $P(S_n < \infty) = P(N_n < \infty) = 1$ is satisfied if and only if $\sum_{n=1}^{\infty} (1/n) P(R_n < 0) = \infty$, see [2, XII.7, Theorem 2]. Sufficient conditions are $E[v_1] < E[u_1]$, or $E[v_1] = E[u_1] < \infty$, see [2, XII.2, Theorem 2]. When $E[v_1] = E[u_1] = \infty$ it is easily seen that $v_1 \leq^D u_1$ is sufficient.

Remark 3 (On $\psi \in \Lambda_0$). Tweedie conjectures in [8] at the end of Section 5 that $E[\bar{\psi}(Y_1^+)] < \infty$ implies $E_x[\bar{\psi}(\tau_{(-\infty,0)})] < \infty$ for $\psi \in \Lambda_0$. This does not hold, however. To see this, suppose it were true. Put $Y_0 = x = 0$ and let $E[\bar{\psi}(Y_1^+)] < \infty$. Observe that $(cR_n)_0^\infty$ is a random walk with increments cY_n, $n \geq 1$, and that $\tau_{(-\infty,0)} = \inf\{n \geq 0: cR_n < 0\}$ for all $c \in (0, \infty)$. Thus $E[\bar{\psi}(1/c(cY_1^+))] = E[\bar{\psi}(Y_1^+)] < \infty$ implies $E[\bar{\psi}((1/c)\tau_{(-\infty,0)})] < \infty$ for all $c \in (0, \infty)$. Now suppose $P(Y_1 > -a) = 1$ for some $a \in (0, \infty)$. Then $Y_1^+ \leq a\tau_{(-\infty,0)}$ implying $E[\bar{\psi}(cY_1^+)] < \infty$ for all $c \in (0, \infty)$. This cannot be true in general since Λ_0 contains functions of the same order as $x \to e^{x^\beta}$ where $\beta \in (0, 1)$. Our conjecture is: If $\psi \in \Lambda_0$ and $E[\psi(Y_1^+)] < \infty$ then there exists an $\varepsilon > 0$ such that $E[\psi(\varepsilon\tau_{(-\infty,0)})] < \infty$.

Acknowledgement

I would like to thank Sören Asmussen for useful comments.

References

[1] S. Asmussen, Some martingale methods in the limit theory of supercritical branching processes, in: Advances in Probability 5 (Dekker, New York, 1978).

[2] W. Feller, An Introduction to Probability Theory and its Applications, Vol. II (Wiley, New York, 1966).
[3] J. Neveu, Discrete-Parameter Martingales (North-Holland, Amsterdam, 1975).
[4] D. Stoyan, Qualitative Eigenschaften und Abschätzungen stochastischer Modelle (Akademie-Verlag, Berlin, 1977).
[5] H. Thorisson, The coupling of regenerative processes. Adv. Applied Probab. 15 (1983) 531–561.
[6] H. Thorisson, The queue $GI/G/k$: Finite moments of the cycle variables and uniform rates of convergence, Stochastic Models (to appear).
[7] H. Thorisson, On regenerative and ergodic properties of the k-server queue with non-stationary Poisson arrivals, J. Applied Probab. (to appear).
[8] R.L. Tweedie, Criteria for rates of convergence of Markov chains, with application to queuing and storage theory. In London Math. Soc. Lect. Note Series 79 (Cambridge University Press, 1983) 260–276.