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# TIGHT IMMERSIONS OF SIMPLICIAL SURFACES IN THREE SPACE

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## 1. INTRODUCTION

In 1960, Kuiper [12, 13] initiated the study of tight immersions of surfaces in three space, and showed that every compact surface admits a tight immersion except for the Klein bottle, the real projective plane, and possibly the real projective plane with one handle. The fate of the latter surface went unresolved for 30 years until Haab recently proved [9] that there is no smooth tight immersion of the projective plane with one handle. In light of this, a recently described simplicial tight immersion of this surface [6] came as quite a surprise, and its discovery completes Kuiper's initial survey. Pinkall [17] broadened the question in 1985 by looking for tight immersions in each equivalence class of immersed surfaces under image homotopy. He first produced a complete description of these classes, and proceeded to generate tight examples in all but finitely many of them.

In this paper, we improve Pinkall's result by giving tight simplicial examples in all but two of the immersion classes under image homotopy for which tight immersions are possible, and in particular, we point out and resolve an error in one of his examples that would have left an infinite number of immersion classes with no tight examples.

## 2. TIGHTNESS AND SIMPLICIAL SURFACES

Classically, for $f\colon M \to \mathbf{R}^3$ a smooth mapping of a closed surface $M$,

$$\tau(f) = \frac{1}{2\pi}\int_M |K|\,dA$$

is the *total absolute curvature* of $M$, where $K$ represents the Gaussian curvature. It is not hard to show [7, 13] that

$$\tau(f) \geqslant 4 - \chi(M)$$

where $\chi(M)$ is the Euler characteristic of $M$. When equality holds, $f$ is called *tight*.

This definition has several geometric consequences, an important one being that a map is tight if, and only if, it has the two-piece property of Banchoff; namely that the preimage of every half-space is connected in $M$ [1], or in other words, every plane in $\mathbf{R}^3$ cuts the surface into at most two parts. For surfaces that are not smooth, the total absolute curvature integral is not defined, but the two-piece property still makes sense, and in particular, for simplicial surfaces, it can be taken as the definition of tightness.

The modern approach to the subject [4, 14, 10] casts the definition in terms of maps on the homology groups of $M$.

*Definition* 2.1. A map $f:M^m \to \mathbf{R}^n$ is *k-tight* if, for all directions $z \in S^{n-1} \subset \mathbf{R}^n$ and heights $c \in \mathbf{R}$, the inclusion map $\{p \in M \mid z \cdot f(p) \leqslant c\} \hookrightarrow M$ induces a monomorphism in the $i$th Čech homology for $0 \leqslant i \leqslant k$.

Here, tightness corresponds to $m$-tightness, while the two-piece property corresponds to 0-tightness. To see the latter, note that $z \cdot f$ is simply the height function in the direction $z$, and so the set $\{p \in M \mid z \cdot f(p) \leqslant c\}$ is the preimage of a half-space; since 0-dimensional homology counts the number of connected components, the fact that inclusion induces a monomorphism implies that there is only one component in the preimage.

This definition is valid for manifolds with boundary, and for both smooth and simplicial manifolds. In the case of closed surfaces without boundary ($m = 2$), 0-tightness and $m$-tightness are equivalent [14, 10], though this is no longer true for surfaces with boundary [10].

We turn now to the specific case of simplexwise linear immersions of simplicial surfaces in three space.

By a *simplexwise-linear map*, we mean a map $f: M \to \mathbf{R}^3$ from a triangulated surface $M$ to $\mathbf{R}^3$ such that the edges and faces of $M$ are mapped as the convex linear combinations of their vertices. More precisely, if $v_1, v_2, v_3$ are vertices of a face $F$ of $M$, then each point of $F$ can be written uniquely in the form $p = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ where $0 \leqslant \alpha_i$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$; if $f$ is simplexwise linear, $f(p) = \alpha_1 f(v_1) + \alpha_2 f(v_2) + \alpha_3 (f(v_3))$. *We assume that such a map is non-degenerate*, that is, it does not reduce the dimension of any simplex of $M$. Two simplices are said to *intersect* if their images intersect other than at a vertex or edge common to both. If $v$ is a vertex of $M$, its *star* is the collection of all simplices containing $v$.

The map $f: M \to \mathbf{R}^3$ is an *embedding* if it is a one-to-one mapping of $M$ into $\mathbf{R}^3$, and it is an *immersion* if it is locally one-to-one (for simplicial surfaces this is enough, although for smooth surfaces one usually requires additional properties that guarantee tangent planes at every point).

If $f$ is a simplexwise-linear map, then each face is mapped one-to-one by $f$, hence the interiors of faces are immersed. Since $f$ is linear, if two triangles with a common vertex intersect, they do so in every neighborhood of the vertex, so if the vertices of an edge are immersed, then the two triangles sharing the edge cannot intersect, and their union is mapped one-to-one. Thus, the interior of an edge is immersed whenever its vertices are. Thus, $f$ is an immersion if and only if no two faces of $M$ with a common vertex intersect, which we state as follows.

LEMMA 2.2. *A simplexwise-linear map $f: M \to \mathbf{R}^3$ is an immersion if and only if the star of every vertex of $M$ is embedded by $f$.*

A vertex $v$ of $M$ is called a *local extreme vertex* if $f(v)$ is a vertex of the convex hull of the image of the star of $v$ (that is, it is an isolated local maximum for the height function on $f(M)$ in some direction), and it is a *(global) extreme vertex* if $f(v)$ is a vertex of the convex hull of $f(M)$.

Since the number of components of a polyhedron (or its intersection with a half-space) depends only on its 1-skeleton, a simplexwise-linear surface has the two-piece property if, and only if, its 1-skeleton does. This provides a useful characterization of tightness for simplicial surfaces.

LEMMA 2.3. *A simplexwise-linear map $f: M \to \mathbf{R}^3$ of a compact, connected surface $M$ is tight if, and only if,*

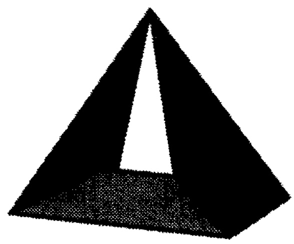


*Fig.* 1. A sphere with two points touching that satisfies the first two conditions of Lemma 2.3 but not the third. It is not tight since a horizontal plane just below the upper vertex cuts $M$ into three pieces.

Fig. 2. The centers of two non-planar triangles are removed (left) and the holes are attached by a triangular tube (right). The resulting handle is tight and immersed.

(1) *every local extreme vertex is a global extreme vertex,*
(2) *every edge of the convex hull of* $f(M)$ *is contained in* $f(M)$,
(3) *every vertex of the convex hull of* $f(M)$ *is the image of a single vertex of* $M$.

This lemma can be found in the literature ([10], for example) as a result for embedded surfaces, without the third condition. This condition is required for immersions, however, as shown by Fig. 1, which is a sphere with two points touching. It satisfies (1) and (2), but it is not tight since a horizontal plane just below the upper vertex cuts off two pieces of $M$ at the top.

Given a tight surface $M$, it is always possible to add a handle to $M$ while maintaining tightness: take any two non-coplanar faces of $M$, remove a triangle from the interior of each, and connect the two holes by a tube (see Fig. 2). This does not change the convex hull, so properties (2) and (3) still hold, and new vertices are not local extreme vertices, so (1) still holds. If $M$ is immersed, then so is the modified surface since the stars of the new vertices are embedded and the stars of the original vertices remain essentially unchanged.

The added handle can be orientable (Fig. 5, middle) or sometimes non-orientable (Fig. 5, right). Any number of handles can be added to a tight immersion in this way to obtain a tight immersion of arbitrarily large genus.

### 3. IMAGE HOMOTOPY OF IMMERSIONS

Given two immersions $f,g: M \to \mathbf{R}^3$, one can ask whether one immersion can be smoothly transformed into the other. Such a transformation is a smooth homotopy that is an immersion at very step, and is called a *regular homotopy* between $f$ and $g$. For example, the first two immersions of the torus shown in Fig. 3 are not regularly homotopic, and represent significantly different immersions. On the other hand, the last two tori are also not regularly homotopic, even though their images are identical. This is due to a change in parametrization between the two rather than a difference in the immersion itself, which motivates the following definition: two immersions are *image homotopic* if there exists

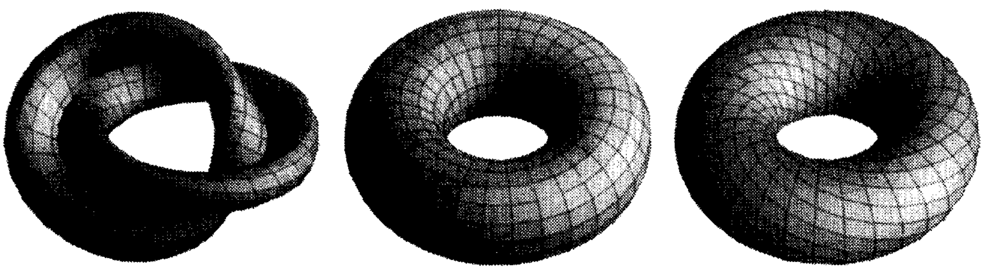


Fig. 3. The immersions of the torus, all distinct under regular homotopy, but the last two are the same under image homotopy. The first is the non-standard, or "twisted" torus.

a diffeomorphism $\phi\colon M \to M$ such that $f$ and $g \circ \phi$ are regularly homotopic. In this way, the last two tori of Fig. 3 are image homotopic, while the first two are not.

For simplicial surfaces, the notion of diffeomorphism is replaced by that of symmetry. A mapping $\phi\colon M \to M$ of a triangulated surface to itself is a *symmetry* of the triangulation if it is one-to-one, onto mapping that preserves the dimension of simplices (i.e., it maps vertices to vertices, edges to edges and faces to faces), and we have the following.

*Definition* 3.1. Two immersions $f, g\colon M \to \mathbf{R}^3$ of a triangulated surface are *image homotopic* if there is a symmetry $\phi$ and a homotopy $H\colon M \times [0,1] \to \mathbf{R}^3$ such that $H_t(p) = H(p,t)$ is an immersion for each $t \in [0,1]$, and such that $H_0 = f$ and $H_1 = g \circ \phi$. (If $f$ and $g$ are immersions of different triangulations of $M$, one must pass first to a common refinement of these triangulations.)

In [16], Pinkall determines the structure of the set of all immersions of surfaces under image homotopy (he defines an *immersed surface* to be an equivalence class of surfaces under diffeomorphism and then considers regular homotopy on these classes; this is equivalent to considering image homotopy as we use it here). He shows that image homotopy is closely tied to the idea of *cobordism*.

*Definition* 3.2. Two immersions $f\colon M \to \mathbf{R}^3$ and $g\colon N \to \mathbf{R}^3$ are *cobordant* if there exists a 3-manifold $X$ having as boundary the disjoint union of $M$ and $N$ and an immersion $h\colon X \to \mathbf{R}^3 \times [0,1]$ such that $h$ is transversal to $\mathbf{R}^3 \times \{0,1\}$ with $h|_M = f \times \{0\}$ and $h|_N = g \times \{1\}$.

Pinkall shows [16, Theorem 8] that two immersions $f$ and $g$ of a given surface $M$ are image homotopic if, and only if, they are cobordant. Wells [18] determined that the equivalence classes of immersions of surfaces in $\mathbf{R}^3$ under cobordism form a group isomorphic to $\mathbf{Z}_8$, with the Boy surface (an immersion of the real projective plane) as generator under the operation of connected sum. We can interpret Pinkall's result in terms of the cobordism group by breaking each element of $\mathbf{Z}_8$ into its different topological types, as in Table 1.

Here $S$ is the standard embedded torus (Fig. 3, center), $T$ is the "twisted' torus (Fig. 3, left), $\bar{B}$ and $B$ are left- and right-handed immersions of the Boy surface, $K_0$ is the standard immersion of the Klein bottle with reflective symmetry (Fig. 4 left), and $K_-$ and $K_+$ are left- and right-handed versions of the "twisted" Klein bottle (Fig. 4, right). Note that $K_0 = B \# \bar{B}$, $K_- = \bar{B} \# \bar{B}$ and $K_+ = B \# B$.

Table 1. The eight elements of the cobordism group of immersed surfaces broken down into their topological types.

| $\mathbf{Z}_8$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ |
|---|---|---|---|---|---|---|---|---|
| | $K_+ \# \bar{B}$ | $K_-$ | $\bar{B}$ | $\left\{\begin{matrix} S \\ K_0 \end{matrix}\right\}$ | $B$ | $K_+$ | $K_+ \# B$ | $\left\{\begin{matrix} T \\ K_0 \# T \end{matrix}\right\}$ |
| | $\# S$ | $\# S$ | $\# S$ | $\# S$ | $\# S$ | $\# S$ | $\# S$ | $\# S$ |
| | $\# 2S$ | $\# 2S$ | $\# 2S$ | $\# 2S$ | $\# 2S$ | $\# 2S$ | $\# 2S$ | $\# 2S$ |
| | $\# 3S$ | $\# 3S$ | $\# 3S$ | $\# 3S$ | $\# 3S$ | $\# 3S$ | $\# 3S$ | $\# 3S$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Note: Each entry is a unique class under image homotopy. Moving down a column corresponds to adding a handle; moving right or left corresponds to adding a right- or left-hand Möbius band. Columns 0 and 4 contain both orientable and non-orientable members; these each form two distinct columns under image homotopy.

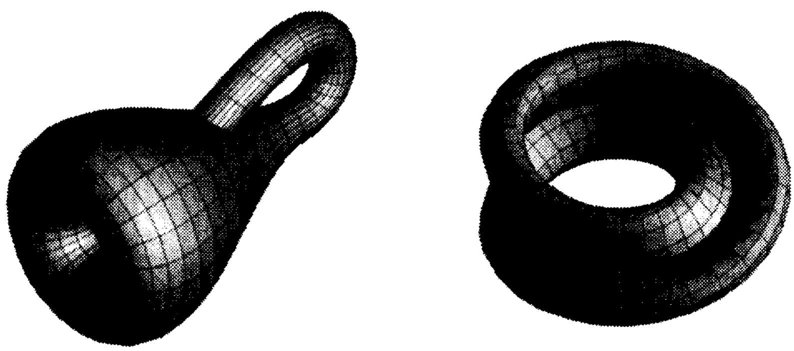


Fig. 4. Two distinct immersions of the Klein bottle: the standard immersion with reflective symmetry (left) and the "twisted" version with rotational symmetry (right).

In Table 1, each column represents one equivalence class under cobordism, but a family of related classes under image homotopy. Moving down a column corresponds to adding a handle and moving left or right to connected sum with $\bar{B}$ or $B$. Columns 0 and 4 each represent two families of immersions under image homotopy: one orientable (based on $S$ or $T$) and one non-orientable (based on $K_0$ or $K_0 \# T$).

#### 4. IMAGE HOMOTOPY AND TIGHTNESS

As a result of Pinkall's work [16], we see that every immersion class under image homotopy can be written as one of ten basic surfaces together with zero or more handles (see Table 1). Given a specific tight immersion of a surface in some column of Table 1, we can always add handles as outlined in Section 2 to obtain tight immersions of all the surfaces below it in that column. When looking for tight immersions, the idea is to find examples in each column as high up as possible.

Note that the columns to the left of zero are simply mirror reflections of the corresponding columns to the right, so a tight immersion in one immediately yields a tight immersion in the other; thus, we need only consider the columns labelled with non-negative numbers.

Figure 5 shows a tight torus (left). Adding handles (middle) gives tight examples of all the orientable surfaces in column 0. Adding a non-orientable handle (right) gives $K_0 \# S$, to which additional handles can be added. Since the Klein bottle itself cannot be tightly immersed [15], this completes the non-orientable surfaces in column 0 [13].

Table 2. A listing of Pinkall's tight immersions.

| $\mathbf{Z}_8$ | 0<br>$K_0$ | 1<br>$B$ | 2<br>$K_+$ | 3<br>$K_+ \# B$ | 4<br>$K_0 \# T$ | 0<br>$S$ | 4<br>$T$ |
|---|---|---|---|---|---|---|---|
| | $\times$ | $\times$ | $\times$ | $\times$ | $\bigcirc$ | $\surd$ | $\times$ |
| $\# S$ | $\surd$ | $\not\surd$ | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ | $\surd$ | $\bigcirc$ |
| $\# 2S$ | $\surd$ | $\surd$ | $\bigcirc$ | $\bigcirc$ | $\underline{\surd}$ | $\surd$ | $\bigcirc$ |
| $\# 3S$ | $\surd$ | $\surd$ | $\bigcirc$ | $\bigcirc$ | $\surd$ | $\surd$ | $\underline{\bigcirc}$ |
| $\# 4S$ | $\surd$ | $\surd$ | $\surd$ | $\underline{\surd}$ | $\surd$ | $\surd$ | $\underline{\bigcirc}$ |
| $\# 5S$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\surd$ | $\underline{\bigcirc}$ |
| | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Note: A circle represents a missing example, while a cross indicates a surface that cannot be tightly immersed. Underlined entries are corrections due to the erroneously identified surface $T \# S \# S \# S$. The check with a slash represents the newly discovered tight polyhedral $B \# S$, which was unknown to Pinkall, and for which no smooth version is possible.

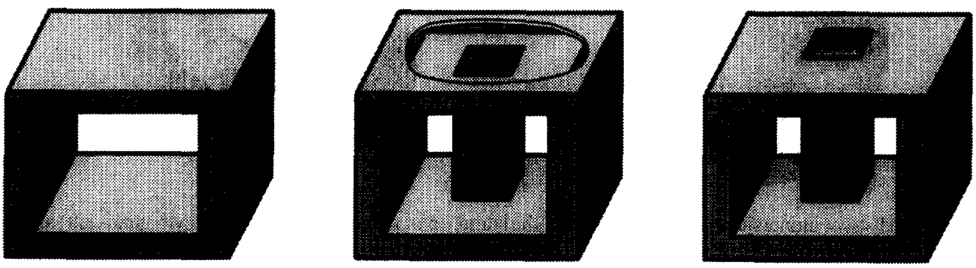


Fig. 5. A tightly-immersed torus (left) can have a handle added (center) to form $S \# S$, or a non-orientable handle added (right) to form $K_0 \# S$.

Kuiper showed that the projective plane cannot be tightly immersed in $\mathbf{R}^3$, but he described a tight immersion of $B \# S \# S$ [13]. In [11], Kühnel and Pinkall give an explicit polyhedral version of $B \# S \# S$ with 3-fold symmetry. Tight handles can be added to this as above. Recently, Haab [9] showed that no smooth tight immersion of $B \# S$ exists, and while his proof is valid only for the smooth case, one might conjecture that the same holds for polyhedral surfaces as well. Surprisingly, this is not the case. The newly discovered tight simplicial immersion of $B \# S$ [6] represents an important example where the smooth and simplicial theories differ in a significant way. This example can be used to complete column 1.

Pinkall [17] investigated the situation for the remaining classes of immersions. First he showed that no tight immersion exists for $T$ or $K_+ \# B$. He then constructed a polyhedral tight immersion of $T \# S \# S \# S$, added a non-orientable handle to get $K_0 \# T \# 3S$, combined it with the polyhedral version of $B \# S \# S$ to get $K_0 \# B \# 5S$, and combined two copies of $B \# S \# S$ to get $K_+ \# 4S$, thus obtaining entries in each of the other columns. Unfortunately, his model of $T \# S \# S \# S$ is in error (in a way that will be made clear in the next section). It is, in fact, a model of $K_0 \# T \# S \# S$, a non-orientable surface. This gives slightly better results in columns 3 and 4, but leaves no known examples of tight orientable immersions based on $T$ (see Table 2).

## 5. SOME NEW TIGHT IMMERSIONS

Pinkall's results [17] leave several missing examples of tight immersions, and in particular, an infinite family of surfaces based on the twisted torus, $T$. In this section, we find tight immersions that improve the results for all columns where there are missing examples, including a model that replaces Pinkall's incorrect one.

We begin by constructing a simplicial version version of the twisted torus of Fig. 3. A neighborhood of its self-intersection is formed by two intersecting twisted strips as shown in Fig. 6 (each strip is twisted since its two boundary curves are linked). To generate such a configuration simplicially, we begin by making one twisted strip, starting with two linked triangular paths (for the boundary curves of the twisted strip) and filling in triangles between these two, as in Fig. 7 (left).

Duplicating this twisted strip in place and moving the vertices of the duplicate away from the originals slightly, we form a second twisted strip that intersects the first in its interior (Fig. 7, middle); this is the desired neighborhood of the self-intersection. To complete the model, it only remains to connect the boundaries of the two strips with two additional bands of triangles, as in Fig. 7 (right). The resulting model is an immersed torus of the non-standard type, $T$, with three axes of 2-fold rotational symmetry (Fig. 8).

As expected, this model is not tight, but it can be made tight by combining it with its *convex envelope* (the boundary of its convex hull). This is accomplished as follows: take the convex envelope and remove from it and the original model any faces they have in common, then glue the envelope to the model along the boundaries of these removed regions (this is

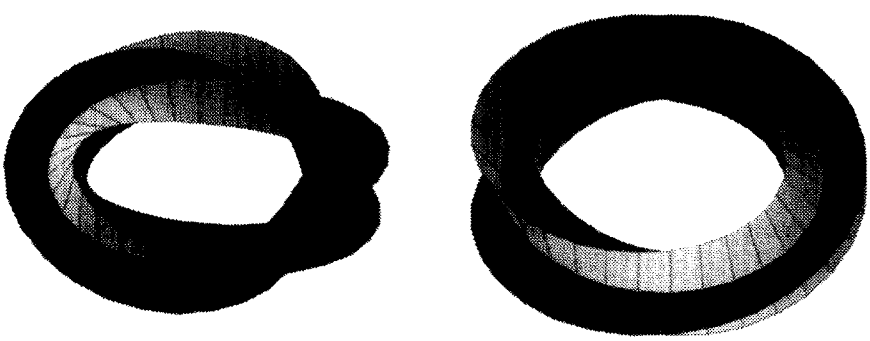


Fig. 6. The neighborhoods of the double curves of the twisted torus (left) and twisted Klein bottle (right). The first is formed by two intersecting twisted strips, and the second by two intersecting Möbius bands.

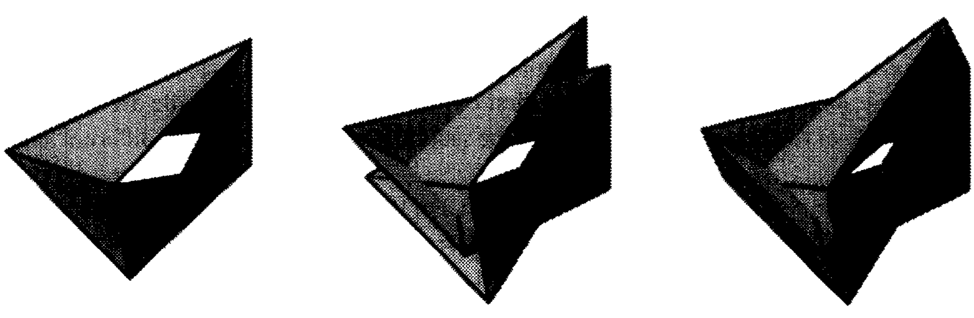
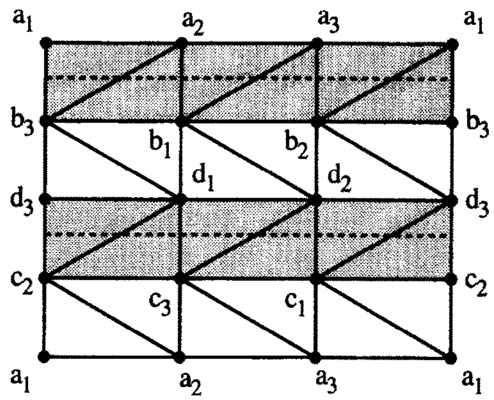


Fig. 7. Starting with a twisted band (left), each triangle is duplicated and the vertices separated to form two intersecting twisted bands (center). Triangles are added between the boundaries of these strips, forming a 12-vertex twisted torus (right).




$a_1 \to (1, -c, -c)$
$a_2 \to (-2, -a-b, -b)$
$a_3 \to (-2, a-b, -b)$

$b_1 \to (-1, -c, -c)$
$b_2 \to (2, b, a+b)$
$b_3 \to (2, b, -a+b)$

$c_1 \to (1, c, c)$
$c_2 \to (-2, -a+b, b)$
$c_3 \to (-2, a+b, b)$

$d_1 \to (-1, c, -c)$
$d_2 \to (2, -b, a-b)$
$d_3 \to (2, -b, -a-b)$

$a = \sqrt{3}$
$b = \frac{1}{3}$
$c = \frac{a-2b}{4}$

Fig. 8. The triangulation for the example of Fig. 7 and its vertex mapping. The intersecting twisted strips are shaded, and a homologically different twisted strip is formed by the triangles $a_3a_1c_1$, $c_1a_1c_2$, $c_1c_2d_3$, $d_3c_2d_1$, $d_3d_1b_3$, $d_3b_3b_2$, $b_2b_3a_1$, $b_2a_1a_3$. The dotted line represents the self-intersection.

called the *mod 2 sum* of the surface and its convex envelope). Under the right conditions, the result will be tight.

Our model of $T$ and its convex envelope have two planar parallelograms in common (they are $a_2c_2c_3a_3$ and $b_2d_2d_3b_3$; see Fig. 8). Adding the remainder of the convex hull to the model with these faces removed yields a tight immersion. To see this, note that the four vertices $a_1$, $b_1$, $c_1$, and $d_1$ that are not on the convex hull are not locally extreme, and all the edges of the convex hull are part of the surface (this is the reason for adding the convex envelope). Finally, the self-intersection does not pass through any vertices, so all the conditions of Lemma 2.3 are satisfied.

What surface does this represent? The convex envelope with two parallelograms removed is topologically a sphere minus two disks, i.e. a cylinder, so this process adds a handle to $T$. We expect this to be $T \# S$, then, but this is not the case. It turns out that the handle is a non-orientable one of the type shown in Fig. 5. To see this, color the two sides of the torus with different colors: due to the self-intersection, part of the visible surface of the model will show one color and part the other. Note that the two removed parallelograms show different colors (Fig. 7). When the rest of the convex envelope is added, this effectively attaches the inside to the outside, forming a Möbius band. The resulting surface is $K_0 \# T$. For the same reason, Pinkall's example of $T \# S \# S \# S$ in [17] is actually the non-orientable surface $K_0 \# T \# S \# S$.

To find a tight, orientable immersion based on $T$ we must use a technique that not only produces a tight surface, but also maintains orientability. One such method is suggested by the procedure pictured in Fig. 5. Here, a tight torus is modified by the addition of a handle in one case and by a non-orientable handle in the other. Both handles are shaped the same, but their attachment points differ: in the first case, both ends of the handle are attached to the same side of the original torus, while in the second, they are attached to different sides of the torus.

In our present situation, we can use the model of the twisted torus (Fig. 7) in place of the new handle of Fig. 5 and attach it as in the non-orientable case. The attachment sites are on opposite sides of the original torus, but the attachment points on the twisted torus are also on opposite sides, so orientability is maintained. The first attachment forms the connected sum $T \# S$ and the second adds one more handle, forming $T \# S \# S$. The resulting surface is tight. Additional orientable handles can be introduced as usual to obtain tight immersions of any higher genus.

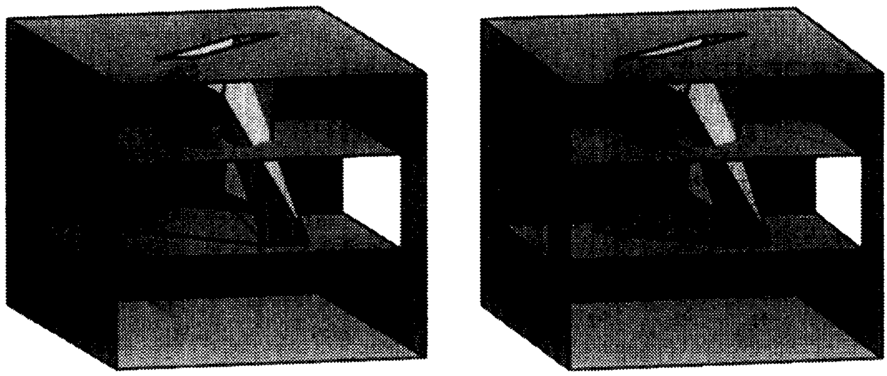


Fig. 9. A stereo pair showing the tight immersion of $T \# S \# S$. Two faces are removed to allow the interior structure to be seen.

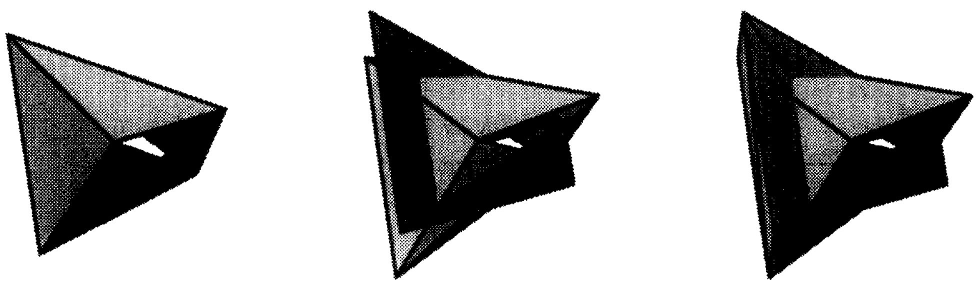


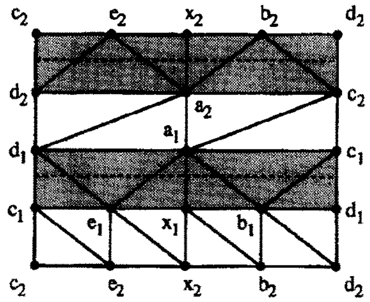
Fig. 10. A 5-vertex Möbius band (left) is duplicated and the vertices separated to form two intersecting Möbius bands (center). Triangles are added between the boundaries of these bands to form a 10-vertex twisted Klein bottle (right).

A stereo pair picturing the tight immersion of $T \# S \# S$ is given in Fig. 9. Two faces have been removed (the front of the large cube, and one wall of the tube running through it) so that the interior structure is visible.

Our tight immersions based on $T$ improve Pinkall's results in two columns of Table 2 (the ones labelled 4), leaving two columns remaining to be considered. The approach used to generate $T$ also proves fruitful in the case of $K_+$. From Fig. 6 (right), we see that a neighborhood of the self-intersection for $K_+$ is formed by two intersecting Möbius bands. To find a simplicial version of this surface, we begin with a 5-vertex Möbius band, as shown in Fig. 10 (left).

Duplicating the band and moving the vertices of the duplicate away from the originals as before yields two Möbius bands that intersect in their interiors (Fig. 10, middle). Adding a strip between the boundaries of the bands completes the figure as a twisted Klein bottle, as shown (right).

We would like to add the convex envelope as we did before, Unfortunately, it shares a non-planar region with the surface (formed by the triangles $b_1c_1b_2$, $b_1b_2e_2$, $e_1b_2e_2$, and $e_1e_2d_2$; see Fig. 11). This can be rectified by the addition of two new vertices at the midpoints of the edges $b_1e_1$ and $b_2e_2$; if these vertices are moved toward vertices $a_1$ and $a_2$, then the parallelogram $b_1e_1e_2b_2$ bends so that it no longer lies on the convex envelope (Fig. 11). Taking the mod 2 sum of this surface with its convex envelope, we achieve an immersion with six interior vertices, none of which is locally extreme. By Lemma 2.3, the resulting surface is tight. As before, the convex envelope has two disks removed, so the sum effectively adds a handle, yielding $K_+ \# S$. This completes column 2 in the table.




$a_1 \to (0,-a,-2a)$
$b_1 \to (-8,11-a,-2a)$
$c_1 \to (9+a,0,5+2a)$
$d_1 \to (9-3a/2,0,-5+2a)$
$e_1 \to (-8,-11-a,-2a)$

$a_2 \to (0,a,-2a)$
$b_2 \to (-8,11+a,2a)$
$c_2 \to (9-3a/2,0,5-2a)$
$d_2 \to (9+a,0,-5-2a)$
$e_2 \to (-8,-11+a,2a)$

$x_1 \to (-6,0,-a)$
$x_2 \to (-6,0,a)$ $\quad a=1$

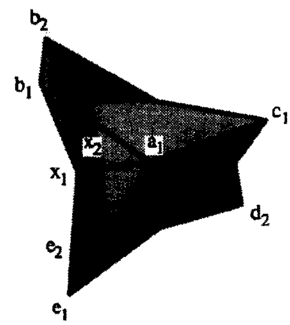



Fig. 11. The Klein bottle that forms the central core of the tight immersion of $K_+ \# S$. The two intersecting Möbius bands are shaded, and the dotted lines represent the self-intersection.

Table 3. A listing of tight immersions.

| $\mathbf{Z}_8$ | 0 $K_0$ | 1 $B$ | 2 $K_+$ | 3 $K_+ \# B$ | 4 $K_0 \# T$ | 0 $S$ | 4 $T$ |
|---|---|---|---|---|---|---|---|
| | × | × | × | × | ⊙ | √ | × |
| $\# S$ | √ | √̸ | ⊙ | ○ | ⊙ | √ | ○ |
| $\# 2S$ | √ | √ | ⊙ | ⊙ | √ | √ | ⊙ |
| $\# 3S$ | √ | √ | ⊙ | ⊙ | √ | √ | ⊙ |
| $\# 4S$ | √ | √ | √ | √ | √ | √ | ⊙ |
| $\# 5S$ | √ | √ | √ | √ | √ | √ | ⊙ |
| | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |

Note: A circle represents a missing example, while a cross indicates a surface that cannot be tightly immersed. A dot in a circle represents a new polyhedral example developed in this section. The check with a slash represents the polyhedral tight immersion of $B \# S$, for which no smooth version is possible.

The final column is the one based on $K_+ \# B$. We have just created a tight model of $K_+ \# S$, and in [6] there is a tight simplicial imersion if $B \# S$. Since tightness is a projective property [12, 14], with the appropriate choice of scalings and skewings, these two models can be combined (via connected sum) to form a tight immersion of $K_+ \# B \# S \# S$. This gives an example in the $\# 2S$ row of column 3.

The results of these new examples are summarized in Table 3.

Note that there are only two surfaces for which no tight immersion is known. The missing $T \# S$ seems difficult to obtain, and the author conjectures that no such immersions exists. As for $K_+ \# B \# S$, the author suspects that a simplicial tight immersion of this surface does exist.

In [17], Pinkall uses a smoothing algorithm to convert his polyhedral tight immersions into smooth ones. His algorithm has rather strict valence conditions, however, which the models presented here do not satisfy. It may be possible to modify these examples (or the algorithm itself) to fit, but we have already seen that there is no guarantee that this is possible, as in the case of $B \# S$, so at the moment these results are purely simplicial.

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