Certain Sufficient Conditions for Univalency of the Class $C'$

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1. INTRODUCTION

In a recent paper [1] the authors obtained sufficient conditions for functions of the class $C'$ to be univalent and spirallike in the unit disc $U$. In this note we obtain sufficient conditions for univalency for a much larger subclasses of functions in $C'(U)$. Special cases of our results can be found in [1–5].

2. PRELIMINARIES

Let $f$ be a complex function defined in the unit disc $U$. For $z = x + iy \in U$ we put

$$f(z) = u(x, y) + iv(x, y).$$

We say that $f$ belongs to the class $C'(U)$ if the real functions $u(x, y) = \text{Re } f(z)$, $v(x, y) = \text{Im } f(z)$ of the real variables $x$ and $y$ have continuous first order partial derivatives in $U$. For $f \in C'(U)$ we let

$$Df = z \frac{\partial f}{\partial z} - \bar{z} \frac{\partial f}{\partial \bar{z}},$$

$$\mathcal{Q}f = z \frac{\partial f}{\partial z} + \bar{z} \frac{\partial f}{\partial \bar{z}}.$$
where
\[
\frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

It is easy to verify the following useful formulas:
\[
\begin{align*}
Df &= \overline{D\bar{f}}, \\
D \text{Re } f &= i \text{ Im } Df, \\
D \text{Im } f &= -i \text{ Re } Df, \\
D |f| &= i |f| \text{ Im } \frac{Df}{f} , \\
D \text{arg } f &= -i \text{ Re } \frac{Df}{f}, \\
\frac{\partial f}{\partial \theta} &= i Df, \\
\frac{\partial f}{\partial r} &= \frac{1}{r} \frac{\partial f}{\partial f}, \quad \text{where } z = re^{i\theta}.
\end{align*}
\]

Hence
\[
\begin{align*}
\frac{\partial |f|}{\partial \theta} &= -|f| \text{ Im } \frac{Df}{f}, \\
\frac{\partial |f|}{\partial r} &= |f| \text{ Re } \frac{Df}{f}, \\
\frac{\partial \text{arg } f}{\partial \theta} &= \text{Re } \frac{Df}{f}, \\
\frac{\partial \text{arg } f}{\partial r} &= \frac{1}{r} \text{ Im } \frac{Df}{f},
\end{align*}
\]

The Jacobian of $f$ is given by
\[
J_f = \left| \frac{\partial f}{\partial z} \right|^2 - \left| \frac{\partial f}{\partial \bar{z}} \right|^2.
\]

If $J_f(z) > 0$ for all $z \in U$, then $f$ is locally homeomorphic preserving the orientation.

3. Results

For the proofs of our results we follow the approach of Rakhmanov (the geometrical families) as outlined and applied in [3, 4] for the case of analytic functions.
THEOREM 1. Let \( f \in C'(U) \) and let \( F \) be a real continuous function in the interval \((0, +\infty)\). If the following conditions are satisfied:

(i) \( f(0) = 0, f(z) \neq 0 \) for all \( z \in U \setminus \{0\} \),
(ii) \( Jf(z) > 0 \) for all \( z \in U \),
(iii) \( \text{Re} \{ |1 + iF(|f(z)|)| \frac{Df(z)}{f(z)} \} > 0 \) for all \( z \in U \setminus \{0\} \),

then \( f \) is univalent in \( U \).

Proof. Let \( \Phi : (0, +\infty) \rightarrow \mathbb{R} \) such that

\[ t\Phi'(t) = -F(t). \] (3)

Consider the family of Jordan arcs \( (\Gamma_\phi), \phi \in [0, 2\pi) \), where \( \Gamma_\phi \) has the parametric equation

\[ \Gamma_\phi : w = w_\phi(t), \quad t \in (0, +\infty), \]

where

\[ w_\phi(t) = te^{i(\Phi(t) + \phi)}. \] (4)

It is obvious that through each point \( w \in C \setminus \{0\} \) passes a Jordan arc \( \Gamma_\phi \) and only one.

For \( z = re^{i\theta}, 0 < r < 1, 0 \leq \theta < 2\pi \), the equation

\[ f(z) = w_\phi(t) \] (5)

determines a unique value of \( \phi = \phi(r, \theta) \) and a unique value of \( t = t(r, \theta) \). From (4) and (5) we get

\[ t = |f(z)|, \]
\[ \Phi(t) + \phi = \text{arg} \ f(z). \] (7)

We therefore have

\[ \phi = \text{arg} \ f(z) - \Phi(|f(z)|). \] (8)

Let \( U_r = \{ z : |z| < r \} \) and \( C_r = f(\partial U_r) \). In order to show that \( f \) is univalent in \( U \), it is sufficient to prove that \( C_r \) are nonintersecting Jordan curves, \( r \in (0, 1) \). We first show that \( C_r \) are Jordan curves for \( r \in (0, 1) \) by showing that \( \frac{\partial \phi}{\partial \theta} > 0 \) and \( \text{Var}_{0 < \theta < 2\pi} \phi(r, \theta) = 2\pi \), where \( \text{Var} \) stands for total variation.
Differentiating (8) and using (1) and (2) we obtain

\[
\frac{\partial \phi}{\partial \theta} = \frac{\partial}{\partial \theta} \arg f(z) - \frac{\partial}{\partial \theta} \Phi(|f(z)|) = \text{Re} \left( \frac{Df(z)}{f(z)} + |f(z)| \Phi'(|f(z)|) \text{Im} \frac{Df(z)}{f(z)} \right) = \text{Re} \left\{ |1 + iF(|f(z)|)| \frac{Df(z)}{f(z)} \right\} > 0.
\]

Also by (i) the curves \( C_r \) are homotopic in \( \mathbb{C} \setminus \{0\} \). Therefore, \( C_r \) have the same index with respect to the origin. By (ii) the function \( f \) is univalent in a neighborhood of the origin. Hence there exists \( r_0 \in (0, 1) \) such that \( \text{ind}_0 C_r = 1 \) for \( r < r_0 \); thus \( \text{ind}_0 C_r = 1 \) for all \( r \in (0, 1) \). That is, the \( \text{Var}_{0 < \theta < 2\pi} \arg f(z) \) along \( C_r \) is \( 2\pi \). Consequently (8) yields

\[
\text{Var}_{0 < \theta < 2\pi} \phi(r, \theta) = \text{Var}_{0 < \theta < 2\pi} f(re^{i\theta}) = 2\pi.
\]

With this, we have verified that \( C_r \) is a Jordan curve for each \( r \in (0, 1) \).

Now we show that \( C_r \cap C_{r'} = \emptyset \) for \( r \neq r' \), \( r, r' \in (0, 1) \). Fix a value \( \phi \in [0, 2\pi) \). The system

\[
f(z) = w_\phi(t), \quad |z| = r, \quad 0 < r < 1
\]

yields a unique point \( z = re^{i\theta}, \theta = \phi(r) \), and a unique \( t = t(r) \). We need only show

\[
\frac{dt}{dr} > 0 \quad \text{for} \quad r \in (0, 1).
\] (9)

From (6) and (7) and using (1) and (2) we get

\[
\frac{dt}{dr} = \frac{\partial}{\partial r} |f(z)| + \frac{\partial}{\partial \theta} |f(z)| \frac{d\theta}{dr} = |f(z)| \left( \frac{1}{r} \text{Re} \frac{Df(z)}{f(z)} - \frac{d\theta}{dr} \text{Im} \frac{Df(z)}{f(z)} \right),
\]

and

\[
\Phi'(|f(z)|) \frac{dt}{dr} = \frac{\partial}{\partial r} \arg f(z) + \frac{\partial}{\partial \theta} \arg f(z) \cdot \frac{d\theta}{dr} - \frac{1}{r} \text{Im} \frac{Df(z)}{f(z)} + \frac{d\theta}{dr} \text{Re} \frac{Df(z)}{f(z)}.
\]
By eliminating \(d\theta/dr\) we deduce

\[
\frac{dt}{dr} \Re \left\{ \left[ 1 + iF(|f(z)|) \right] \frac{Df(z)}{f(z)} \right\} = \frac{|f'(\tau)|}{r} \Re \frac{Df(\tau)}{f(\tau)} \frac{\partial f(\tau)}{\partial \tau} = \frac{r}{|f(z)|} Jf(z).
\]

Finally, (ii), (iii) and the above yield (9). This completes the proof of our main theorem.

The same method yields this result:

**Theorem 2.** Let \(f \in C'(U)\) and let \(F\) be a real continuous function in the interval \((0, +\infty)\). If the following conditions are satisfied:

(i) \(f(0) = 0\) and \(f(z) \neq 0\) for all \(z \in U \setminus \{0\}\),

(ii) \(Jf(z) > 0\) for all \(z \in U\),

(iii) \(\Re \{|F(|f(z)|) + i|Df(z)/f(z)|\} > 0\) for all \(z \in U \setminus \{0\}\),

then \(f\) is univalent in \(U\).

4. Special Cases

Several applications can be made of Theorems 1 and 2 through making different choices of \(F\). We shall mention the following cases:

(a) If \(F \equiv 0\), then Theorem 1 reduces to the case of starlike functions \([1, \text{Corollary 1; 2}].\)

(b) If \(F \equiv \tan \gamma, |\gamma| < \pi/2\), then Theorem 1 reduces to the case of spirallike functions \([1, \text{Theorem 1}].\)

(c) If \(F = -t\), then Theorem 1 reduces to \([1, \text{Theorem 2, 1}].\)

(d) If \(F = t\), then Theorem 2 reduces to \([1, \text{Theorem 3, 1}].\)

It is also noteworthy that when \(f\) is analytic in the unit disc, then cases (a), (b), (c) and (d) above will correspond to the well-known cases of starlike functions, the logarithmic spirallike of type \(\gamma\) due to Spaček \([5]\), the Archimedean spirallike \([3]\) and the hyperbolic spirallike \([4]\), respectively.

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REFERENCES


