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Linear bilevel programs with multiple objectives at the upper level[☆]

Herminia I. Calvete[a,∗], Carmen Galé[b]

[a] Dpto. de Métodos Estadísticos, Universidad de Zaragoza, Pedro Cerbuna 12, 50009 Zaragoza, Spain

[b] Dpto. de Métodos Estadísticos, Universidad de Zaragoza, María de Luna 3, 50018 Zaragoza, Spain

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ABSTRACT

Bilevel programming has been proposed for dealing with decision processes involving two decision makers with a hierarchical structure. They are characterized by the existence of two optimization problems in which the constraint region of the upper level problem is implicitly determined by the lower level optimization problem. Focus of the paper is on general bilevel optimization problems with multiple objectives at the upper level of decision making. When all objective functions are linear and constraints at both levels define polyhedra, it is proved that the set of efficient solutions is non-empty. Taking into account the properties of the feasible region of the bilevel problem, some methods of computing efficient solutions are given based on both weighted sum scalarization and scalarization techniques. All the methods result in solving linear bilevel problems with a single objective function at each level.

1. Introduction

General bilevel programming problems with a single objective function at each level can be formulated as:

$$\min_{x_1,x_2} \quad f_1(x_1,x_2), \tag{1a}$$

$$\text{s.t.} \quad (x_1,x_2) \in R \tag{1b}$$

where x_2 solves

$$\min_{x_2} \quad f_2(x_1,x_2) \tag{1c}$$

$$\text{s.t.} \quad (x_1,x_2) \in S \tag{1d}$$

where $x_1 \in \mathbb{R}^{n_1}$ are the upper level variables, which are controlled by the leader or upper level decision maker; $x_2 \in \mathbb{R}^{n_2}$ are the lower level variables, which are controlled by the follower or lower level decision maker; $f_1, f_2 : \mathbb{R}^n \longrightarrow \mathbb{R}$, $n = n_1 + n_2$, are the upper level and lower level objective functions, respectively; and $R, S \subseteq \mathbb{R}^n$ are the sets defined by the upper level and the lower level constraints, respectively.

These mathematical programs provide an appropriate model for hierarchical decision processes with two decision makers, the leader and the follower, each controlling part of the variables and having his own objective function and constraints. Bilevel problems have been increasingly addressed in the literature. Dempe [1] and Vicente and Calamai [2] provide surveys on the subject. Bard [3], Dempe [4] and Shimizu et al. [5] are good textbooks on this topic.

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∗ Corresponding author. Tel.: +34 976 762210; fax: +34 976 761115.
E-mail addresses: herminia@unizar.es (H.I. Calvete), cgale@unizar.es (C. Galé).

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Optimization of a single objective function often oversimplifies processes involved in real systems. Most of these entail either a decision maker trying to get several goals or several decision makers each of them with his own objective. With regard to the last topic, Calvete and Galé [6] consider a linear bilevel problem with one leader and multiple followers which are assumed to be independent. In this problem all involved functions are linear and the objective function and the set of constraints of each follower only include the leader's variables and his own variables. They show that this problem can be reformulated into a linear bilevel problem with a single objective function at each level.

In order to deal with multiple and conflicting objectives, when there is a single level of decision making, several approaches have been proposed in the literature. Multiobjective programming refers to mathematical programs involving several objectives. This has been a very productive research field in mathematical programming during recent decades. Ehrgott [7], Ehrgott and Gandibleux [8] and Figueira et al. [9] provide a comprehensive overview of the literature.

Regarding the use of multiple objective functions in bilevel problems, Bonnel and Morgan [10] consider a semivectorial bilevel optimization problem in which the upper level is a scalar optimization problem, the lower level is a vector optimization problem and the constraints define Hausdorff topological spaces. They show that this bilevel problem with multiple objectives at the lower level can be approached using an exterior penalty method.

In this paper we are concerned by bilevel problems with multiple linear objective functions at the upper level and a linear objective function at the lower level. The constraint regions are assumed to be polyhedra. This model was motivated by a production–distribution planning problem in a supply chain. The distribution company, at the upper level, aims to minimize transportation cost as well as satisfy the preferences of retailers. Manufacturing plants, at the lower level, aim to minimize their own operation costs. First we reformulate the bilevel problem as a standard multiobjective problem with linear objective functions over a non-convex region. Next, we approach it from the multiobjective programming point of view, aiming to analyze its efficient set. We prove that the efficient set is non-empty and give some methods based on both weighted sum scalarization and scalarization techniques to obtain efficient points. Besides, several examples illustrate the complexity of the problem and show that some important properties held by linear multiobjective programs with a single level of decision making are no longer true, due to the lack of convexity. The paper is organized as follows. Section 2 states the problem. In Section 3 the existence of non-dominated points is proved. Sections 4 and 5 provide the main results on finding efficient and weakly efficient solutions. Finally, Section 6 concludes the paper with a summary.

2. Linear bilevel programs with multiple objective functions at the upper level

The linear multiobjective/linear bilevel programming (LMOLBP) problem can be formulated as:

$$\min_{x_1,x_2} \quad (d_1(x_1,x_2),\ldots,d_k(x_1,x_2)) \tag{2a}$$

$$\text{s.t.} \quad A_1^1x_1 + A_2^1x_2 \leqslant b^1 \tag{2b}$$

$$x_1 \geqslant 0 \tag{2c}$$

where x_2 solves

$$\min_{x_2} \quad c_2x_2 \tag{2d}$$

$$\text{s.t.} \quad A_1^2x_1 + A_2^2x_2 \leqslant b^2 \tag{2e}$$

$$x_2 \geqslant 0, \tag{2f}$$

where $d_i(x_1,x_2) = d_{i1}x_1 + d_{i2}x_2$; $d_{i1} : 1 \times n_1$; $d_{i2} : 1 \times n_2$, $i = 1,\ldots,k$; $c_2 : 1 \times n_2$; $A_1^1 : m_1 \times n_1$; $A_1^2 : m_2 \times n_1$; $A_2^1 : m_1 \times n_2$; $A_2^2 : m_2 \times n_2$; $b^1 : m_1 \times 1$; $b^2 : m_2 \times 1$. We assume that the polyhedron R defined by the upper level constraints (2b) and (2c) is non-empty and the polyhedron S defined by the lower level constraints (2e) and (2f) is non-empty and bounded. The polyhedron defined by all constraints is called the constraint region of the (LMOLBP) problem and will be denoted by T. We assume that T is non-empty.

For a given $\tilde{x}_1$, the follower solves the lower level linear programming problem:

$$LP(\tilde{x}_1): \quad \begin{array}{ll} \min\limits_{x_2} & c_2x_2 \\ \text{s.t.} & A_2^2x_2 \leqslant b^2 - A_1^2\tilde{x}_1 \\ & x_2 \geqslant 0. \end{array} \tag{3}$$

Let $M(\tilde{x}_1)$ be the set of optimal solutions to (3):

$$M(\tilde{x}_1) = \left\{ \tilde{x}_2 \in \mathbb{R}^{n_2} : \tilde{x}_2 \in \operatorname*{argmin}_{x_2}\{c_2x_2 : A_2^2x_2 \leqslant b^2 - A_1^2\tilde{x}_1, x_2 \geqslant 0\} \right\}.$$

The feasible region of problem (2), called the inducible region, is implicitly defined as:

$$\mathrm{IR} = \{(x_1,x_2) : (x_1,x_2) \in T,\ x_2 \in M(x_1)\}.$$

Any point of IR is a bilevel feasible solution. Taking into account previous notations, the LMOLBP problem formulated in (2) can be equivalently written as:

$$\begin{aligned} \min_{x_1,x_2} \quad & (d_1(x_1,x_2),\ldots,d_k(x_1,x_2)) \\ \text{s.t.} \quad & (x_1,x_2) \in \mathrm{IR}. \end{aligned} \tag{4}$$

By looking at the objective space, we denote by $\mathcal{Y}$ the image of IR under the objective function mapping $(d_1,\ldots,d_k)$:

$$\mathcal{Y} = \{y \in \mathbb{R}^k : y = (d_1(x_1,x_2),\ldots,d_k(x_1,x_2))^t, (x_1,x_2) \in \mathrm{IR}\},$$

where the superscript t stands for transposition.

We are interested in finding the best of the bilevel feasible solutions according to all objective functions. However, as they usually conflict, there is no well-defined optimal solution. The symbol 'min' will be understood as finding mathematically equally good solutions, called efficient solutions, according to the following definition:

Definition 1. A bilevel feasible solution $(\tilde{x}_1,\tilde{x}_2) \in \mathrm{IR}$ is an efficient solution of problem (2) if there is no $(x_1,x_2) \in \mathrm{IR}$ such that $d_{i1}x_1 + d_{i2}x_2 \leqslant d_{i1}\tilde{x}_1 + d_{i2}\tilde{x}_2$ for $i = 1,\ldots,k$ and $d_{j1}x_1 + d_{j2}x_2 < d_{j1}\tilde{x}_1 + d_{j2}\tilde{x}_2$ for some $j \in \{1,\ldots,k\}$.

A bilevel feasible solution $(\tilde{x}_1,\tilde{x}_2) \in \mathrm{IR}$ is a weakly efficient solution of problem (2) if there is no $(x_1,x_2) \in \mathrm{IR}$ such that $d_{i1}x_1 + d_{i2}x_2 < d_{i1}\tilde{x}_1 + d_{i2}\tilde{x}_2$ for $i = 1,\ldots,k$.

If $(\tilde{x}_1,\tilde{x}_2), (\hat{x}_1,\hat{x}_2) \in \mathrm{IR}$ and $d_{i1}\tilde{x}_1 + d_{i2}\tilde{x}_2 \leqslant d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2$ for $i = 1,\ldots,k$ and $d_{j1}\tilde{x}_1 + d_{j2}\tilde{x}_2 < d_{j1}\hat{x}_1 + d_{j2}\hat{x}_2$ for some $j \in \{1,\ldots,k\}$ we say that $(\tilde{x}_1,\tilde{x}_2)$ dominates $(\hat{x}_1,\hat{x}_2)$.

If $(\tilde{x}_1,\tilde{x}_2)$ is an efficient solution, $\tilde{y} = (d_1(\tilde{x}_1,\tilde{x}_2),\ldots,d_k(\tilde{x}_1,\tilde{x}_2))^t$ is a non-dominated point. Similarly, if $(\tilde{x}_1,\tilde{x}_2)$ is a weakly efficient solution, $\tilde{y} = (d_1(\tilde{x}_1,\tilde{x}_2),\ldots,d_k(\tilde{x}_1,\tilde{x}_2))^t$ is a weakly non-dominated point.

In other words, a bilevel feasible solution is efficient if it is not possible to move feasibly from it to decrease one objective function without increasing at least one of the others. Obviously, a bilevel feasible solution which is not efficient should not represent an alternative of interest to the leader.

The set of all efficient solutions is denoted by IR_E and is called the efficient set. The set of all non-dominated points is denoted by $\mathcal{Y}_N$ and is called the non-dominated set. The non-dominated points are located in the 'lower left part' of $\mathcal{Y}$. In fact, $\mathcal{Y}_N$ is a subset of the boundary of $\mathcal{Y}$ and $\mathcal{Y}_N = (\mathcal{Y} + \mathbb{R}^k_{\geqq})_N$, where $\mathbb{R}^k_{\geqq}$ denotes the non-negative orthant of $\mathbb{R}^k$, i.e. $\mathbb{R}^k_{\geqq} = \{y = (y_1,\ldots,y_k)^t \in \mathbb{R}^k : y \geqq 0\}$, where $y \geqq 0$ stands for $y_i \geqslant 0$, $i = 1,\ldots,k$. Similarly, IR_{wE} and $\mathcal{Y}_{wN}$ denote the sets of weakly efficient solutions and weakly non-dominated points, respectively.

In order to gain an insight into the meaning of previous definitions we consider the following examples in which variables x and y are controlled by the leader and variable z is controlled by the follower. Let S be the convex hull of points $A = (0,0,0)$; $B = (1,0,0)$; $C = (1,1,0)$; $D = (0,1,0)$; $E = (0,0,2)$; $F = (1,0,3)$; $G = (2,0,2)$; $H = (2,2,2)$; $I = (1,2,3)$; $J = (0,2,2)$.

Example 1.

$$\begin{aligned} \min_{x,y,z} \quad & (d_1(x,y,z) = x + 2y + 3z,\ d_2(x,y,z) = -x - y), \text{ where } z \text{ solves} \\ \min_{z} \quad & z \\ \text{s.t.} \quad & (x,y,z) \in S. \end{aligned}$$

Since the lower level minimizes z,

$$\mathrm{IR} = \mathrm{conv}\,(A,B,C,D) \cup \mathrm{conv}\,(B,C,G,H) \cup \mathrm{conv}\,(C,D,H,J).$$

where conv denotes convex hull. The left part of Fig. 1 displays the polyhedron S and the inducible region IR (in grey) of Example 1. Notice that IR is the union of faces of the polyhedron S and is non-convex. Figs. 2 and 3 show the images of S and IR under (d_1,d_2), respectively. Points $\widetilde{A}$ to $\widetilde{J}$ are, respectively, the images of points A to J. The image of IR, $\mathcal{Y}$, is non-convex. The set of non-dominated points $\mathcal{Y}_N$ is the union of segments $\widetilde{A} - \widetilde{B}$, $\widetilde{B} - \widetilde{C}$ and $\widetilde{C} - \widetilde{H}$. Notice that $\mathcal{Y}_N$ is connected.

Example 2.

$$\begin{aligned} \min_{x,y,z} \quad & (d_1(x,y,z) = x + 2y + 3z,\ d_2(x,y,z) = -x - y), \text{ where } z \text{ solves} \\ \min_{z} \quad & -z \\ \text{s.t.} \quad & (x,y,z) \in S. \end{aligned}$$

In this case, the lower level maximizes z, thus $\mathrm{IR} = \mathrm{conv}\,(F,G,H,I) \cup \mathrm{conv}\,(E,F,I,J)$. The right part of Fig. 1 displays the inducible region IR (in grey) of Example 2. Fig. 4 shows the image of IR under (d_1,d_2). Now the set of non-dominated points is the union of segment $\widetilde{E} - G'$, open at G', and segment $\widetilde{G} - \widetilde{H}$. Notice that in this example $\mathcal{Y}_N$ is non-connected.

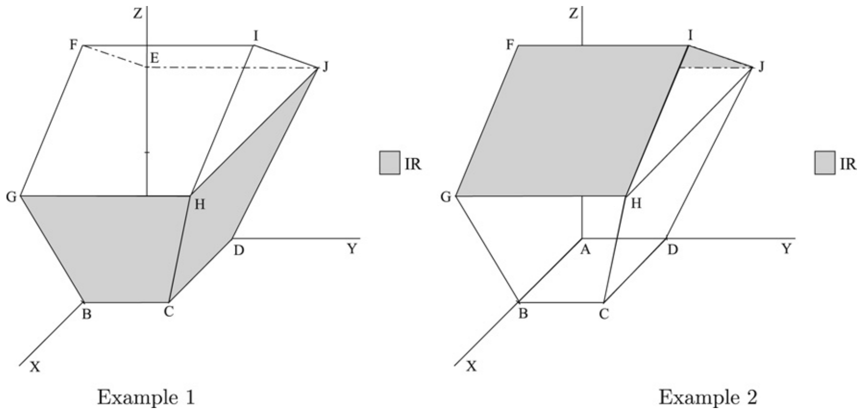

Fig. 1. Constraint region and inducible region of examples.

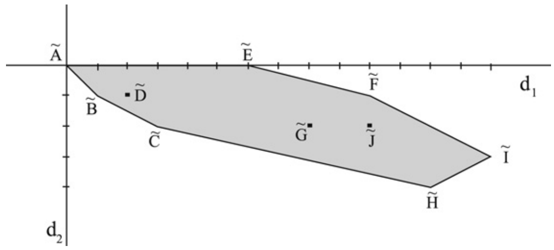

Fig. 2. Examples 1 and 2: Image of S under (d_1, d_2).

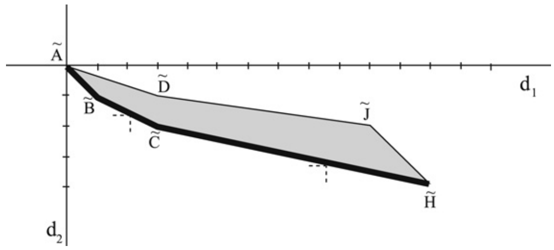

Fig. 3. Example 1: Image of IR under (d_1, d_2). The thick line refers to the non-dominated set.

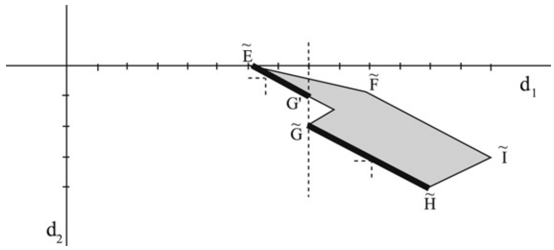

Fig. 4. Example 2: Image of IR under (d_1, d_2). The thick line refers to the non-dominated set.

Remark 2. Before analyzing the non-dominated set in the following Sections, we draw attention to the possible relationship between the non-dominated points of the LMOLBP problem (2) and the non-dominated points of the so-called relaxed problem (5), in which the lower level objective function has been removed:

$$\begin{aligned} \min_{x_1,x_2} \quad & (d_1(x_1,x_2),\ldots,d_k(x_1,x_2)) \\ & A_1^1x_1 + A_2^1x_2 \leqslant b^1 \\ & A_1^2x_1 + A_2^2x_2 \leqslant b^2 \\ & x_1 \geqslant 0, \qquad x_2 \geqslant 0. \end{aligned} \tag{5}$$

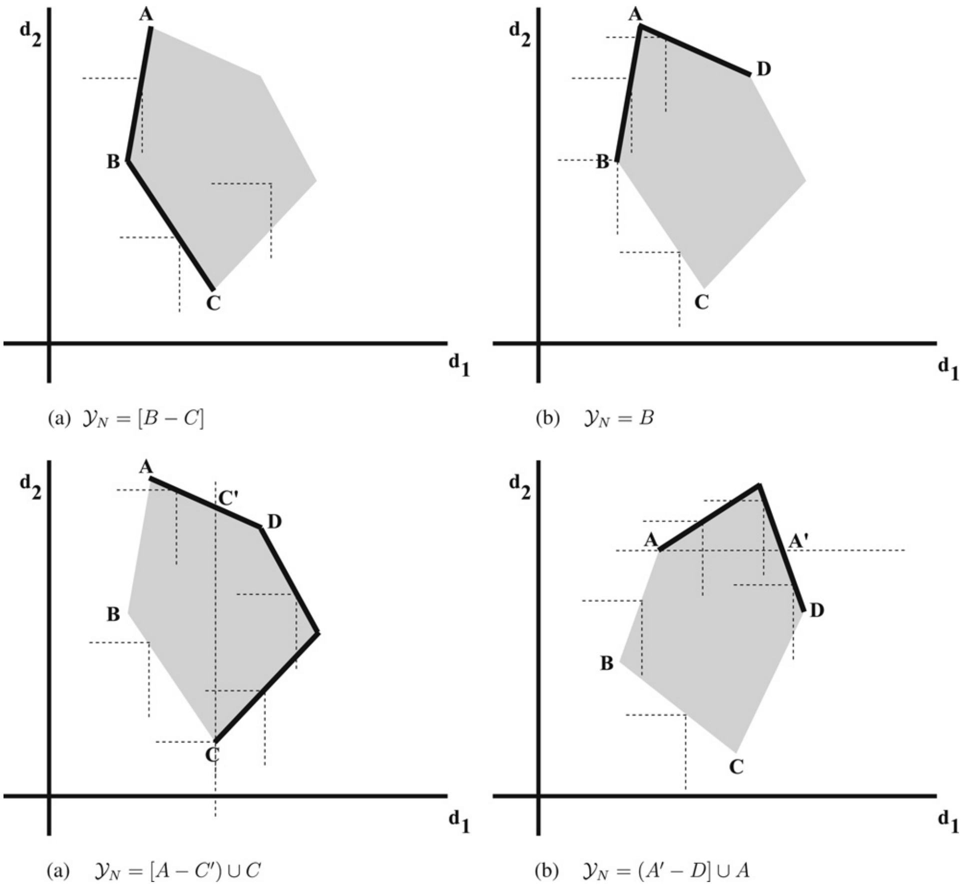

Fig. 5. Examples of non-dominated sets. The thick line represents the image of IR under the objective function mapping.

This is a linear multiobjective problem over the polyhedron T. The set of its efficient solutions is denoted by T_E.

By looking at Example 1, the relaxed problem is:

$$\begin{array}{ll} \min\limits_{x,y,z} & (d_1(x,y,z) = x + 2y + 3z,\ d_2(x,y,z) = -x - y) \\ \text{s.t.} & (x,y,z) \in S. \end{array}$$

The non-dominated set is the union of segments $\widetilde{A} - \widetilde{B}$, $\widetilde{B} - \widetilde{C}$ and $\widetilde{C} - \widetilde{H}$. Hence, for this example, non-dominated sets of problem (2) and problem (5) are equal. In contrast, if we consider Example 2, the single point in common is $\widetilde{H}$.

Let us consider four hypothetical LMOLBP problems with only two variables and two objectives. For every problem, Fig. 5 displays in grey the image of the constraint region under the objective function mapping (d_1, d_2). The thick black line is the image of IR under (d_1, d_2). In all cases, the non-dominated set of the corresponding relaxed problem is the segment $B - C$. However, if we take into consideration the multiobjective bilevel problem, things are very different. In example (a) the non-dominated set is also the segment $B-C$. In example (b), this set is comprised only of point B. The set of non-dominated points of example (c) is the segment $A - C'$, open at C', together with the point C. Finally, in example (d) this set is the segment $A' - D$, open at A', together with the point A. Hence, the non-dominated sets of both the multiobjective bilevel problem and the relaxed problem can be equal, as in the top left corner of Fig. 5, or have nothing in common, as in the bottom right corner of it.

Remark 3. Finally in this Section, it is worth mentioning that bilevel problems are very sensitive to the existence of upper level constraints involving upper level and lower level variables. As a matter of fact, IR could be non-connected and even be an empty set although T is a non-empty compact set [4,11].

If we slightly modify Example 2 to include the upper level constraint $z \leqslant 2$, then IR is the union of segments $G - H$ and $E - J$ and so it is non-connected. On the other hand, if we add the upper level constraints $z \leqslant 1$, then IR is empty. Hence, from now on, we assume that IR is non-empty.

Moreover, shifting the upper level constraints to the lower level optimization problem completely changes the problem [6,4,11]. For instance, if we consider Example 2 with upper level constraint $z \leqslant 2$, the non-dominated set is the union of segment $\widetilde{E} - G'$, open at G', and segment $\widetilde{G} - \widetilde{H}$. But, shifting this constraint to the lower level results in

$IR = \text{conv}\{E, G, H, J\}$. Hence the non-dominated set is the union of segments $\widetilde{E} - \widetilde{G}$ and $\widetilde{G} - \widetilde{H}$. Notice that this problem provides efficient points which are not even bilevel feasible solutions of the original problem.

3. Existence of non-dominated points

It is well known in multiobjective programming that even for convex feasible region and continuous functions, the non-dominated set might be empty. The properties of IR, although it is not necessarily convex, and the fact that upper level objective functions are linear allow us to prove that $\mathcal{Y}_N$ is non-empty.

Definition 4. A point-to-set mapping $\Gamma : \mathbb{R}^{n_1} \to 2^{\mathbb{R}^{n_2}}$ is called polyhedral if its graph grph $\Gamma = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : x_2 \in \Gamma(x_1)\}$ is equal to the union of a finite number of convex polyhedra.

Theorem 5. *The point-to-set mapping M defined by the set of optimal solutions of the lower level problem is polyhedral.*

Proof. Similar to Theorem 2.5 in Savard [11], taking into account that IR is non-empty and bounded. □

In other words, this Theorem asserts that IR is the union of a finite number of polyhedra. Hence, it is not necessarily convex. In fact,

$$IR = T \cap \{(x_1, x_2) : x_1 \in S_1, c_2 x_2 = v(x_1)\},$$

where S_1 is the projection of S onto $\mathbb{R}^{n_1}$ and $v(x_1)$ denotes the optimal value function of lower level problem $LP(x_1)$. Taking into account that S is compact, $v(x_1)$ is finite, $\forall x_1 \in S_1$. Moreover, since $LP(x_1)$ is a linear programming problem, $v(x_1)$ is piecewise linear.

Theorem 6. *$\mathcal{Y}_N$ is non-empty.*

Proof. As IR is a union of a finite number of non-empty bounded polyhedra, its image $\mathcal{Y}$ is also a union of non-empty bounded polyhedra. Hence, $\mathcal{Y}$ is non-empty and compact. Let $y^0 \in \mathcal{Y}$.

We consider the section of $\mathcal{Y}$ defined by y^0 as:

$$\mathcal{Y}^0 = (y^0 - \mathbb{R}^k_{\geqq}) \cap \mathcal{Y} = \{y \in \mathcal{Y} : y \leqq y^0\}.$$

Since $\mathcal{Y}$ is compact, so is $\mathcal{Y}^0$. As a result, $\mathcal{Y}$ has a compact section and the conclusion of the Theorem follows [12]. □

Corollary 7. *$\mathcal{Y}_{wN}$ is non-empty.*

Proof. It is a consequence of $\mathcal{Y}_N \subset \mathcal{Y}_{wN}$. □

Remark 8. Notice that k efficient points can be obtained by solving the following bilevel problems, for $i = 1, \ldots, k$:

$$\begin{aligned} \min_{x_1, x_2} \quad & d_i(x_1, x_2) \\ \text{s.t.} \quad & A_1^1 x_1 + A_2^1 x_2 \leqslant b^1 \\ & x_1 \geqslant 0 \end{aligned} \tag{6a}$$

where x_2 solves

$$\begin{aligned} \min_{x_2} \quad & c_2 x_2 \\ \text{s.t.} \quad & A_1^2 x_1 + A_2^2 x_2 \leqslant b^2 \\ & x_2 \geqslant 0. \end{aligned} \tag{6b}$$

Let (x_1^i, x_2^i) be an optimal solution of problems (6). If it is unique, then it is an efficient solution. Otherwise, from the set of optimal solutions to (6) it is always possible to select an efficient solution.

Let $y^i = d_i(x_1^i, x_2^i)$, $i = 1, \ldots, k$. In multiobjective programming, the point $y^I = (y^1, \ldots, y^k)$ is called the ideal point and provides a lower bound on non-dominated points.

Problems (6) are standard linear bilevel problems, that is to say, bilevel problems in which both objective functions are linear and the constraint region is a polyhedron. Linear bilevel problems have been much studied in the literature [3,4]. Although even this 'simple' version of bilevel problems is (strongly) NP-hard [13], an important property of these problems is that their solution set contains at least one extreme point of the constraint region. This important property allows us, amongst other things, to develop enumerative algorithms to solve them. Other approaches replace the lower level problem by its Karush–Kuhn–Tucker conditions, use penalty functions, use gradient methods, etc. [3,4]. However, most of these algorithms are far from being efficient in terms of computational time involved when solving large problems. Calvete et al. [14] propose a metaheuristic algorithm which combines classical enumeration techniques that search for extreme points with genetic algorithms, which has proved to provide near-optimal solutions in reasonable computational time.

In the following Sections, we investigate to what extent the efficient points of the LMOLBP problem can be found by solving standard linear bilevel problems.

4. Finding efficient points by weighted sum scalarization

When linear multiobjective problems with a single level of decision making are considered, all efficient solutions can be obtained by solving a linear programming problem whose objective function is a linear combination with positive coefficients of the objective functions. Concerning LMOLBP problems, below we prove that optimal solutions of scalarized problems with positive weights are always efficient. But, we will show with an example that there can be efficient solutions which cannot be obtained in this way.

For a fixed $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_i \in \mathbb{R}$, $i = 1, \ldots, k$, let us consider the linear bilevel problem:

$$\begin{aligned} \text{WSS-LMOLBP}(\lambda): \quad & \min_{x_1,x_2} \quad \sum_{i=1}^{k} \lambda_i (d_{i1}x_1 + d_{i2}x_2) \\ & \text{s.t.} \quad A_1^1 x_1 + A_2^1 x_2 \leqslant b^1 \\ & \qquad\ x_1 \geqslant 0 \end{aligned} \tag{7a}$$

where x_2 solves

$$\begin{aligned} \min_{x_2} \quad & c_2 x_2 \\ \text{s.t.} \quad & A_1^2 x_1 + A_2^2 x_2 \leqslant b^2 \\ & x_2 \geqslant 0. \end{aligned} \tag{7b}$$

This problem is called a weighted sum scalarization of the LMOLBP problem. By looking at the objective space, problem (7) can be formulated as:

$$\begin{aligned} \min_{y} \quad & \sum_{i=1}^{k} \lambda_i y_i \\ \text{s.t.} \quad & y \in \mathcal{Y}. \end{aligned} \tag{8}$$

Theorem 9. *Let $(x_1^\star, x_2^\star)$ be an optimal solution of the WSS-LMOLBP problem (7) for a given $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_i \geqslant 0$, $i = 1, \ldots, k$, $\sum_{i=1}^{k} \lambda_i = 1$. Then $(x_1^\star, x_2^\star) \in \mathrm{IR}_{wE}$.*

Proof. By hypothesis, $(x_1^\star, x_2^\star) \in \mathrm{IR}$ and

$$\sum_{i=1}^{k} \lambda_i (d_{i1}x_1^\star + d_{i2}x_2^\star) \leqslant \sum_{i=1}^{k} \lambda_i (d_{i1}x_1 + d_{i2}x_2), \quad \forall\, (x_1, x_2) \in \mathrm{IR}.$$

Suppose $(x_1^\star, x_2^\star) \notin \mathrm{IR}_{wE}$. Hence, there must be $(\hat{x}_1, \hat{x}_2) \in \mathrm{IR}$ such that $d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2 < d_{i1}x_1^\star + d_{i2}x_2^\star$, $i = 1, \ldots, k$. Multiplying componentwise by non-negative weights λ_i and taking into account that at least one of them is different from zero, we get

$$\sum_{i=1}^{k} \lambda_i (d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2) < \sum_{i=1}^{k} \lambda_i (d_{i1}x_1^\star + d_{i2}x_2^\star),$$

which contradicts the optimality of $(x_1^\star, x_2^\star)$. $\square$

A similar Theorem can be proved for efficient solutions if all λ_i are assumed to be positive.

Theorem 10. *Suppose that $(x_1^\star, x_2^\star)$ is an optimal solution of the WSS-LMOLBP problem (7) for a given $\lambda = (\lambda_1, \ldots, \lambda_k)$, $\lambda_i > 0$, $i = 1, \ldots, k$, $\sum_{i=1}^{k} \lambda_i = 1$. Then $(x_1^\star, x_2^\star) \in \mathrm{IR}_E$.*

Proof. By hypothesis, $(x_1^\star, x_2^\star) \in \mathrm{IR}$ and

$$\sum_{i=1}^{k} \lambda_i (d_{i1}x_1^\star + d_{i2}x_2^\star) \leqslant \sum_{i=1}^{k} \lambda_i (d_{i1}x_1 + d_{i2}x_2), \quad \forall\, (x_1, x_2) \in \mathrm{IR}.$$

Suppose $(x_1^\star, x_2^\star) \notin \mathrm{IR}_E$. Hence, there must be $(\hat{x}_1, \hat{x}_2) \in \mathrm{IR}$ which dominates $(x_1^\star, x_2^\star)$, i.e., $d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2 \leqslant d_{i1}x_1^\star + d_{i2}x_2^\star$, $i = 1, \ldots, k$, and $d_{j1}\hat{x}_1 + d_{j2}\hat{x}_2 < d_{j1}x_1^\star + d_{j2}x_2^\star$ for some $j \in \{1, \ldots, k\}$. Multiplying componentwise by weights λ_i and taking into account that $\lambda_i > 0$, $\forall i = 1, \ldots, k$, we get

$$\sum_{i=1}^{k} \lambda_i (d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2) < \sum_{i=1}^{k} \lambda_i (d_{i1}x_1^\star + d_{i2}x_2^\star),$$

which contradicts the optimality of $(x_1^\star, x_2^\star)$. $\square$

Remark 11. The WSS-LMOLBP problem associated with Example 2 is:

$$\begin{array}{lll} \text{WSS-LMOLBP}(\lambda): & \min & \lambda_1(x+2y+3z)+\lambda_2(-x-y), \text{ where } z \text{ solves} \\ & \min & -z \\ & \text{s.t.} & (x,y,z)\in S. \end{array}$$

For instance, taking $\lambda = (0.34, 0.66)$ we get that G is efficient. This example also illustrates that there are efficient solutions that cannot be obtained by solving problem (7). Taking into account that this problem is equivalent to minimizing a linear function over a union of polyhedra, it is not possible to get any of the efficient solutions which are points of the segment $E - J$ (except point E) by solving weighted sum scalarization problems. Fig. 4 shows that no non-dominated point in the segment $\widetilde{E} - G'$, open at G', (except point $\widetilde{E}$) is an optimal solution of the corresponding problem (8) for some (λ_1, λ_2).

Theorem 12. *Let F be a face of T such that $F \subset \mathrm{IR}$. Let $(\tilde{x}_1, \tilde{x}_2) \in ri\,F$, where ri F denotes the relative interior of F. If $(\tilde{x}_1, \tilde{x}_2)$ is an efficient solution of the relaxed problem (5), then $F \subset \mathrm{IR}_E$.*

Proof. Taking into account that problem (5) is a linear multiobjective program, the fact that $(\tilde{x}_1, \tilde{x}_2) \in T_E$ implies that $F \subset T_E$. Moreover, since $\mathrm{IR} \subset T$, then $\mathrm{IR} \cap T_E \subset \mathrm{IR}_E$. Hence the assertion of the Theorem follows. □

Now consider the linear multiobjective problem (9) in which the upper level linear multiobjective function is minimized over the convex hull of the inducible region:

$$\begin{array}{lll} \text{CHLMP}: & \displaystyle\min_{x_1,x_2} & (d_1(x_1,x_2),\ldots,d_k(x_1,x_2)) \\ & \text{s.t.} & (x_1,x_2)\in \operatorname{conv}(\mathrm{IR}). \end{array} \tag{9}$$

Let $\operatorname{conv}(\mathrm{IR})_E$ be its efficient set.

Theorem 13. *Suppose that $(\tilde{x}_1, \tilde{x}_2)$ is an extreme point of* $\operatorname{conv}(\mathrm{IR})$ *such that $(\tilde{x}_1, \tilde{x}_2) \in \operatorname{conv}(\mathrm{IR})_E$. Then $(\tilde{x}_1, \tilde{x}_2) \in \mathrm{IR}_E$.*

Proof. As $(\tilde{x}_1, \tilde{x}_2)$ is an extreme point of $\operatorname{conv}(\mathrm{IR})$, $(\tilde{x}_1, \tilde{x}_2) \in \mathrm{IR}$.

Suppose $(\tilde{x}_1, \tilde{x}_2) \notin \mathrm{IR}_E$. Hence, there must be $(\hat{x}_1, \hat{x}_2) \in \mathrm{IR}$ which dominates $(\tilde{x}_1, \tilde{x}_2)$, i.e., $d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2 \le d_{i1}\tilde{x}_1 + d_{i2}\tilde{x}_2$, $i = 1, \ldots, k$, and $d_{j1}\hat{x}_1 + d_{j2}\hat{x}_2 < d_{j1}\tilde{x}_1 + d_{j2}\tilde{x}_2$ for some $j \in \{1, \ldots, k\}$.

Taking into account that $\mathrm{IR} \subset \operatorname{conv}(\mathrm{IR})$, $(\hat{x}_1, \hat{x}_2) \in \operatorname{conv}(\mathrm{IR})$. Therefore, $(\tilde{x}_1, \tilde{x}_2) \notin \operatorname{conv}(\mathrm{IR})_E$. Contradiction. □

Theorem 14. *Let F be a face of* $\operatorname{conv}(\mathrm{IR})$ *and $(\tilde{x}_1, \tilde{x}_2) \in ri\,F$. If $(\tilde{x}_1, \tilde{x}_2) \in \operatorname{conv}(\mathrm{IR})_E \cap \mathrm{IR}$ then $F \subset \mathrm{IR}_E$.*

Proof. Since $(\tilde{x}_1, \tilde{x}_2) \in \mathrm{IR}$, then $F \subset \mathrm{IR}$. Moreover, taking into account that problem CHLMP is a linear multiobjective program, as a result of $(\tilde{x}_1, \tilde{x}_2) \in \operatorname{conv}(\mathrm{IR})_E$, $F \subset \operatorname{conv}(\mathrm{IR})_E$. Hence, $F \subset \mathrm{IR} \cap \operatorname{conv}(\mathrm{IR})_E \subset \mathrm{IR}_E$. □

5. Finding efficient points by scalarization techniques

In multiobjective programming with a single level of decision making, several other methods have been proposed to cope with the problem of getting efficient solutions or checking efficiency based on transforming objective functions into constraints. They are specially addressed to multiobjective functions in which neither $\mathcal{Y}$ nor $\mathcal{Y} + \mathbb{R}^k_{\geqq}$ are convex sets. In this Section we give an idea of two of these procedures, paying special attention to the fact that, when applied to LMOLBP problems, they result in solving standard linear bilevel problems.

5.1. The ϵ-constraint method

This consists in optimizing one of the original objectives, setting the remaining objectives as constraints [15]. Bearing in mind the LMOLBP problem, the associated jth objective ϵ-constraint problem is the linear bilevel problem:

$$\begin{array}{lll} P_j(\epsilon): & \displaystyle\min_{x_1,x_2} & d_{j1}x_1 + d_{j2}x_2 \\ & \text{s.t.} & d_{i1}x_1 + d_{i2}x_2 \leqslant \epsilon_i, \quad i = 1,\ldots,k,\ i \neq j \\ & & A_1^1 x_1 + A_2^1 x_2 \leqslant b^1 \\ & & x_1 \geqslant 0 \end{array} \tag{10a}$$

where x_2 solves

$$\begin{array}{ll} \displaystyle\min_{x_2} & c_2 x_2 \\ \text{s.t.} & A_1^2 x_1 + A_2^2 x_2 \leqslant b^2 \\ & x_2 \geqslant 0, \end{array} \tag{10b}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_{j-1}, \epsilon_{j+1}, \ldots, \epsilon_k)^t \in \mathbb{R}^{k-1}$. Notice that every feasible solution (x_1, x_2) of problem (10) is a point of IR.

Theorem 15. *Let $(x_1^\star, x_2^\star)$ be an optimal solution of problem $P_j(\epsilon)$ for some j. Then $(x_1^\star, x_2^\star) \in \mathrm{IR}_{wE}$.*

Proof. By hypothesis, $(x_1^\star, x_2^\star) \in \mathrm{IR}$ and $d_{i1}x_1^\star + d_{i2}x_2^\star \leqslant \epsilon_i$, $i = 1, \dots, k$, $i \neq j$.

Suppose $(x_1^\star, x_2^\star) \notin \mathrm{IR}_{wE}$. Hence, there must be $(\hat{x}_1, \hat{x}_2) \in \mathrm{IR}$ such that $d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2 < d_{i1}x_1^\star + d_{i2}x_2^\star \leqslant \epsilon_i$, $i = 1, \dots, k$. Therefore, $(\hat{x}_1, \hat{x}_2)$ is a feasible solution of problem (10). Moreover, $d_{j1}\hat{x}_1 + d_{j2}\hat{x}_2 < d_{j1}x_1^\star + d_{j2}x_2^\star$, which contradicts the optimality of $(x_1^\star, x_2^\star)$. □

Theorem 16. *If $(x_1^\star, x_2^\star)$ is the unique optimal solution of problem $P_j(\epsilon)$ for some j, then $(x_1^\star, x_2^\star) \in \mathrm{IR}_E$.*

Proof. If $(x_1^\star, x_2^\star) \notin \mathrm{IR}_E$, then there must be $(\hat{x}_1, \hat{x}_2) \in \mathrm{IR}$ which dominates $(x_1^\star, x_2^\star)$, i.e., $d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2 \leqslant d_{i1}x_1^\star + d_{i2}x_2^\star$, $i = 1, \dots, k$, and $d_{h1}\hat{x}_1 + d_{h2}\hat{x}_2 < d_{h1}x_1^\star + d_{h2}x_2^\star$ for some $h \in \{1, \dots, k\}$. Hence, $(\hat{x}_1, \hat{x}_2)$ is a feasible solution of problem (10). Besides, $d_{j1}\hat{x}_1 + d_{j2}\hat{x}_2 \leqslant d_{j1}x_1^\star + d_{j2}x_2^\star$. Therefore, $d_{j1}\hat{x}_1 + d_{j2}\hat{x}_2 = d_{j1}x_1^\star + d_{j2}x_2^\star$ and $(\hat{x}_1, \hat{x}_2)$ is also an optimal solution of (10), which contradicts the uniqueness of $(x_1^\star, x_2^\star)$. □

Theorem 17. *A bilevel feasible solution $(x_1^\star, x_2^\star)$ is efficient if and only if there exists $\epsilon^\star \in \mathbb{R}^k$ such that $(x_1^\star, x_2^\star)$ solves problem $P_j(\epsilon^\star)$ for every $j = 1, \dots, k$.*

Proof. Necessity: By taking $\epsilon^\star = (d_{11}x_1^\star + d_{12}x_2^\star, \dots, d_{k1}x_1^\star + d_{k2}x_2^\star)^t$, $(x_1^\star, x_2^\star)$ is a feasible solution of every problem $P_j(\epsilon^\star)$, $j = 1, \dots, k$. Suppose that there is some $h \in \{1, \dots, k\}$ such that $(x_1^\star, x_2^\star)$ is not an optimal solution of $P_h(\epsilon^\star)$. Then, there exists $(\hat{x}_1, \hat{x}_2)$ feasible solution of problem $P_h(\epsilon^\star)$ such that $d_{h1}\hat{x}_1 + d_{h2}\hat{x}_2 < d_{h1}x_1^\star + d_{h2}x_2^\star$ and $d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2 \leqslant \epsilon_i = d_{i1}x_1^\star + d_{i2}x_2^\star$, $i = 1, \dots, k$, $i \neq h$. Therefore, $(\hat{x}_1, \hat{x}_2)$ dominates $(x_1^\star, x_2^\star)$. Contradiction.

Sufficiency: By hypothesis, $(x_1^\star, x_2^\star) \in \mathrm{IR}$. If $(x_1^\star, x_2^\star) \notin \mathrm{IR}_E$, there exists $(\hat{x}_1, \hat{x}_2) \in \mathrm{IR}$ which dominates it, i.e., $d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2 \leqslant d_{i1}x_1^\star + d_{i2}x_2^\star$, $i = 1, \dots, k$, and $d_{h1}\hat{x}_1 + d_{h2}\hat{x}_2 < d_{h1}x_1^\star + d_{h2}x_2^\star$ for some $h \in \{1, \dots, k\}$. Hence, $(\hat{x}_1, \hat{x}_2)$ is a feasible solution of problem $P_h(\epsilon^\star)$ and provides a better value of the objective function than $(x_1^\star, x_2^\star)$. Contradiction. □

5.2. Benson's method

This method allows us to check the efficiency of a feasible solution or, in case of a negative answer, to construct an efficient point [16]. Bearing in mind the LMOLBP problem, for a given $(\tilde{x}_1, \tilde{x}_2) \in \mathrm{IR}$, it consists in solving the linear bilevel problem:

$$\begin{array}{lll} \max\limits_{x_1,x_2,z} & \sum\limits_{i=1}^{k} z_i & \\ \text{s.t.} & d_{i1}x_1 + d_{i2}x_2 + z_i = d_{i1}\tilde{x}_1 + d_{i2}\tilde{x}_2, \quad i = 1, \dots, k & \\ & A_1^1 x_1 + A_2^1 x_2 \leqslant b^1 & \\ & x_1 \geqslant 0, \quad z \geqslant 0 & \end{array} \tag{11a}$$

where x_2 solves

$$\begin{array}{ll} \min\limits_{x_2} & c_2 x_2 \\ \text{s.t.} & A_1^2 x_1 + A_2^2 x_2 \leqslant b^2 \\ & x_2 \geqslant 0, \end{array} \tag{11b}$$

where $z = (z_1, \dots, z_k)^t \in \mathbb{R}^k$. Note that for every feasible solution (x_1, x_2, z) of problem (11), (x_1, x_2) is a point of IR.

Theorem 18. *The bilevel feasible solution $(\tilde{x}_1, \tilde{x}_2)$ is efficient if and only if the optimal objective value of problem (11) is equal to zero.*

Proof. Necessity: If $(\tilde{x}_1, \tilde{x}_2) \in \mathrm{IR}_E$ then there is no bilevel feasible solution which dominates it. Hence, the feasible region of problem (11) only includes points (x_1, x_2, z) such that $(x_1, x_2) \in \mathrm{IR}$ and $z = 0$, and so the optimal objective value is zero.

Sufficiency: Let $(x_1, x_2) \in \mathrm{IR}$ such that $d_{i1}x_1 + d_{i2}x_2 \leqslant d_{i1}\tilde{x}_1 + d_{i2}\tilde{x}_2$, $i = 1, \dots, k$. As the optimal value of problem (11) is zero, $d_{i1}x_1 + d_{i2}x_2 = d_{i1}\tilde{x}_1 + d_{i2}\tilde{x}_2$, $i = 1, \dots, k$. Hence there is no a bilevel feasible solution which dominates $(\tilde{x}_1, \tilde{x}_2) \in \mathrm{IR}$, i.e. $(\tilde{x}_1, \tilde{x}_2) \in \mathrm{IR}_E$. □

Theorem 19. *Let $(x_1^\star, x_2^\star, z^\star)$ be an optimal solution of problem (11). Then $(x_1^\star, x_2^\star) \in \mathrm{IR}_E$.*

Proof. By hypothesis, $(x_1^\star, x_2^\star) \in \mathrm{IR}$. If $(x_1^\star, x_2^\star) \notin \mathrm{IR}_E$ there exists $(\hat{x}_1, \hat{x}_2) \in \mathrm{IR}$ which dominates it, i.e., $d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2 \leqslant d_{i1}x_1^\star + d_{i2}x_2^\star$, $i = 1, \dots, k$, and $d_{j1}\hat{x}_1 + d_{j2}\hat{x}_2 < d_{j1}x_1^\star + d_{j2}x_2^\star$ for some $j \in \{1, \dots, k\}$.

Let $\hat{z}_i = d_{i1}x_1^\star + d_{i2}x_2^\star - d_{i1}\hat{x}_1 + d_{i2}\hat{x}_2 \geqslant 0$, $i = 1, \dots, k$. Then $(\hat{x}_1, \hat{x}_2, \hat{z})$ is a feasible solution of problem (11). Taking into account the selection of $\hat{z}_i$, $\sum_{i=1}^{k} \hat{z}_i > \sum_{i=1}^{k} z_i^\star$, which contradicts the optimality of $(x_1^\star, x_2^\star, z^\star)$. □

6. Conclusions

In this paper we have analyzed general bilevel problems with multiple objectives at the upper level, when all objective functions are linear and constraints at both levels define polyhedra. This problem can be reformulated as a multiobjective problem with linear objective functions over a feasible region which is implicitly defined by a linear optimization problem and, in general, is non-convex. Assuming that the inducible region is non-empty and taking into account its compactness, we have proved that the non-dominated set is non-empty. In order to obtain efficient solutions we use weighted sum scalarization methods and scalarization methods, together with the fact that the feasible region is a union of polyhedra. One of the main characteristics of these methods is that they result in solving linear bilevel problems, that is to say bilevel problems in which both objective functions are linear and the constraint region is a polyhedron. Furthermore, several examples are displayed. They enable us to show the relationship of the efficient set of the LMOLBP problem with the efficient set of the linear multiobjective problem which results in removing the lower level objective function. The examples also show that it is not possible to obtain all the efficient points based on weighted sum scalarization methods.

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