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A Matroid Generalization of a Result on Row-Latin Rectangles

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row or column. We generalize this to matrices with entries in the ground set of a matroid. For such a matrix A, we show that if each row of A forms an independent set, then we can require the transversal to be independent as well. We determine the complexity of an algorithm based on the proof of this result. Finally, we observe that $m \geqslant 2n-1$ appears to force the existence of not merely one but many transversals. We discuss a number of conjectures related to this observation (some of which involve matroids and some of which do not). © 1999 Academic Press

Key Words: row-Latin rectangle; matroid; transversal.

1. INTRODUCTION

We define a *partial transversal* of length k in a matrix A to be a set of k distinct entries of A, no two in the same row or column. A *transversal* is a partial transversal that meets every column. A *Latin square* of order n is an $n \times n$ matrix in which each of the rows and columns is a permutation of $\{1, 2, ..., n\}$. The existence of partial transversals in Latin squares has been discussed in a number of works [1, 2, 5, 6, 8]; see [2] for a survey.

Stein [7] and Erdős *et al.* [5] investigated partial transversals in generalizations of Latin squares. One such generalization is a *row-Latin rectangle*: an $m \times n$ matrix in which each row is a permutation of $\{1, 2, ..., n\}$. Dillon [3] asked for the minimum m so that every $m \times n$ row-Latin rectangle has a transversal. Drisko [4] answered Dillon's question by showing that $m = 2n-1$ suffices. In fact, he proved this for a more general class of matrices. A *row-Latin rectangle based on k* is an $m \times n$ matrix with entries in $\{1, 2, ..., k\}$ so that no entry appears twice in any row.

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$$R_{2,2} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}; \qquad R_{4,3} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}; \qquad R_{6,4} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 2 & 3 & 4 & 1 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

FIG. 1. The matrices $R_{2,2}$, $R_{4,3}$, and $R_{6,4}$ of Example 1.2. None of these matrices has a transversal.

THEOREM 1.1 (Drisko [4, Theorem 1]). *Let A be an $m \times n$ row-Latin rectangle based on k. If $m \geqslant 2n-1$, then A has a transversal.*

The following well known example, based on [4, Example 1], shows that the bound on m in Theorem 1.1 is sharp.

EXAMPLE 1.2. Let m, n be positive integers with $m \geqslant n-1$. We define $R_{m,n}$ to be an $m \times n$ matrix whose first $m-(n-1)$ rows consist of the symbols $1, 2, ..., n$ in order and whose remaining $n-1$ rows have the same symbols in the order $2, 3, ..., n, 1$. Figure 1 shows $R_{2,2}$, $R_{4,3}$, and $R_{6,4}$.

The matrix $R_{2n-2,n}$ is a row-Latin rectangle with no transversals. To see this, assume that $R_{2n-2,n}$ has a transversal. Without loss of generality, we may assume that this transversal has a 1 in the first column. Then it cannot have a 1 in column n, so it must have an n in column n. Similarly, it must have an $n-1$ in column $n-1$, an $n-2$ in column $n-2$, etc., and a 3 in column 3. But this leaves no possible value in column 2, since we have already used each of the first $n-1$ rows; thus, there is no transversal.

Our main result is a matroid generalization of Theorem 1.1. Let A be an $m \times n$ matrix with entries in the ground set of a matroid M. We define an *independent partial transversal* (IPT) of length k in A to be a partial transversal of length k whose elements form an independent set in M. An *independent transversal* (IT) is an IPT that meets every column. Suppose that the entries of each row of A are all distinct and form an independent set. We show that if $m \geqslant 2n-1$, then A has an IT. Theorem 1.1 follows by letting M be a free matroid.

In Section 2 we prove our main result. Our proof can be written as an algorithm to find an IT; in Section 3 we determine the complexity of this algorithm. In Section 4 we observe that $m \geqslant 2n-1$ appears to force the existence of not merely one but many transversals. We discuss a number of conjectures and examples stemming from this observation.

2. THE MAIN RESULT

THEOREM 2.1. *Let A be an $m \times n$ matrix with entries in the ground set of a matroid M. Suppose that the set of entries of each row of A forms an independent set of size n in M. If $m \geqslant 2n-1$, then A has an IT.*

Proof. Our proof is based on a simplification of Drisko's proof of Theorem 1.1 [4, Theorem 1].

Let m, n, A, and M be as in the statement of the theorem. For a subset S of the ground set of M, $r(S)$ denotes the rank of S, and $\sigma(S)$ denotes the span of S. Given a set S of entries of A and an entry a, we write $S+a$ for $S \cup \{a\}$ and $S-a$ for $S-\{a\}$.

We may assume that the entries of A are all distinct. If not, say an element e of the ground set of M occurs several times in A. We can add new elements to the ground set of M, placing them in the same parallel class as e, and replacing each occurrence of e in A by a different element of this parallel class. Now those entries of A formerly equal to e are all distinct, while the positions occupied by IPTs and ITs in A are unchanged.

It suffices to prove the result when $m = 2n-1$. The $n=1$ case is trivial; we proceed by induction on n. We will prove the $n=2$ case as we set up the induction.

We first name a number of entries of $A = (a_{i,j})$. Let b_1 and b_2 denote $a_{1,1}$ and $a_{2,1}$, respectively. For $1 \leqslant i \leqslant n-1$, c_i denotes $a_{i+2,i+1}$. For $1 \leqslant i \leqslant n-3$, d_i denotes $a_{n+i+1,n-i+1}$. Note that if $n = 2, 3$, then we do not define any d_i's. See Fig. 2.

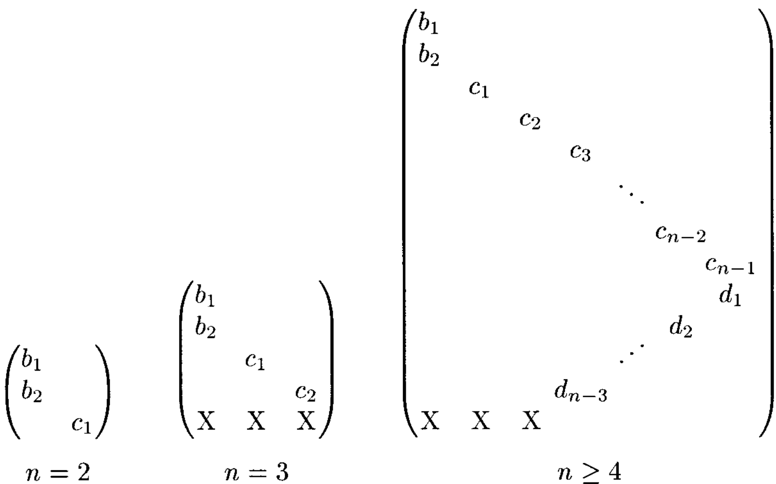

FIG. 2. Various named entries for the proof of Theorem 2.1; X's are in positions $(2n-1, 1)$, $(2n-1, 2)$, and $(2, -1, 3)$.

If, for each column of A, the entries of the column are all parallel, then the main diagonal of A is an IT. Thus, we may assume that some column of A contains two nonparallel entries. Since we can permute the rows and columns of A to place these two entries in positions (1, 1) and (2, 1), we may assume that the two nonparallel entries are b_1 and b_2.

Throughout this proof, we will make assumptions similar to that above based on the fact that we could permute rows and columns to put certain values in the required positions. Some of these permutations may move the entries of previously defined sets; however, we will always ensure that the set of positions occupied by the elements of each such set does not change. For example, if $S=\{c_1, c_2\}$, then the permutation that transposes rows 3 and 4 and transposes columns 2 and 3 moves elements of S. However, the *set* of positions occupied by the elements of S does not change.

Now, $\{b_1, b_2\}$ is an independent set of size 2, and $\{c_1\}$ is an independent set of size 1. We may augment $\{c_1\}$ from $\{b_1, b_2\}$ to produce an IPT of length 2; this is an IT if $n=2$. Thus, we may assume $n \geqslant 3$.

If we delete the first two rows and the first column of A, we obtain a $(2n-3)\times(n-1)$ matrix with entries in the ground set of M in which the elements of each row form an independent set. By the induction hypothesis, this matrix has an IT P_1, which is an IPT of length $n-1$ in A. The set P_1 meets neither the first two rows nor the first column of A. Permuting rows and columns, we may assume that $P_1=\{c_1, c_2, ..., c_{n-1}\}$.

For each set S of entries of A with $c_1, c_2 \in S$ and $b_1, b_2 \notin S$, we define $S' := S - c_1 + b_1$ and $S'' := S - c_2 + b_2$.

If $P_1 + b_1$ is independent in M, then $P_1 + b_1$ is an IT, and we are done. A similar argument applies to $P_1 + b_2$, and so we may assume that $P_1 + b_1$ and $P_1 + b_2$ are dependent. Thus, there is a unique circuit C_1 with $b_1 \in C_1 \subseteq P_1 + b_1$. Similarly, there is a unique circuit C_2 with $b_2 \in C_2 \subseteq P_1 + b_2$. Consider $(C_1 \cup C_2) \cap P_1$. If this set contains only one element, then, by circuit elimination, $(C_1 \cup C_2) - P_1$ contains a circuit. However, $(C_1 \cup C_2) - P_1 = \{b_1, b_2\}$ is independent. Thus, there exist $c_i, c_j \in P_1$ with $i \neq j$, $c_i \in C_1$, and $c_j \in C_2$. Permuting rows and columns in such a way as not to change the set of positions occupied by P_1, we may assume that $i=1$ and $j=2$. Now, $P_1 + b_1$ is a dependent set containing a unique circuit C_1, which contains c_1. Thus, $P_1 + b_1 - c_1 = P_1'$ is an IPT. Similarly, $P_1 + b_2 - c_2 = P_1''$ is an IPT.

Since $P_1 + b_1$ is a dependent set of rank $n-1$ containing two independent sets P_1 and P_1', both of size $n-1$, we must have $\sigma(P_1) = \sigma(P_1')$. Similarly, $\sigma(P_1) = \sigma(P_1'')$.

Now we have constructed P_1 and determined some of its properties. Based on the assumption that A does not have an IT, we show that there is a permutation of the rows and columns of A for which the following claim holds. We will then use the claim to verify that A does have an IT, thus provides the theorem.

CLAIM. *For $1 \leqslant k \leqslant n-2$, there exists a set P_k of entries of A such that*

(1) $\{c_1, ..., c_{n-k}\} \subseteq P_k \subseteq \{c_1, ..., c_{n-1}, d_1, ..., d_{k-1}\}$ (*where* $\{d_1, ..., d_{k-1}\} = \varnothing$ *if* $k=1$),

(2) P_k, P'_k, *and* P''_k *are IPTs of length* $n-1$ *in* A, *and*

(3) $\sigma(P_k) = \sigma(P'_k) = \sigma(P''_k)$.

Furthermore,

(4) $r[\bigcap_{i=1}^{k} \sigma(P_i)] = n-k$.

Proof of Claim. We have already defined P_1. By the earlier discussion, the claim holds for $k=1$. If $n=3$, then the claim is proven; we may assume $n \geqslant 4$. We proceed by induction on k. Let $1 \leqslant k \leqslant n-3$, and suppose the claim holds for 1, ..., k. We wish to define P_{k+1} so that the claim holds for $k+1$.

Since $r[\bigcap_{i=1}^{k} \sigma(P_i)] = n-k$, there is an entry x of row $n+k+1$ that lies in the first $n-k+1$ columns and is not in $\bigcap_{i=1}^{k} \sigma(P_i)$. There must exist t, $1 \leqslant t \leqslant k$, with $x \notin \sigma(P_t)$. Now, P_t, P'_t, and P''_t have the same span and miss columns 1, 2, and 3, respectively. Thus, if x lies in one of the first 3 columns, then one of $P_t + x$, $P'_t + x$, or $P''_t + x$ is an IT, and so we may assume that x does not lie in the first three columns. Permuting rows and columns, we may assume that $x = d_k$; note that we can choose the permutations in such a way that the sets of positions occupied by P_i, P'_i, and P''_i remain unchanged, for $1 \leqslant i \leqslant k$.

Let $P_{k+1} = P_t - c_{n-k} + d_k$. Then P_{k+1}, P'_{k+1}, P''_{k+1} are IPTs, since $d_k \notin \sigma(P_t) = \sigma(P'_t) = \sigma(P''_t)$. We see that Statements (1) and (2) in the claim hold for $k+1$. It remains to verify Statements (3) and (4).

We may assume that $P_{k+1} + b_1$ is dependent; otherwise, it is an IT. So $P_{k+1} + b_1$ is a dependent set of rank $n-1$ containing two independent sets of size $n-1$: P_{k+1} and P'_{k+1}. We conclude that $\sigma(P_{k+1}) = \sigma(P'_{k+1})$. Similarly, $\sigma(P_{k+1}) = \sigma(P''_{k+1})$, and so Statement (3) holds.

To see Statement (4), none that $\{c_1, c_2, ..., c_{n-k}\}$ forms an independent set of rank $n-k$. Hence,

$$\begin{aligned}
\bigcap_{i=1}^{k+1} \sigma(P_i) &= \left[\bigcap_{i=1}^{k} \sigma(P_i)\right] \cap \sigma(P_{k+1}) \\
&= \sigma(c_1, c_2, ..., c_{n-k}) \cap \sigma(P_{k+1}) \qquad \text{by (1) and (4)} \\
&= \sigma(c_1, c_2, ..., c_{n-k-1}) \qquad \text{since} \quad d_k \notin \sigma(c_1, ..., c_{n-k}).
\end{aligned}$$

Thus, Statement (4) holds. By induction, the claim is proven.

Now we use the claim to prove the theorem. We apply the claim with $k=n-2$. Since

$$r\left[\bigcap_{i=1}^{k}\sigma(P_i)\right]=n-k=2,$$

there is an entry x of row $2n-1$ that lies in the first 3 columns and is not in $\bigcap_{i=1}^{k}\sigma(P_i)$. There must exist t, $1\leqslant t\leqslant k$, with $x\notin\sigma(P_t)$. Since x lies in one of the first 3 columns, one of P_t+x, P'_t+x, or P''_t+x is an IT, and the theorem is proven. ▮

3. ALGORITHMIC COMPLEXITY

Drisko [4] noted that his proof of Theorem 1.1 [4, Theorem 1] would lead to a recursive algorithm to find a transversal. He calculates the time complexity of this algorithm to be

$$\tfrac{1}{6}n^4+n^3-\tfrac{13}{6}n^2+3n-2=O(n^4),$$

where n is the number of columns of the given matrix.

Our proof of Theorem 2.1 is based on the above-mentioned proof of Drisko; like that proof, it can be phrased as a recursive algorithm. We briefly examine the complexity of this algorithm.

The algorithm follows the steps of the proof, stopping if it finds an IT. It is given an $m\times n$ matrix ($m\geqslant 2n-1$) with entries in the ground set of a matroid M, in which the entries of each row form an independent set of size n in M. It returns an IT. We assume that $m=2n-1$, and that the matroid M is accessed via an independence oracle.

The algorithm begins by checking whether the entries of each column are all parallel. Thus, for each of n columns, we search the entries in rows 2 through $2n-1$ for an entry that is not parallel to the entry in row 1. This requires $O(n^2)$ calls to the independence oracle.

We permute rows and columns so that two nonparallel entries are in the positions of b_1 and b_2 and recursively call the algorithm to find the IPT P_1. The row and column permutations are not a major factor in the complexity of the algorithm; the recursive call will be discussed later.

Next we determine which entries should be c_1 and c_2. This requires $O(n)$ calls to the oracle.

Finally, we construct the IPTs P_2 through P_{n-2} and return the IT. For each of these IPTs, we find the entry that will be d_i using $O(n)$ calls to the oracle; then we determine which of the previous IPTs to make the new transversal out of using another $O(n)$ calls to the oracle. Thus, in constructing these IPTs, we make $O(n^2)$ calls to the oracle.

Overall, running the algorithm requires $O(n^2)$ calls to the oracle and 1 recursive call. Since the recursion has depth $O(n)$, the entire algorithm makes $O(n^3)$ calls to the oracle.

In the case when M is a free matroid, a multiset is independent precisely when its elements are all distinct; this can be determined for a multiset of n elements in $O(n)$ time. Thus, the algorithm corresponding to Drisko's result (Theorem 1.1) runs in $O(n^4)$ time; this agrees with Drisko's calculation.

4. OPEN PROBLEMS

Recall the matrices $R_{m,n}$ of Example 1.2. We noted that $R_{2n-2,n}$ has no transversals. By Theorem 1.1, $R_{2n-1,n}$ has a transversal. For example, we can see in Fig. 3 that the main diagonal of $R_{5,3}$ is a transversal.

However, although $R_{4,3}$ has no transversals at all, $R_{5,3}$ has many transversals: in the first three rows there are three transversals, no two of which share any positions. The matrix $R_{6,3}$ has four such transversals. Generally, while $R_{m,n}$ has no transversals if $m < 2n-1$, it has $m-(n-1)$ transversals, no two of which share any positions, if $m \geqslant 2n-1$. We conjecture that this holds for more general matrices.

Conjecture 4.1. Let A be an $m \times n$ row-Latin rectangle based on k. If $m \geqslant 2n-1$, then A has $m-(n-1)$ transversals, no two of which share any positions in A.

Example 1.2 shows that Conjecture 4.1 is best-possible.

A stronger statement than that made above holds for $R_{5,3}$: this matrix has three rows that together are the union of three transversals. Similarly, for $m \geqslant 2n-1$, $R_{m,n}$ has n rows that are the union of n transversals. However, more general matrices may not have this property. Below, we construct $(n^2-1) \times n$ matrices in which the entries of each row are all distinct, but no n rows are the union of n transversals. We conjecture that these are the largest such matrices, that is, that n^2 rows force the existence of n transversals whose union is n rows.

$$R_{4,3} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}; \qquad R_{5,3} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}; \qquad R_{6,3} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \end{pmatrix}.$$

FIG. 3. Matrices from Example 1.2: $R_{4,3}$, which has no transversals, and $R_{5,3}$ and $R_{6,3}$, which have many transversals.

$$T_2 = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 1 \end{pmatrix}; \qquad T_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 3 & 4 \\ 3 & 4 & 1 \\ 3 & 4 & 1 \\ 4 & 1 & 2 \\ 4 & 1 & 2 \end{pmatrix}; \qquad T_4 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 \\ 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 \\ 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 \\ 5 & 1 & 2 & 3 \end{pmatrix}.$$

FIG. 4. The matrices T_2, T_3, and T_4 of Example 4.2.

EXAMPLE 4.2. For each integer $n \geqslant 2$ we define T_n to be an $(n^2-1) \times n$ matrix as follows. Begin with an $(n^2-1) \times (n+1)$ matrix whose rows are $n-1$ copies of each of the cyclic permutations of 1, 2, ..., $n+1$. Delete the last column of this matrix to obtain T_n.

Thus, the first $n-1$ rows of T_n are all 1, 2, 3, ..., n. The next $n-1$ rows are all 2, 3, ..., n, $n+1$, and so on. Each row omits exactly one element of $\{1, 2, ..., n+1\}$.

The first three matrices T_n are shown in Fig. 4.

PROPOSITION 4.3. *There do not exist n rows of T_n that together are the union of n transversals.*

Proof. Assume for a contradiction that there are n rows of T_n that together are the union of n transversals. Form a matrix $A = (a_{i,j})$ with these n rows, and consider the above-mentioned transversals as transversals of A. Each row of A and each transversal of A omit exactly one element of $\{1, 2, ..., n+1\}$. The multiset union of the n rows and the multiset union of the n transversals are the same. Equivalently, the multiset of elements omitted from the n rows is equal to the multiset of elements omitted from the n transversals. Thus, there is a one-to-one correspondence between the set of rows and the set of transversals so that a corresponding row-transversal pair each omit the same number. Number the transversals from 1 to n so that for each i, transversal i omits the same number as row i.

For the remainder of this proof, arithmetic will be modulo $n+1$.

Let b_i denote the number omitted from row i and transversal i ($1 \leqslant i \leqslant n$), so that $a_{i,j} = b_i + j$. Let $t_{i,j}$ denote the element of transversal i that lies in column j, and let $r_{i,j}$ denote the row that this entry lies in, so that $t_{i,j} = a_{r_{i,j},j} = b_{r_{i,j}} + j$. For any given i, since row i and transversal i omit the same number, we have

$$\sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} t_{i,j}.$$

As stated above, $a_{i,j} = b_i + j$, and $t_{i,j} = b_{r_{i,j}} + j$. Subtracting, we obtain $a_{i,j} - t_{i,j} = b_i - b_{r_{i,j}}$, and so

$$0 = \sum_{j=1}^{n} (a_{i,j} - t_{i,j}) = \sum_{j=1}^{n} (b_i - b_{r_{i,j}}) = nb_i - \sum_{k=1}^{n} b_k = -b_i - \sum_{k=1}^{n} b_k,$$

since n is congruent to -1 modulo $n+1$. Hence, we have

$$b_i = -\sum_{k=1}^{n} b_k, \qquad 1 \leqslant i \leqslant n.$$

Since the right-hand side does not depend on i, all of the b_i's are equal, and so all n rows must omit the same number. But by the definition of T_n, at most $n-1$ rows all omit the same number. By contradiction, the proposition is proven. ∎

Conjecture 4.4. Let A be an $m \times n$ row-Latin rectangle based on k. If $m \geqslant n^2$, then there exist n rows of A that together are the union of n transversals.

The bound on m in Conjecture 4.4 is best-possible, by Proposition 4.3.

If Conjecture 4.4 is proven, then we need only verify Conjecture 4.1 for $m < n^2$; we can then prove Conjecture 4.1 by an inductive argument. Hence, Conjecture 4.1 can be verified for any given value of n by a bounded search.

We used the matrices $R_{2n-2,n}$ of Example 1.2 to show that a $(2n-2) \times n$ matrix in which the entries of each row are all distinct need not have a transversal. Drisko [4] conjectured that these are essentially the only such matrices without transversals.

Conjecture 4.5 (Drisko [4, Conjecture 2]). Let $n \geqslant 2$. Let A be a $(2n-2) \times n$ row-Latin rectangle based on k. Then either A has a transversal, or A can be transformed into $R_{2n-2,n}$ by permuting rows, columns, and symbols.

The corresponding statement for Conjectures 4.1 and 4.4 is false; that is, the matrices $R_{2n-2,n}$ and T_n are not unique. For example, the left-hand matrix in Fig. 5 is a 4×3 row-Latin rectangle in which there do *not* exist 3 transversals, no two of which share any positions. This matrix cannot be transformed into $R_{4,3}$ by permuting rows, columns, and symbols. The

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 2 & 3 & 1 \\ 2 & 3 & 1 \\ 1 & 3 & 2 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

FIG. 5. A 4×3 matrix, different from $R_{4,3}$, in which no 3 transversals occupy pairwise disjoint sets of positions, and an 8×3 matrix, different from T_3, in which no 3 rows are the union of 3 transversals.

right-hand matrix is an 8×3 row-Latin rectangle in which no 3 rows are the union of 3 transversals. This matrix cannot be transformed into T_3 by permuting rows, columns and symbols.

We generalize Conjectures 4.1 and 4.4 to matrices with entries in the ground set of a matroid.

Conjecture 4.6. Let A be an $m \times n$ matrix with entries in the ground set of a matroid M. Suppose that the set of entries of each row of A forms an independent set of size n in M. If $m \geqslant 2n-1$, then A has $m-(n-1)$ ITs, no two of which share any positions in A.

Conjecture 4.7. Let A be an $m \times n$ matrix with entries in the ground set of a matroid M. Suppose that the set of entries of each row of A forms an independent set of size n in M. If $m \geqslant n^2$, then there exist n rows of A that together are the union of n ITs.

Conjectures 4.6 and 4.7 imply Conjectures 4.1 and 4.4, respectively, by letting M be a free matroid. As above, Conjecture 4.6 and the bound on m in Conjecture 4.7 are best-possible. If Conjecture 4.7 is proven, then we need only verify Conjecture 4.6 for $m < n^2$.

REFERENCES

1. A. E. Brouwer, A. J. de Vries, and R. M. A. Wieringa, A lower bound for the length of partial transversals in a Latin square, *Nieuw Arch. Wisk. (3)* **26** (1978), 330–332.
2. J. Dénes and A. D. Keedwell, "Latin Squares: New Developments in the Theory and Applications," Annals of Discrete Mathematics, Vol. 46, North-Holland, Amsterdam, 1991.
3. J. F. Dillon, Transversal query, message ⟨199611271503.KAA17678@afterlife.ncsc.mil⟩, mailing list *COMB-L*, November 27, 1996.
4. A. A. Drisko, Transversals in row-Latin rectangles, *J. Combin. Theory Ser. A* **84** (1998), 181–195.

5. P. Erdős, D. R. Hickerson, D. A. Norton, and S. K. Stein, Has every Latin square of order n a partial Latin transversal of size $n-1$? *Amer. Math. Monthly* **95** (1988), 428–430.
6. P. W. Shor, A lower bound for the length of a partial transversal in a Latin square, *J. Combin. Theory Ser. A* **33** (1982), 1–8.
7. S. K. Stein, Transversals of Latin squares and their generalizations, *Pacific J. Math.* **59** (1975), 567–575.
8. D. E. Woolbright, An $n \times n$ Latin square has a transversal with at least $n-\sqrt{n}$ distinct symbols, *J. Combin. Theory Ser. A* **24** (1978), 235–237.