Adjacent integrally closed ideals in dimension two*

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Communicated by C.A. Weibel
Received 20 November 1991

Abstract

Let I be an m-primary integrally closed ideal in a 2-dimensional regular local ring $R$. Zariski proved that I can be uniquely factored into a product of simple integrally closed ideals, and Lipman later proved that the Hilbert function $H_i(n) = \lambda(R/I^n)$ of I is a polynomial for all $n \geq 1$.

By using these results with many others, we study various properties of adjacent integrally closed ideals in 2-dimensional regular local rings. In particular, multiplicities, factorizations, minimal reductions, and Rees valuations of adjacent integrally closed ideals are studied.

Introduction

Let $(R, m, k)$ be a 2-dimensional regular local ring (RLR for short) with maximal ideal $m$ and algebraically closed residue field $k$. The theory of complete (= integrally closed) ideals in 2-dimensional RLRs was founded by Zariski in [10] and developed further in Appendix 5 of [11]. In [11, Appendix 5], among many others, Zariski proved two beautiful theorems concerning the structure of complete ideals in 2-dimensional RLRs. One of them, Zariski's Unique Factorization Theorem (ZUFT) [11, Theorem 3, Appendix 5], asserts that every complete ideal can be uniquely factored into a product of simple complete ideals. The other theorem, called Zariski's Product Theorem (ZPT) [11, Theorem 2', Appendix 5], says that any product of complete ideals is complete.

An ideal $I$ is simple if $I \neq R$ and if whenever $I = JL$ for ideals $J, L$ of $R$ then $J = R$ or $L = R$. Let $v$ be a valuation of the quotient field $K$ of $R$ and let

* A part of this work was presented in AMS Special Session, 845th meeting, Lawrence, Kansas in October 1988.
Let \((V, m(V), k(v))\) be the corresponding valuation ring. Then \(v\) is called a **prime divisor of the second kind** if \(v\) dominates \(R\) (i.e., \(V \supset R\) and \(m(V) \cap R = m\)), and such that the transcendence degree of \(k(v)\) over \(k\) (denoted by \(\text{trdeg}_k k(v)\)) is 1. In [11] Zariski also set up a one-to-one correspondence between the set of prime divisors of the second kind of \(R\) and the set of simple complete \(m\)-primary ideals of \(R\) [11, Theorem (E), Appendix 5].

Therefore, if \(I\) is an \(m\)-primary complete ideal of \(R\) and \(I = l_1^{k_1} l_2^{k_2} \cdots l_n^{k_n}\) is the unique factorization of \(I\), then there are prime divisors \(v_1, \ldots, v_n\) associated to simple factors \(l_1, \ldots, l_n\). This set of prime divisors \(\{v_i | i = 1, \ldots, n\}\) associated to \(I\) is the set of the Rees valuations \(T(I)\) of \(I\).

In [5] and [6] Lipman further showed that the multiplicity \(e(I)\) of \(I\) is \(\sum_{i=1}^{n} k_i v_i(I)\) ([6, (21.4), (23.3)], [7, (1.9), (3.8)]). In this connection Huneke raised a question of how much the multiplicity can differ for two adjacent complete ideals. Two ideals \(I \supset J\) are said to be **adjacent** if their lengths differ by one.

Let \(I \supset J\) be adjacent \(m\)-primary complete ideals and let \(I = l_1^{k_1} l_2^{k_2} \cdots l_n^{k_n}\) and \(J = j_1^{r_1} j_2^{r_2} \cdots j_m^{r_m}\) be their unique factorizations, where \(l_i\) and \(j_i\) are simple complete ideals associated to prime divisors \(v_i\) and \(w_i\) respectively for \(1 \leq i \leq n\) and \(1 \leq j \leq m\).

In this paper we answer the above question of Huneke by obtaining a formula for \(e(J) - e(I)\) with respect to the prime divisors associated to \(J\) (Section 2) and we also obtain various properties of adjacent complete ideals.

One of the properties concerns results of Hoskin and Lipman. In [2, Theorem 3.1], Hoskin showed that if \(J\) is a simple \(v\)(valuation)-ideal and \(I\) is the immediate predecessor of \(I\) for a 0-dimensional valuation (see Section 1 for the definition) \(v\) of \(K\), then \(I\) is either simple or a product of two simple \(v\)-ideals. In [5], Lipman generalized this to arbitrary complete ideals; which shows that if \(J\) is a simple complete ideal, then a complete ideal \(I\) right above \(J\) is either simple or a product of two simple complete ideals. We further show that there is a unique such complete ideal \(I\) right above \(J\). We also show that this fails if \(J\) is not simple (Section 3). In addition to these results, we briefly summarize the results in each section.

In Section 1, we first show the existence of adjacent complete ideals right above and below a given complete ideal. Then for given adjacent complete ideals \(I \supset J\), we compare the orders \(o(I)\) and \(o(J)\), and the characteristic forms \(c(I)\) and \(c(J)\) of \(I\) and \(J\). We also compare the \(v\)-values \(v(I)\) and \(v(J)\) for a Rees valuation \(v \in T(I) \cup T(J)\).

In Section 2, we obtain a formula for \(e(J) - e(I)\) and generalize it to arbitrary adjacent \(m\)-primary ideals. We obtain upper bounds for \(e(J) - e(I)\). We also consider how minimal reductions of \(I\) and \(J\) are related.

In Section 3, we determine the number of complete ideals right above a given complete ideal. We classify which adjacent ideals \((J'', f)\) right above \(J''\) are complete for a simple complete ideal \(J\) for any \(n \geq 1\). At the end of Section 3, we
compute the exact difference \( w(J) - w(I) \) for adjacent complete ideals \( I \supset J \) and for \( w \in T(J) \setminus T(I) \) in various cases. We also leave several questions on factorizations of two adjacent complete ideals.

1. Backgrounds and preliminaries

Let us begin with some definitions and notations. Throughout this paper \((R, m, k)\) is a 2-dimensional RLR with algebraically closed residue field if it is not said to be otherwise. The multiplicity of \( R \) or an ideal \( I \) of \( R \) is denoted by \( e(R) \) or \( e(I) \). The number of elements in a minimal generating set of an ideal \( I \) is denoted by \( \mu(I) \). The length of an \( R \)-module \( M \) is denoted by \( \lambda(M) \). Two ideals \( I \supset J \) are said to be adjacent if \( \lambda(I/J) = 1 \). The integral closure of an ideal \( I \) is denoted by \( \overline{I} \). The order \( o(I) \) of an ideal \( I \) is \( r \) if \( I \subseteq m^r \), but \( I \not\subseteq m^{r+1} \). An ideal \( I \) is said to be contracted if there is an element \( x \in m \setminus m^2 \) such that \( IR[m/x] \cap R = I \), in which case we say that \( I \) is contracted from \( R[m/x] \).

Since every complete ideal is an intersection of valuation ideals \([11, p. 353]\), it is natural to study valuation ideals related to a given complete ideal. An ideal \( I \) is said to be a valuation ideal if \( I \) is contracted from a valuation ring \( V \) containing \( R \), in which case we say that \( I \) is a \( v \)-ideal if \( v \) is the corresponding valuation \([11, p. 340]\). A valuation \( v \) is called 0-dimensional (1-dimensional respectively) if \( \text{trdeg}_k k(v) \) is 0 (1 respectively). One example of an infinite sequence of adjacent complete ideals was given by Zariski in \([10]\), which are 0-dimensional valuation ideals. If \( I \) is a 0-dimensional valuation ideal, then there exist adjacent complete ideals right above and below \( I \) \([10, \text{Theorem 1}]\). We now show the existence of adjacent complete ideals for an arbitrary complete ideal. We include a proof given by Lipman which shows the existence of a complete ideal right below a given complete ideal.

**Lemma 1.1.** Let \( I \) be an \( m \)-primary complete ideal. Then there exist adjacent complete ideals right above and below \( I \).

**Proof.** (i) (Lipman) Let \((S, n)\) be a 2-dimensional RLR containing \( R \) in \( K \) such that \( S \) is not a base point \( I \) (see Section 2 for the definition). Let \( v \) be the \( n \)-adic order valuation of \( S \) and let \( I_v = \{ a \in R \mid v(a) > v(I) \} \). Then \( I_v \) is a \( v \)-ideal and \( IS \) is principal. Say \( IS = xS \), and define a map \( f : I \to S/n \) by \( f(y) = y/x + n \) for \( y \in I \). Then \( \ker(f) = I_v \), i.e., \( I/I_v = S/n \cong k \). Hence \( I_v \) is a complete ideal right below \( I \).

(ii) Let \( T(I) = \{ v_1, v_2, \ldots, v_n \} \) be the Rees valuations of \( I \). Since \( I = \overline{I} \), \( I = \bigcap_{i=1}^n (IW_i \cap R) \). Let \( w \) be a composite of \( v_i \) and a valuation ring of \( k(v_i) \) such that \( k(w) \) is algebraic over \( k \), i.e., \( k(w) = k \). We can find such \( w \) since \( \text{trdeg}_k k(v_i) \) is finite and \( k \) is algebraically closed. Then \( \bigcap_{i=1}^n (IW_i \cap R) = I \) since \( W_i \subseteq V_i \) and \( I = \overline{I} \). Suppose \( I = IW_i \cap R \) for some \( i \), i.e., assume \( I \) is a \( w \)-ideal. Then there exist \( w \)-ideals \( K \supset I \supset J \) such that \( w_i(K) = w_i(I) + 1 \), \( w_i(J) = w_i(I) + 1 \), and \( w_i(I) \).
1, and \( \lambda(K/I) = \lambda(I/J) = 1 \) since \( w_i \) is a 0-dimensional valuation [10, Theorem 1]. Hence \( K \) and \( J \) are adjacent complete ideals right above and below \( I \). Now suppose \( IW_i \cap R \neq I \) for all \( i \). Let \( K_i \supset IW_i \supset J_i \) be adjacent ideals in \( W_i \). Let \( I_i - IW_i \cap R, M - K_i \cap R \), and \( N - J_i \cap R \). Then we have \( \lambda(M/I_i) = \lambda(I_i/N) = 1 \). Let \( L = \bigcap_{i \neq j} (IW_i \cap R) \). Then

\[
M \cap L/I \simeq M \cap L/I_i \cap L = (M \cap L) + I_i/I_i = M/I_i = k
\]

since \( M \supset I_i \) are adjacent. Therefore \( M \cap L \) is a complete ideal right above \( I \). Similarly,

\[
I/N \cap L \simeq I_i/N \cap L = (I_i \cap L) + N/N = I/I = k 
\]

hence \( N \cap L \) is a complete ideal right below \( I \). \( \square \)

Zariski defined the characteristic form \( c(I) \) of an \( m \)-primary ideal \( I \) of order \( r \) to be the greatest common divisor of the elements in \( (I + m')/m'^{r+1} \) in the UFD \( \text{gr}_m(R) = R/m \oplus m/m^2 \oplus \cdots \) [11, p. 363]. If \( o(I) = r \), then \( \deg(c(I)) \leq r \) always holds.

Zariski showed that for a contracted \( m \)-primary ideal \( I, m \mid I \) if and only if \( o(I) > \deg(c(I)) \). Moreover, if \( o(I) - \deg(c(I)) = s \), then \( I = m^s (I : m^s) \), and \( m^s I : m^s \) [11, Proposition 3, Appendix 5].

For adjacent ideals \( I \supset J, o(J) \) is either \( o(I) \) or \( o(I) + 1 \) since \( ml \subset J \). Now we compare the orders and the characteristic forms of \( I \) and \( J \).

**Lemma 1.2.** Let \( I \supset J \) be adjacent \( m \)-primary contracted ideals of \( R \). If \( o(I) = o(J) \), then the following three cases are possible.

1. \( m \mid I, m \mid J \).
2. \( m \not\mid I, m \not\mid J \).
3. \( m \mid I, m \mid J \).

If \( o(J) = o(I) + 1 \), then \( m \mid J, m \not\mid I \) and \( I : m = J : m \).

**Proof.** Let \( o(I) = r \). Suppose \( o(I) = o(J) \). Let \( m \not\mid I \). Then \( \deg(c(I)) = o(I) \) by [11, Proposition 3, Appendix 5]. Since \( o(I) = o(J), \deg(c(J)) \geq \deg(c(I)) \), hence \( \deg(c(J)) = o(J) \). Therefore, \( m \not\mid J \). Suppose \( o(J) = r + 1 \). Then \( \lambda(I : m/I) = o(I) = r \) and \( \lambda(J : m/J) = o(J) = r + 1 \) (cf. [4, Theorem 2.1]). Since \( ml \subset J \), \( I \subset J : m \). Therefore \( I : m = J : m \) since \( \lambda(I/J) = 1 \). Since \( ml \subset m(I : m) = \)
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m(J : m) ⊆ J, o(I : m) = r and hence μ(I : m) = λ(I : m/m(I : m)) = r + 1 [7, (3.2)]. Hence J = m(I : m) = m(J : m), i.e., m | J and m/I again by [11, Proposition 3, Appendix 5]. □

For ideals I ⊇ J, J is called a reduction of I if there exists some n ≥ 1 such that Jn = I^n+1. Further if J does not contain any other reduction of I, then it is called a minimal reduction of I. In a Noetherian local ring with infinite residue field, every m-primary ideal has a minimal reduction generated by a system of parameters. In [8, Proposition 5.5], Lipman and Teissier showed that if (a, b) is a minimal reduction of a complete ideal I of a 2-dimensional RLR, then I^2 = (a, b)I. Zariski showed that if I is a simple complete ideal associated to the prime divisor v and if (a, b) is a minimal reduction of I, then k(v) is purely transcendental over k and generated by the image of b/a over k. We also modify this for an arbitrary complete ideal I. The following well-known fact also shows the converse of [9, Lemma 1.9], as well as of [4, Remark 3.6].

**Lemma 1.3.** Let I be a complete m-primary ideal and let (a, b) be a minimal reduction of I. Then the image of b/a in k(v) is transcendental over k for all v ∈ T(I), i.e., v(a) − v(b) − v(I) for all v ∈ T(I).

**Proof.** Let v ∈ T(I). Since I = (a, b), v(I) = v(a, b) = min{v(a), v(b)}. Say v(I) = v(a). Let A = R[b/a] and B = [I/a]. Then B is normal since R and I are normal. Furthermore B = A since I = (a, b). Since a/b is a regular sequence, A = R[X]/(aX − b) for X a variable. Since aX − b ⊆ mR[X], A/mA = R[X]/(aX − b, mR[X]) = k[X]. Therefore, mA is a height-one prime of A and hence A_mA is a 1-dimensional local domain. Since v ∈ T(I), V = B_{P_i} for some P_i ∈ Min(B/aB) = {P_1, . . . , P_n}. Let S = A − mA. Then B_s = A_mA and B_s is a 1-dimensional semi-local ring with the maximal ideals P_sB_s, . . . , P_sB_s. Furthermore, V_s = B_s = (B_s)_{P_sB_s} for each i. Since P_sB_s ∩ A_mA = mA_{mA}, B_s/P_sB_s is algebraic over A_{mA}/mA_{mA}. On the other hand, k(v_i) = B_i/P_iB_i = B_i/I/P_iB_i for each i. Therefore, k(v_i) is algebraic over A_{mA}/mA_{mA}. However, A_mA/mA_{mA} = k(b/a*) = k(b/a) is the image of b/a in k(v_i). Since k(v_i) is purely transcendental over k, b/a is transcendental over k. Hence v_i(b/a) = 0, i.e., v_i(a) = v_i(b) for all i. □

2. Multiplicities and reductions of adjacent ideals

Let us begin with the definition of the intersection multiplicity which is due to Lipman. For a detailed explanation we refer to [7]. For a 2-dimensional RLR S containing R in the quotient field K (call them points), there is always a unique finite quadratic sequence of RLRs from R to S [1, Theorem 3]. For an ideal I, if the transform I_s in S is not the unit ideal, then S is called a base point of I. Let
Let \( P(I) \) denote the set of the base points of \( I \). Then the point basis \( B(I) \) of \( I \) is defined to be the set of nonnegative integers \( \{o_S(I_S)\} \), where \( o_S \) is the \( m(S) \)-adic order valuation of a base point \( (S, m(S)) \) of \( I \). If \( I \) is \( m \)-primary, then \( B(I) \) is finite [7, Theorem 3.1].

For two \( m \)-primary ideals \( I \) and \( J \), the intersection multiplicity \( (I \cdot J) \) is defined to be \( \sum o_S(I_S)o_S(J_S) \), where the sum is over the base points of \( I \) and \( J \). Let us further assume that \( I \) and \( J \) are complete. Lipman showed that if \( I \) and \( J \) are simple ideals associated to the prime divisors \( v \) and \( w \), then \( (I \cdot J) = v(J) = w(I) \) (reciprocity formula [6, (21.4)]). By this theorem, we can see that \( (m \cdot J) = o(J) \) for any complete ideal \( J \). And if \( I = I_1 \cdots I_n \) is the unique factorization of \( I \) with \( T(I_i) = \{v_i\} \) for each \( i \), then \( (I \cdot J) = \sum_{v_i \in T(I)} k_i v_i(J) \). Therefore, \( e(I) = (I \cdot I) = \sum_{v_i \in T(I)} k_i v_i(J) \) (multiplicity formula, cf. [6, (23.3)], [7, (3.8)]). He also showed that \( (I \cdot J) = \lambda(R/II) - \lambda(R/I) - \lambda(R/J) \) (cf. [7, (3.7), (3.8)]). From these, we have the mixed multiplicity formula

\[
e(I'J') = r^2 e(I) + 2rs(I \cdot J) + s^2 e(J)
\]

since \( B(IJ) = B(I) + B(J) \) (cf. [7, (1.9)]).

From the above reciprocity and the multiplicity formulas of Lipman, we can find a formula that describes the difference between the multiplicities of adjacent integrally closed ideals with respect to the Rees valuations of the smaller ideal.

**Theorem 2.1.** Let \( I \subseteq J \) be adjacent complete \( m \)-primary ideals. Let \( J = J_1 \cdots J_m \) be a unique factorization of \( J \) and \( T(J_j) = \{w_j\} \) for \( j = 1, \ldots, m \). Then

\[
e(J) - e(I) = \sum_{w_j \in T(J) \setminus T(I)} s_j(w_j(J) - w_j(I))
\]

**Proof.** Let \( I = I_1 \cdots I_n \) be a unique factorization of \( I \) and \( T(I) = \{v_i \mid v_i = T(I_i)\} \) for \( i = 1, \ldots, n \). Choose \( a \in J \) so that \( v_i(a) = v_i(J) \) for all \( i \) and \( w_j(a) = w_j(J) \) for all \( j \). Then \( v_i(I) = v_i(J) \) for all \( i \) [9, Lemma 1.6]. Therefore \( a \) is part of a minimal generating set for a minimal reduction of \( I \) as well as of \( J \) [9, Lemma 1.9]. Let \( (a, b) \) be a minimal reduction of \( I \). Then \( b \not\in I \) since \( I \) is not integral over \( J \). Since \( I \) is integrally closed, \( \mu(I) = r + 1 \) if \( o(I) = r \) ([4, Theorem 2.1], [7, (3.2)]). Choose a minimal generating set \( a_1, a_2, \ldots, a_r = a, b \) for \( I \) so that \( J = (a_1, \ldots, a_r) + mI \). Since \( b \not\in J \), there exists some \( j \) such that \( w_j(b) < w_j(J) \). For such \( j \), we have \( w_j(b) = w_j(I) \). Therefore, by Lipman’s multiplicity formula, the reciprocity formula, and Lemma 1.6 in [9], we have

\[
e(I) = \sum_{i=1}^n k_i v_i(I) = \sum_{i=1}^n k_i v_i(J)
\]

\[
= \sum_{i=1}^m k_i \left( \sum_{j=1}^m s_j w_j(J_j) \right) = \sum_{i=1}^m k_i \left( \sum_{j=1}^m s_j w_j(I_i) \right)
\]

\[
= \sum_{j=1}^m s_j w_j(I).
\]
On the other hand, \( e(J) = \sum_{j=1}^{m} s_j w_j(J) \). Therefore,

\[
e(J) - e(I) = \sum_{j=1}^{m} s_j (w_j(J) - w_j(I))
= \sum_{w_j \in T(J) \setminus T(I)} s_j (w_j(J) - w_j(I)). \quad \square
\]

Sam Huckaba asked whether Theorem 2.1 is still true for arbitrary adjacent \( m \)-primary ideals. We show in Theorem 2.2 this is indeed the case.

**Theorem 2.2.** Let \( I \supset J \) be adjacent \( m \)-primary ideals of \( R \). Then

\[
e(J) - e(I) = \sum_{w_j \in T(J) \setminus T(I)} s_j (w_j(J) - w_j(I)),
\]

where \( \tilde{I} = J_1^* J_2^* \cdots J_m^* \) and \( T(J_j) = \{ w_j \} \) for \( j = 1, \ldots, m \).

**Proof.** Let \( \tilde{I} = I_1^* \cdots I_n^* \) and \( T(I_j) = \{ v_j \} \) for \( i = 1, \ldots, n \). Choose a minimal reduction \( (a, b) \) of \( I \). Since \( \lambda(I/J) = 1 \), we may assume that one of \( \{ a, b \} \), say \( a \), is in \( J \). Then \( v_j(a) \geq v_j(J) \geq v_j(I) \) implies that \( v_j(a) = v_j(I) = v_j(J) \) for all \( v_j \in T(I) \) since \( a \) is part of a minimal generating set for a minimal reduction of \( I \) (Lemma 1.3). Therefore,

\[
e(I) = e(\tilde{I}) = \sum_{i=1}^{n} k_i(v_i(\tilde{I})) = \sum_{i=1}^{n} k_i(v_i(I))
= \sum_{i=1}^{n} k_i(v_i(J)) = \sum_{i=1}^{n} k_i(v_i(\tilde{J}))
= \sum_{i=1}^{n} k_i \left( \sum_{j=1}^{m} s_j v_j(J_j) \right) = \sum_{i=1}^{n} k_i \left( \sum_{j=1}^{m} s_j w_j(I) \right) \quad \text{(by reciprocity)}
= \sum_{j=1}^{m} s_j w_j(\tilde{I}) = \sum_{j=1}^{m} s_j w_j(I).
\]

On the other hand, \( e(J) = e(\tilde{J}) = \sum_{j=1}^{m} s_j w_j(J) \). Therefore,

\[
e(J) - e(I) = \sum_{w_j \in T(J) \setminus T(I)} s_j (w_j(J) - w_j(I)). \quad \square
\]

Note that adjacent \( m \)-primary ideals \( I \supset J \) do not necessarily have adjacent integral closures \( \tilde{I} \supset \tilde{J} \) as we can see in the following example.

**Example 2.3.** Let \( m = (x, y) \), \( I = (x^2, y^2) \), and \( J = (x^2, xy^2, y^3) \). Then \( \lambda(I/J) = 1 \). However, \( \tilde{I} = (x, xy, y^2) \), \( \tilde{J} = J \), and \( \lambda(\tilde{I}/\tilde{J}) = 2 \). In general, \( \lambda(I/J) = 1 \) implies that \( ml \not\subseteq J \). Therefore, \( ml \not\subseteq J \). Therefore, \( \lambda(\tilde{I}/\tilde{J}) \leq \lambda(\tilde{I}/ml) = o(\tilde{I}) + 1 = o(I) + 1 \). If \( o(J) = o(I) + 1 \), then \( \lambda(\tilde{I}/\tilde{J}) \leq o(J) \). If \( o(J) = o(I) \), then \( ml \not\subseteq J \).
Hence $\lambda(\tilde{I}/\tilde{J}) < \lambda(\tilde{I}/\tilde{m}\tilde{I}) = o(I) + 1$ which imply that $\lambda(\tilde{I}/\tilde{J}) \leq o(I) \leq o(I)$. In either case, $\lambda(\tilde{I}/\tilde{J}) \leq o(J)$. Note that $o(I) = o(I)$ since $(a, b)\tilde{I} = \tilde{I}^2$ for a minimal reduction $(a, b)$ of $I$ [8, Proposition 5.5].

As a corollary of the above theorems we obtain an upper bound for $e(J) - e(I)$ by the reciprocity formula.

**Corollary 2.4.** Let $I \supset J$ be adjacent $m$-primary ideals of $R$. Then $e(J) - e(I) \leq o(J)$.

**Proof.** Let $\tilde{J}$ be factored as in Theorem 2.2. Since $\lambda(I/J) = 1$, $mI \subseteq J$. Therefore, $w_j(m) + w_j(I) \geq w_j(J)$ for all $j$. Hence $w_j(J) - w_j(I) \leq w_j(m)$ for all $j$. However, by the reciprocity formula of Lipman, we have $w_j(m) = o(J_j)$. Therefore, $w_j(J) - w_j(I) \leq o(J_j)$ for all $j$. Therefore,

$$e(J) - e(I) = \sum_{i=1}^{m} s_j(w_j(J) - w_j(I))$$

$$\leq \sum_{j=1}^{m} s_j o(J_j)$$

$$= o(\tilde{J}) = o(J).$$

In the following theorem we obtain a sharper upper bound for $e(J) - e(I)$ for adjacent contracted ideals of the same order by using the intersection multiplicity.

**Theorem 2.5.** Let $I \supset J$ be adjacent contracted ideals of the same order. Then $e(J) - e(I) \leq \deg(c(J))$, where $c(J)$ is the characteristic form of $J$.

**Proof.** Let $o(I) = o(J) = r$. If $m \nmid J$, then $\deg(c(J)) = r$ by [11, Proposition 3, Appendix 3]. Hence $\deg(c(J)) = o(J)$, thus $e(J) - e(I) \leq \deg(c(J))$ by Corollary 2.4. Therefore assume $m \mid J$. Then $m \mid I$ by Lemma 1.2 since $o(I) = o(J)$. Let $s = o(J) - \deg(c(J))$ and $t = o(I) - \deg(c(I))$. Then $I = m'(I : m'), m \nmid I : m'$ and $J = m'(J : m'), m \nmid J : m'$ by [11, Proposition 3, Appendix 3]. Since $I \supset J$ and $o(I) = o(J)$, $\deg(c(J)) \geq \deg(c(I))$. Hence $t \geq s$, and $I = m(I : m) = m^2(I : m^2) = \cdots = m^s(I : m^s)$. Consider

$$I : m^s \supset J : m^s \supset I = m^s(I : m^s) \supset J = m^s(J : m^s).$$

Let us compute $\lambda(J : m^s/J)$ and $\lambda(I : m^s/I)$.

$$\lambda(J : m^s/J)$$

$$= \sum_{i=0}^{s-1} \lambda(m'(J : m')/m^{i+1}(J : m')) = \sum_{i=0}^{s-1} \mu(m'(J : m'))$$

$$= \sum_{i=0}^{s-1} [o(m'(J : m')) + 1] = \sum_{i=0}^{s-1} (i + r - s + 1),$$
since $J : m^t$ is an integrally closed ideal of order $r - s$. Note that $m'(J : m^t)$ is contracted since it is a product of two contracted ideals (ZPT [11, p. 379]). By the same argument we can show

$$\lambda(\mathcal{I} : m^\ell / m'(\mathcal{I} : m^\ell)) = \lambda(\mathcal{J} : m^\ell / m'(\mathcal{J} : m^\ell)).$$

Since $\lambda(\mathcal{I}/\mathcal{J}) = 1$, we have $\lambda(\mathcal{I} : m^\ell / \mathcal{J} : m^t) = 1$, i.e., $\mathcal{I} : m^t$ and $\mathcal{J} : m^t$ are adjacent integrally closed ideals of the same order $r - s$. Therefore,

$$e(\mathcal{J} : m^t) - e(\mathcal{I} : m^t) \leq o(\mathcal{J} : m^t) - r - s,$$

by Corollary 2.4. However,

$$e(\mathcal{J}) = e(m'(\mathcal{I} : m^t)) = e(\mathcal{J} : m^t) + 2s((\mathcal{J} : m^t) \cdot m) + s^2 e(m),$$

and

$$e(\mathcal{I}) = e(m'(\mathcal{J} : m^t)) = e(\mathcal{I} : m^t) + 2s((\mathcal{I} : m^t) \cdot m) + s^2 e(m).$$

By (21.4) of [6], $((\mathcal{I} : m^t) \cdot m) = o(\mathcal{I} : m^t) = r - s$. Similarly, $((\mathcal{J} : m^t) \cdot m) = r - s$. Hence we have

$$e(\mathcal{J}) - e(\mathcal{I}) = e(\mathcal{J} : m^t) - e(\mathcal{I} : m^t) \leq o(\mathcal{J} : m^t) = r - s = \deg(c(\mathcal{J})).$$

**Remark 2.6.** (1) Lipman also gave a different formula for $e(\mathcal{J}) - e(\mathcal{I})$. He claimed and proved that $e(\mathcal{J}) - e(\mathcal{I}) = \dim_k(\mathcal{J}/\mathcal{J}^2 : \mathcal{I}) - 1$ without the assumption that $k$ is algebraically closed. This claim shows that $e(\mathcal{J}) - e(\mathcal{I}) \leq \dim_k(\mathcal{J} / \mathcal{mJ}) - 1 = \mu(\mathcal{J}) - 1 = o(\mathcal{J})$ since $\mathcal{mJ} \subseteq \mathcal{J} : \mathcal{I} \subseteq \mathcal{J}$.

(2) In his personal letter to me, Rees included the results bounding $e(\mathcal{J}) - e(\mathcal{I})$ by using degree functions in a $d$-dimensional quasi-unmixed local domain. In the case of 2-dimensional RLRs, Rees’ result also implies Theorem 2.2 and Corollary 2.4 of this paper.

In the proof of Theorem 2.1, we showed that for adjacent complete ideals $\mathcal{I} \supseteq \mathcal{J}$, we can choose an element $a \in \mathcal{J}$ so that $a$ is part of minimal generating sets of minimal reductions for both $\mathcal{I}$ and $\mathcal{J}$. Now we further ask if $a \in \mathcal{J}$ is part of a minimal generating set of a minimal reduction of $\mathcal{I}$, then is it also part of a minimal generating set for a minimal reduction of $\mathcal{J}$?

This seems to be not true in general as we can see in Example 2.9. However, we give a positive answer to the above question in certain cases. Note that $a \in \mathcal{I}$ is
part of a minimal reduction of \( I \) if and only if \( \nu(a) = \nu(I) \) for all \( \nu \in \mathcal{T}(I) \) (cf. Lemma 1.3 and [9, Lemma 1.9]). The following is a preparation work to prove Theorem 2.8.

**Lemma 2.7.** Let \( J \) be a simple complete ideal with \( B(J) = \{ r_i \}_{i=0}^{t} \). Let \( I \) be a complete ideal right above \( J^n \) for \( n \geq 1 \). Then

(i) \( \nu(I) = \nu(J^n) \),

(ii) there exists \( a \in J^n \) which is part of minimal generating sets for minimal reductions of \( I \) and \( J^n \), and

(iii) \( P(J) \subseteq P(I) \), and \( nr_i - 1, nr_i \in B(I) \) for \( i = 0, \ldots, t - 1 \).

**Proof.** (i) It follows from Lemma 1.2.

(ii) Let \( T(J) = \{ w \} \). Choose \( a \in J^n \) such that \( w(a) = w(J^n) \) and \( \nu(a) = \nu(J^n) \) for all \( \nu \in \mathcal{T}(I) \). Let \( (c, d) \) be a minimal reduction of \( I \). Since \( \lambda(I/J^n) = 1 \), we may assume that either \( c \) or \( d \) is in \( J^n \), say \( c \in J^n \). Then \( \nu(c) \geq \nu(J^n) = \nu(a) \geq \nu(I) = \nu(c) \) by Lemma 1.3. Hence \( \nu(a) = \nu(I) \), i.e., \( a \) is part of a minimal generating set for a minimal reduction of \( I \) as well as of \( J^n \) by [9, Lemma 1.9].

(iii) Let \( \nu(I) = \nu(J^n) = r \). As in (ii), we can choose \( a \in J^n \) so that \( a \) is a part of minimal generating sets for minimal reductions of \( I \) and \( J^n \) and \( \nu(a) = r \). Let \( (a, b) = I \). Then \( b \not\in J^n \) since \( I \) is not integral over \( J^n \).

Let

\[
(R, m) \subset (R_1, m_1) \subset \cdots \subset (R_t, m_t)
\]

be the quadratic sequence along \( w \). Suppose \( R_1 = R[m/x]_N \) for some \( x \in m \setminus m^2 \) and \( N = \text{Max}(R[m/x]) \). Since \( J \) is simple, the transform of \( J \) in \( R[m/x] \) is also simple and contained in a unique maximal ideal \( (y/x - a, x) \) in \( R[m/x] \) for some unit \( a \) of \( R \) (cf. [3, Remark 3.8]). Hence we may assume that \( m_1 = (y/x, x) \) by replacing \( y \) by \( y - ax \). Since \( m \not\in J^n \) and \( \nu(a) = r \), we have \( c(J^n) = a^* \) for the image \( a^* \) of \( a \) in \( gr_n(R) \). Since \( \nu(I) = r \) and \( a \in I \), \( c(I) \mid a^* \). Therefore \( I \) is also contracted from \( R_1 \) (cf. [3, Proposition 2.3]). Choose a minimal generating set \( a_1, \ldots, a_r, a, b \) of \( I \) such that \( J^n = (a_1, \ldots, a_r) + ml \). Note that \( \mu(I) = \mu(J^n) = r + 1 \). Since \( b \not\in J^n \) and \( J^n \) is a \( w \)-ideal [11, Theorem (D), Appendix 5], \( w(bx) = w(b) + w(x) < w(J^n) + w(m) = w(mJ^n) \). Therefore \( bx \not\in mJ^n \) since \( mJ^n \) is a \( w \)-ideal by [9, Theorem 2.4]. Let \( m = (x, y) \). Then \( w(y) = w(x) - r \). Since \( \lambda(ml/mJ^n) = (m \cdot J^n) - (m \cdot I) + \lambda(I/J^n) = \nu(J^n) - \nu(I) + 1 = 1 \), we may assume that \( b \in mJ^n \). Let \( I_1, J_1 \) be the transforms of \( I, J \) in \( R_1 \). Since \( \nu(I) = \nu(J^n) = r \), \( I_1 = (a_1/x', \ldots, a_r/x', b/x') \) and \( J_1 = (a_1/x', \ldots, a_r/x', bx/x') \). Since \( m_1 = (y/x, x) \), we can see that \( m_1I_1 \subseteq J_1^n \), i.e., \( I_1 \supseteq J_1^n \) are adjacent in \( R_1 \). Furthermore, \( o_1(I_1) = o_1(J_1^n) \) by Lemma 1.2, where \( o_1 \) is the \( m_1 \)-adic order valuation of \( R_1 \). Inductively we can show that \( \lambda(I_i/J_i^n) = 1 \) for \( i = 0, \ldots, t \) and \( o_1(I_i) = o_1(J_i^n) \) for \( i = 0, \ldots, t - 1 \) and \( o_1(I_i) = o_1(J_i^n) - 1 \). Therefore, \( P(J) \subseteq P(I) \) and \( n - 1, nr_i \in B(I) \) for \( i = 0, \ldots, t - 1 \). \( \Box \)
Theorem 2.8. Let $I \supset J^n$ be adjacent complete ideals, where $J$ is simple. Let $a \in J^n$. If $(a, b) = I$ for some $b \in I$, then there exists $c \in J^n$ such that $(a, c) = J^n$.

Proof. Let $w$ be the Rees valuation of $J$ and $o(J) = r$. Since $m \not\mid J^n$, $o(I) = o(J^n) = nr$ by Lemma 1.2. Let

$$(R = R_0, m = m_0) \subset (R_1, m_1) \subset \cdots \subset (R_t, m_t)$$

be the quadratic sequence along $w$, where $m_i$, $o_i$, $I_i$, and $J_i$ denote the maximal ideal, the $m_i$-adic order valuation, the transform of $I$, and the transform of $J$ in $R_i$ for $i = 0, \ldots, t$. We further assume that $m_{i-1} = (x_{i-1}, y_{i-1})$ and $R_i = R_{i-1}[m_{i-1}/x_{i-1}, y_{i-1}/x_{i-1}, y_{i-1}, u_{i-1}, v_{i-1}]$ for $i = 1, \ldots, t$. In the proof of Lemma 2.7, we showed that $\lambda(I_i/J_i^n) = 1$ for $i = 0, \ldots, t$. Let $o_i(J_i) = r_i$ for $i = 0, \ldots, t$. Since $o(a, b) = o(I) = nr$, $(a, b)R_i = x^{nr}(a, b)R_i$, where $a_i = a/x^{nr}$ and $b_i = b/x^{nr}$. Then $(a_i, b_i)$ is also a minimal reduction of $I_i$ since $IR_i = x^{nr}I_i$ is integral over $(a, b)R_i = x^{nr}(a_i, b_i)R_i$. Therefore, $o_i(a_i, b_i) = o_i(I_i) = nr_i$ and $a_i \in J_i$. By the same argument we obtain a minimal reduction $(a_i, b_i)$ of $I_i$ such that $a_i = a_i/x_{i-1}^{nr_i}$ and $b_i = b_i/x_{i-1}^{nr_i}$, for $i = 1, \ldots, t$. Then $o_i(a_i, b_i) = nr_i$ and $a_i \in J_i^n$ for all $i$.

Now compute $w(a)$ and $w(J^n)$.

$$w(a) = w(x^{nr}) + w(a_i) = nrw(x) + w(a_i)$$
$$= nr^2 + w(a_i) = nr^2 + w(x^{nr}) + w(a_2)$$
$$= nr^2 + nr_i^2 + w(a_2) = nr^2 + nr_i^2 + \cdots + nr_{i-1}^2 + w(a_i)$$
$$= n \left[ \sum_{i=0}^{t-1} r_i^2 \right] + w(a_i),$$

$$w(J^n) = nw(J) = n \sum_{i=0}^{t} r_i^2$$
$$= n \left[ \sum_{i=0}^{t-1} r_i^2 \right] + nw(J_i) = n \left[ \sum_{i=0}^{t-1} r_i^2 \right] + n$$

since $J_i = m_i$ and $w$ is the $m_i$-adic order valuation. Since $I_i = (a_i, b_i)$ and $a_i \in J_i^n$, $a_i$ is part of a minimal generating set of $I_i$. Hence $a_i \in m_i^n \setminus m_i^{n+1}$, i.e., $w(a_i) = o_i(a_i) = n$. Therefore, $w(a) = w(J^n)$, i.e., $a$ is part of a minimal generating set for a minimal reduction of $J^n$ by [9, Lemma 1.9].

Theorem 2.8 is not true if $I$ is a complete ideal right above a complete ideal with two distinct Rees valuations.

Example 2.9. (1) Let $m = (x, y)$, $I = m(y^2 - x^3, x^4, x^2y)$, and $J = (y, x^2)(x^3, x^2y, y^5)$. Then $y^5 \in J$ is part of a minimal reduction for $I$, but not for $J$. 

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since \( w(y') = 6 > w(J) = 5 \) for the Rees valuation \( w \) of \( J \) corresponding to the simple factor \( (y, x^2) \).

(2) Let \( m = (x, y) \), \( I = m^2 \) and \( J = (x^2, y)m \). Then \( y^2 \in J \) is part of a minimal reduction for \( I \), but not for \( J \) since \( w(y^2) = 4 > w(J) = 3 \) for the Rees valuation \( w \) of \( J \) corresponding to the simple factor \( (x^2, y) \).

We will discuss more about \( e(J) - e(I) \) at the end of Section 3.

### 3. Factorizations of adjacent integrally closed ideals

In Lemma 1.1, we showed that for any complete ideal \( I \), there exist adjacent complete ideals right above and below \( I \). If \( J \) is a simple complete ideal, then a complete ideal \( I \) right above \( J \) is either simple or the product of two simple complete ideals (cf. [2, Theorem 3.1], [5, Corollary 4.4]). For a prime divisor \( w \) of the second kind on \( R \), the set of \( w \)-ideals form an infinite descending chain of \( m \)-primary ideals in \( R \). For more properties of the sequence of valuation ideals of prime divisors of the second kind we refer to [9]. Now we show that if \( J \) is a simple complete ideal with the prime divisor \( w \) and \( I \) is a complete ideal right above \( J \), then \( I \) is the unique complete ideal right above \( J \) which is the immediate predecessor of \( J \) as a \( w \)-ideal. Later in this section we further show that this is not true if \( J \) is not simple.

**Theorem 3.1.** Let \( J \) be a simple integrally closed ideal. Then there is a unique integrally closed ideal right above \( J \).

**Proof.** Let \( I \) be a complete ideal right above \( J \). Let \( w \) be the prime divisor associated to \( J \) and \((W, m(W), k(w))\) be the valuation ring of \( w \). Since \( J \) is simple, \( o(I) = o(J) \) by Lemma 1.2. Let \( o(I) = r \). Choose an element \( a \in J \) of order \( r \) which is part of minimal generating sets of minimal reductions of \( I \) and \( J \) as in Lemma 2.7. In Lemma 2.7, we also showed that \( I \) and \( J \) go along the same quadratic sequence

\[
(R, m) = (R_0, m_0) \supseteq (R_1, m_1) \supseteq \cdots \supseteq (R_r, m_r),
\]

where

- \( m_i = \max(R_i) \),
- \( I_i = \) transform of \( I \) in \( R_i \),
- \( J_i = \) transform of \( J \) in \( R_i \),
- \( o_i = m_i \)-adic order valuation,
- \( J_r = m_r \),
- \( o_r = w \).
Furthermore, $\lambda(I_i/J_i) = 1$ for $i = 0, \ldots, t - 1$ and $I_{t-1} = m_{t-1}$. By Lemma 1.11 in [7],

$$w(I) = w(I_t) + \sum_{i=0}^{t-1} o_i(I_i) w(m_i),$$
and

$$w(J) = w(J_t) + \sum_{i=0}^{t-1} o_i(J_i) w(m_i).$$

Since $o_i(I_i) = o_i(J_i)$ for $i = 0, \ldots, t - 1$, we have

$$w(J) - w(I) = w(J_t) - w(I_t) = 1 - 0.$$

Therefore, $I \subseteq L$, where $L$ is the immediate predecessor of $J$ as a $w$-ideal. Since $\lambda(L/J) = 1$ (cf. [9, Theorem A.2]), we have $I = L$. Therefore, $I$ is the $w$-ideal such that $w(I) - w(J) = 1$, i.e., the immediate predecessor of $J$. Therefore, $I$ is the unique complete ideal right above $J$. □

Now we show that Theorem 3.1 cannot be generalized for higher powers of $J$. In fact we show that there are infinitely many complete ideals right above $J^n$ for $n \geq 2$.

**Theorem 3.2.** Let $J$ be a simple integrally closed ideal. Then there are infinitely many integrally closed ideals right above $J^n$ for $n \geq 2$.

**Proof.** Let $w$ be the prime divisor associated to $J$ and $(W, m(W), k(w))$ be the corresponding valuation ring of $w$. Suppose $J = m$ and $m = (x, y)$. Then for some unit $\alpha$, $I_n = (m^n, (x - \alpha y)^{n-1})$ is an integrally closed ideal by Proposition 2.2 in [4]. Clearly $mI_n \subseteq m^n$ since $o((x - \alpha y)^{n-1}) = n - 1$. Therefore, $\lambda(I_n/m^n) = 1$. Hence there are infinitely many such $I_n$ right above $m^n$. Now assume $J \neq m$. Take the quadratic sequence of $R$ along $W$

$$(R, m) = (R_0, m_0) \subseteq (R_1, m_1) \subseteq \cdots \subseteq (R_t, m_t),$$

where $w$ is the $m_i$-adic order valuation and $J_i = m_i$. Let $m_i = (x_i, y_i)$. Let $v_\alpha$ be the prime divisor of $R_i$ associated to the simple integrally closed ideal $J_\alpha = (m_i^{\alpha+1}, (x_i - \alpha y_i)^{\alpha})$ for some unit $\alpha$. Then $v_\alpha(x_i - \alpha y_i) = n + 1$ and $v_\alpha(y_i) = n$. Note that $J_\alpha$ is simple integrally closed by [4, Proposition 2.2]. By Corollary 1.3 in [9], $m^n_i$ is a $v_\alpha$-ideal of $v_\alpha$-value $n^2$. Consider the ideal $K_\alpha = (m^n_i, (x_i - \alpha y_i)^{n-1})$. Then $v_\alpha(K_\alpha) = n^2 - 1 = v_\alpha(m^n_i) - 1$ and $\lambda(K_\alpha/m^n_i) = 1$. Since the sequence of $v_\alpha$-ideals from $J_\alpha$ to $m_i$ is saturated [9, Theorem A.2], $K_\alpha$ is the immediate predecessor of $m^n_i$ as a $v_\alpha$-ideal. Let $J_\alpha$ be the inverse transform of $J_\alpha$ in $R$. Since $(J_i)^n$ is a $v_\alpha$-ideal, $J^n$ is a $v_\alpha$-ideal by [11, Theorem (D), p. 390]. $J_\alpha$ is the simple
integrim closed ideal in $R$ associated to $u_a$ and $J^n \supseteq \tilde{J}_n$ by [11, Theorem (D), p. 390]. Since the sequence of all the $u_a$-ideals from $m$ to $J_a$ is saturated [9, Theorem A.2], there exists a $u_a$-ideal $I_a \supseteq J^n$ such that $\lambda(I_a/J^n) = 1$. In the proof of Lemma 2.7, we showed that $I_a$ and $J^n$ go along the same quadratic sequence $R \supset R_1 \supset \cdots \supset R_t$ as follows: If $(I_a)_i$ and $J_i$ are the ideal transforms of $I_a$ and $J$ in $R_i$, then $\lambda((I_a)_i/J_i)^n = 1$ for $0 \leq i \leq t$. By [11, Theorem (D), p. 390], $(I_a)_i$ and $(J_i)^n$ are $u_a$-ideals for $0 \leq i \leq t$. Since $(J_i)^n = m_i^n$, $(I_a)_i$ is the $u_a$-ideal right above $m_i^n$, i.e., $(I_a)_i = (m_i^n, (x_i - \alpha y_i)^{n-1}) = K_a$. Since $I_a$ is a $u_a$-ideal, $I_a$ is complete. By taking different $\alpha$’s, we can see that there are infinitely many integrally closed ideals right above $J^n$ if $n \geq 2$. □

From now on we consider the cases when $J$ has more than one simple factor. For two complete ideals $I \supset J$ and for an arbitrary complete ideal $L$, we can see that $\lambda(IL/JL) = (J \cdot L) - (I \cdot L) + \lambda(I/J)$ by [7, Corollary 3.7]. Therefore, for adjacent complete ideals $I \supset J$, $\lambda(IL/JL) = 1$ if and only if $((I \cdot L) = (J \cdot L)$, i.e., $u(I) = u(J)$ for all $u \in T(L)$. In Lemma 1.6 of [9], we showed that for adjacent complete ideals $I \supset J$ and for $u \in T(J)$, we have $u(I) = u(J)$ if $u \in T(J)$. Further we ask what $u(J) - u(I)$ is for $u \not\in T(I) \cup T(J)$? We give an equivalent condition for $I \supset J$ and $L$ to have $(J \cdot L) = (I \cdot L)$ for the case $J$ is simple.

**Lemma 3.3.** Let $I \supset J$ be adjacent complete ideals and $J$ be simple and $T(J) = \{w\}$. Let $L$ be another simple complete ideal and $T(L) = \{u\}$. Then $(I \cdot L) = (J \cdot L)$ if and only if $J$ is not a $u$-ideal.

**Proof.** ($\Rightarrow$) Suppose $J$ is a $u$-ideal. Let

$$(R, m) \subset (R_1, m_1) \subset \cdots \subset (R_s, m_s)$$

be the quadratic sequence along $u$, where $u$ is the $m_s$-adic order valuation. Let $I_j$, $L_j$, and $J_j$ be the transforms of $I$, $L$, and $J$ in $R_j$ for each $j$. Let

$$L = L^{(s)} \subset L^{(s-1)} \subset \cdots \subset L^{(1)} \subset m$$

be all the simple $u$-ideals in $R$ as in [11, Theorem (F), Appendix 5]. Then $J = L^{(i)}$ for some $i \leq n - 1$ since $J$ is a $u$-ideal. Furthermore, $J_i = m_i$ and $J_j = R_j$ for $j \geq i + 1$, i.e., $P(J) = \{R_0, R_1, \ldots, R_t\}$. However, $L_j \neq R_j$ for $j \geq i + 1$, i.e., $o_j(L_j) \neq 0$ for all $j$. By Theorem 3.1, $P(I) \subseteq P(J)$ and

$$(I \cdot L) = \sum_{0 \leq j \leq i} o_j(I_j) o_j(L_j), \quad (J \cdot L) = \sum_{0 \leq j \leq i} o_j(J_j) o_j(L_j).$$

Since $o_j(L_j) > 0$ and $o_j(I_j) = o_j(J_j) - 1$ (Lemma 2.7), therefore $(J \cdot L) \neq (I \cdot L)$.

($\Leftarrow$) Suppose $(J \cdot L) \neq (I \cdot L)$. Let
be the quadratic sequence along \( w \). Since \( B(J) = B(I) \cup \{1\} \), \( (I \cdot L) \neq (J \cdot L) \) implies that \( L_i \neq R_i \), where \( L_i \) is the ideal transform of \( L \) in \( R_i \). Since \( L \) is simple, \( L_i \) is a simple integrally closed ideal associated to the prime divisor \( u \) of \( R_i \) and \( o_i(L_i) > 0 \). Since \( u \) dominates \( R_i \), therefore \( m_i \) is a \( u \)-ideal. Since \( J \) is the inverse transform of \( m_i \) in \( R \), \( J \) is also a \( u \)-ideal by [11, Theorem (D), Appendix 5].

By Lemma 3.3, we give a sufficient condition for \( J \) to have more than one complete ideal right above \( J \) when \( J \) has more than one simple factors.

**Theorem 3.4.** Let \( J = J_1 J_2 \ldots J_n \) be a unique factorization of \( J \). If \( J_i \) is not a \( w_i \)-ideal for any \( j \neq i \), then there exist at least \( n \) integrally closed ideals right above \( J \).

**Proof.** Let \( I_1, \ldots, I_n \) be the unique integrally closed ideals right above \( J_1, \ldots, J_n \) respectively. Since \( J' = J_1' \ldots J_n' R' \) for any base point \( R' \) of \( J \), \( (I_1 \cdot (J_2 \ldots J_n)) = \sum_{i=2}^{n} (I_1 \cdot J_i) \). Similarly, \( (J_1 \cdot (J_2 \ldots J_n)) = \sum_{i=2}^{n} (J_i \cdot J_1) \). Therefore,

\[
\lambda(I_1 J_2 \ldots J_n / J_1 J_2 \ldots J_n) = (J_1 \cdot (J_2 \ldots J_n)) - (I_1 \cdot (J_2 \ldots J_n)) + \lambda(I_1 / J_1) = \left[ \sum_{i=2}^{n} ((J_1 \cdot J_i) - (I_1 \cdot J_i)) \right] + 1.
\]

Therefore, for \( i = 2, \ldots, n \),

\[
\lambda(I_1 J_2 \ldots J_n / J_1 J_2 \ldots J_n) = 1 \iff (I_1 \cdot J_i) = (J_1 \cdot J_i) \\
\iff J_1 \text{ is not a } w_i \text{-ideal} ,
\]

by Lemma 3.3. Similarly, we can see that

\[
\lambda(J_1 J_2 \ldots I_j \ldots J_n / J_1 J_2 \ldots J_i \ldots J_n) = 1 \iff J_i \text{ is not a } w_j \text{-ideal} ,
\]

for all \( j \neq i \). For \( i \neq j \), \( J_1 \ldots I_i \ldots J_n \neq J_1 \ldots J_i \ldots J_n \). Since if it were, then \( J_i \mid I_i \) and \( J_i \mid I_j \) since \( J_i \neq J_j \), which contradicts to \( I_i \supset J_i \) and \( I_j \supset J_j \). Therefore,

\[
J_1 \ldots I_i \ldots J_n \neq J_1 \ldots J_i \ldots J_n ,
\]

for \( i \neq j \). Therefore, there are at least \( n \) integrally closed ideals right above \( J = J_1 J_2 \ldots J_n \).
**Question 3.5.** What if $J = J_1^{s_1} J_2^{s_2} \cdots J_n^{s_n}$ with some $s_i > 1$ in Theorem 3.4? By Theorem 3.2, we can see that there are infinitely many complete ideals right above $J_1^{s_1}$ if $s_1 > 1$. Assume $J_1 \neq m$ and $s_1 \geq 2$. Let $I$ be one of the complete ideals right above $J_1^{s_1}$. Let $L = J_2^{s_2} \cdots J_n^{s_n}$. Then

$$\lambda (IL/J) = \lambda (IL/J_1^{s_1} L)$$

$$= (J_1^{s_1} \cdot L) - (I \cdot L) + \lambda (I/J_1^{s_1})$$

$$= \left[ \sum_{i=2}^{n} \left( (J_1^{s_1} \cdot J_i^{s_i}) - (I \cdot J_i^{s_i}) \right) \right] + 1$$

$$= \left[ \sum_{i=2}^{n} \left( s_i (J_1^{s_1} \cdot J_i^{s_i}) - s_i (I \cdot J_i^{s_i}) \right) \right] + 1.$$ 

Therefore, $\lambda (IL/J) = 1 \Leftrightarrow (J_1^{s_1} \cdot J_i) = (I \cdot J_i) \Leftrightarrow w_i (I) = w_i (J_i^{s_i})$ for $T(J_i) = \{ w_i \}$ and for $i = 2, \ldots, n$. Hence we leave a question: for adjacent complete ideals $I \supset J$, what is $v(J^n) - v(I)$ for a prime divisor $v \not\in T(I) \cup T(J)$? We answered the question for the case $n = 1$ in Lemma 3.3.

From now on we consider the complete ideals which are divisible by the maximal ideal $m$. Since we already showed that there are infinitely many complete ideals right above $m^n$ in Theorem 3.2, we consider the complete ideals which have more simple factors in addition to the maximal ideal. Let $I$ be a simple complete ideal and consider the complete ideals right above $m^n J$. The orders of those are either $\omega (m^n J)$ or $\omega (m^n J) - 1$. We consider each case separately.

**Lemma 3.6.** Let $I$ be a simple integrally closed ideal of order $r$. Then there is a unique integrally closed ideal of order $n + r$ right above $m^n J$ for all $n \geq 1$.

**Proof.** Let $I$ be an integrally closed ideal of order $n + r$ such that $\lambda (I/m^n J) = 1$. Since $\omega (I) = \omega (m^n J)$ and $I$ is simple,

$$\deg (c(I)) \leq \deg (c(m^n J))$$

$$= \deg (c(J))$$

$$= \omega (J) = r.$$ 

Therefore, $I = m^n M$ for some integrally closed ideal $M$ by Lemma 1.2. Hence

$$\lambda (I/m^n J) = \lambda (m^n M/m^n J)$$

$$= (m^n \cdot J) - (m^n \cdot M) + \lambda (M/J)$$

$$= n (\omega (J) - \omega (M)) + \lambda (M/J) = 1.$$
implies that $\lambda(M/J) = 1$ since $o(M) = o(J)$. By Theorem 3.1, $M$ is the unique integrally closed ideal right above $J$. Therefore, $I = m^sM$ is the unique integrally closed ideal right above $m^sJ$ such that $o(I) = o(m^sJ)$. \[\square\]

The above lemma fails if $J$ is not simple as one can see in the Lemma 3.8.

**Corollary 3.7.** Let $J = L_1L_2\cdots L_s$ be the unique factorization of $J$ for $s > 1$. Let $T(L_i) = \{u_i\}$ for $i = 1, \ldots, s$. Assume $m \neq I_i$ for any $i$. If $I_i$ is not a $u_i$-ideal for $j \neq i$, then there exist at least $s$ complete ideals right above $m^sJ$ of order $o(m^sJ)$.

**Proof.** Use Theorem 3.4. \[\square\]

On the contrary to the uniqueness in Lemma 3.6, we show that there are infinitely many complete ideals of order $o(m^sJ)$ right above $mJ^s$.

**Lemma 3.8.** Let $J$ be a simple complete ideal of order $r$. Then there are infinitely many complete ideals of order $o(m^sJ)$ right above $mJ^s$ for all $n \geq 1$.

**Proof.** By Theorem 3.2, there exist infinitely many complete ideals $K_a$ right above $J^s$. Then $o(K_a) = o(J^s)$ by Lemma 1.2. Let $I_a = mK_a$. Then $I_a = \tilde{I}_a$ by ZPT and $\lambda(I_a/mJ^s) = (m \cdot J^s) - (m \cdot K_a) + \lambda(K_a/J^s) = 1$ since $K_a \supset J^s$ are adjacent ideals of the same order. Therefore, $I_a$ is a complete ideal right above $mJ^s$ such that $o(I_a) = nr + 1 = o(m^sJ)$. \[\square\]

In Lemma 3.6 and Lemma 3.8, we considered the ideals right above $m^sJ$ and $mJ^s$ which have the same orders as those of $m^sJ$ and $mJ^s$. We further consider the complete ideals right above $m^sJ$ and $mJ^s$ whose orders are 1 less than these of $m^sJ$ and $mJ^s$.

In Lemma 3.7 of [4], Huneke and Sally showed that if $J$ is a simple complete ideal of order $r$ with the prime divisor $w$, then there exists a minimal generating set $a_1, \ldots, a_{r+1}$ of $J$ such that $(a_i, a_{i+1}) = J$, $w(a_i) = w(a_{i+1}) = w(J)$, and $w(a_i) > w(J)$ for $i = 1, \ldots, r - 1$. In Theorem 1.5 of [9], we showed that we can choose those $a_i$'s so that $w(a_i) = w(J) + i$ for $i = 1, \ldots, r - 1$. Now we further show that we can choose minimal generating sets of $m^sJ$ and $mJ^s$ from $a_i$'s.

**Lemma 3.9.** Let $J$ be a simple complete ideal of order $r$. Then there exist a minimal generating set $a_1, \ldots, a_r = a, b$ of $J$ such that \{(a_1, \ldots, a_r)x^n, b(x, y)^n\} is a minimal generating set of $m^sJ$ and \{(a, b)^{n-1}(a_1, \ldots, a_r)x, b^n(x, y)\} is a minimal generating set of $mJ^n$ for $m = (x, y)$.

**Proof.** We only consider the case $n = 1$ since the same argument works for the cases $n > 1$ for $m^sJ$. Let $w$ be the prime divisor associated to $J$ and let $w(J) = e$. As in the proof of [9, Theorem 1.5], choose $a_i$'s $\in J$ and $x, y \in m \setminus m^2$ so that
$J = (a_1, \ldots, a_r = a, a_{r+1} = b)$ and $m = (x, y)$ as follows; for $i = 1, \ldots, r - 1,$

$$w(x) = r, \quad w(y) = r + s \quad \text{for } s \geq 1,$$

$$w(a_i) = e + i,$$

$$(a, b) = J,$$

$$o(b) = r,$$

$$J_i = (J_{i+1}, a_i),$$

where $J = J_0 \supset J_1 \supset \cdots \supset J_r = mJ$ is the sequence of $w$-ideals from $J$ to $mJ.$ Now we claim that $a_1x, \ldots, a_rx, bx, by$ is a minimal generating set of $mJ.$ Suppose

$$\left[ \sum_{i=1}^{r} \alpha_i(a_i) \right] + \alpha_{r+1}(bx) + \alpha_{r+2}(by) \in mJ.$$

Since $o(by) = r + 1$ and $o(m^2J) = r + 2,$ we have $\alpha_{r+2} \in m.$ Hence we have $\sum_{i=1}^{r+1} \alpha_i(a_i) \in m^2J.$ Since $w(x) = w(m) = r,$ we have

$$w\left( \sum_{i=1}^{r+1} \alpha_i(a_i) \right) = w(m^2J) = e + 2r$$

$$\Rightarrow w\left( \sum_{i=1}^{r+1} \alpha_i(a_i) \right) \geq e + 2r - r = e + r$$

$$\Rightarrow \sum_{i=1}^{r+1} \alpha_i(a_i) \in mJ.$$

Note that $mJ$ is a $w$-ideal of $w$-value $e + r$ [9, Theorem 1.5]. Therefore, $\alpha_i \in m$ for $i = 1, \ldots, r + 1$ since $a_1, \ldots, a_{r+1}$ are a minimal generating set of $J.$ Therefore the images of $a_1x, \ldots, a_rx, bx, by$ in $mJ/m^2J$ are linearly independent. Therefore these $r + 2$ elements $a_1x, \ldots, a_rx, bx, by$ are a minimal generating set of $mJ$ since $\mu(mJ) = o(mJ) + 1 = r + 2$ (cf. [11, ZPT], [4, Theorem 2.1]).

By Proposition 5.5 of [8], we can see that \{(a, b)^{n-1}(a_1, \ldots, a_r), b^n\} is a minimal generating set of $J^n$ since $\mu(J^n) = nr + 1.$ By the same token as above, we can show that \{(a, b)^{n-1}(a_1x, \ldots, a_rx), b^n x, b^ny\} is a minimal generating set of $mJ^n$ since $\mu(mJ^n) = nr + 2,$ $o(b) = r,$ $w(x) = r,$ and $mJ^n$ is a $w$-ideal [9, Theorem 2.4].

By using the above lemma, we can show that there are infinitely many contracted (may or may not be complete) ideals right above $m^nJ$ whose orders are 1 less than that of $m^nJ.$

**Lemma 3.10.** Let $J$ be a simple complete ideal of order $r.$ Then there are infinitely many contracted ideals of order $o(m^nJ) - 1$ right above $m^nJ.$
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Proof. Let \( J = (a_1, \ldots, a_r, a, b) \) and \( m^n J = (a, x^n, \ldots, a, x^n, b(x, y)^n) \) as in Lemma 3.9. Further let \( I_a = (a x^n, \ldots, a x^n, b(x, y) x^n) \) for some unit \( a \) of \( R \). Then \( \lambda(I_a/m^n J) = 1 \) since \( I_a = (m^n J, b(x - \alpha y)^n) \) and \( b(x - \alpha y)^n \cdot m \subseteq m^n J \). Furthermore, \( I_a \) is contracted since \( \mu(I_a) = r + n \) and \( o(I_a) = r + n - 1 \) [4, Theorem 2.1]. □

If \( o(J) = 1 \), then ‘contracted’ can be replaced by ‘complete’ in the statement of Lemma 3.10.

Lemma 3.11. Let \( J \) be a simple complete ideal of order 1. Then there are infinitely many complete ideals of order 1 right above \( mJ \).

Proof. If \( J = m \), then \( I_a = (y - \alpha x, x^2) \), for some unit \( \alpha \) of \( R \), is an integrally closed ideal right above \( m^2 \). Hence we are done. Assume \( J \neq m \). Since \( o(J) = 1 \), either \( J \subseteq (x - \beta y, y^2) \) or \( J \subseteq (y - \alpha x, x^2) \) for some units \( \alpha, \beta \) of \( R \). Since \( J \) is complete, \( \mu(J) = o(J) + 1 = 2 \). Let \( J = (y + \alpha x + \alpha x^2 + \cdots + \alpha_{n-1} x^{n-1} + a_n x^n) \) and \( M = (y + \alpha x + \alpha x^2 + \cdots + \alpha_{n-1} x^{n-1} + a_n x^n, x^{n+1}) \). Then \( M \) is also a complete ideal right below \( J \). Hence \( mJ \subseteq M \subseteq J \) implies that \( \lambda(M/mJ) = 1 \) since \( \mu(J) = 2 \) and \( \lambda(J/M) = 1 \). Therefore there are infinitely many complete ideals right above \( mJ \). We can prove the case \( J \subseteq (x - \beta y, y^2) \) in the same way. □

In Lemma 3.10, we could not show whether the contracted ideal \( I_a \) is complete or not. In Proposition 2.2 in [4], Huneke and Sally completely classify which ideals of the form \( (f, m^n) \) are complete. From this proposition we can see that if \( (f, m^n) \) is adjacent to \( m^n \), then \( (f, m^n) \) is complete. However a contracted ideal right above a complete ideal needs not be a complete ideal in general. We also completely classify which adjacent ideals \( (f, J^n) \) are complete for a simple complete ideal \( J \) and for any \( n \geq 1 \).

Proposition 3.12. Let \( J \) be a simple complete ideal and \( I = (J^n, f) \) be an \( m \)-primary ideal right above \( J^n \) and \( T(J) = \{ w \} \). Then \( I = \tilde{I} \) iff \( w(f) = w(J^n) - 1 \).

Proof. We showed the proposition for the case \( n = 1 \) in the proof of Theorem 3.1. Therefore assume \( n \geq 2 \). Lemma 1.2, \( o(I) = o(J^n) \). Let \( o(J) = r, w(J) = e \). Suppose \( w(f) = ne - 1 \). Let

\[
(R = R_0, m = m_0) \subseteq (R_1, m_1) \subseteq \cdots \subseteq (R_t, m_t)
\]

be the quadratic sequence along \( w \). And let \( I_i, J_i \) be the transforms of \( I, J \) in \( R_i \). Then \( B(J) = \{ r_i = o_i(J_i) \mid i = 0, \ldots, t \} \). Use induction on \( \text{rk}(J) = t \) (cf. [11, Theorem (F), Appendix 5]). Let \( t = 0 \), i.e., \( J = m \) and \( I = (m^n, f) \). Since \( o(f) = n - 1 \), \( I = \tilde{I} \) by [4, Proposition 2.2]. Assume \( t > 0 \) and let \( M \) be a \( w \)-ideal of \( w \)-value \( ne - 1 \), i.e., \( M \) is the predecessor of \( J^n \) as a \( w \)-ideal. Then the transform
$M_1$ of $M$ in $R_1$ is also a $w$-ideal by [11, Theorem (D), Appendix 5]. Since $w(f) = ne - 1$, $f \in M$. Suppose $R_1 - R[m/x]_{(y/x, z)}$. By Theorem 3.2 in [9], $o(M) = o(J^n) = nr$. Since $o(I) = nr$, we have $M_1 \supset I_1 = (J^n_1, f_1) \supset J^n_1$, where $fR_1 = x^n f, R_1$. Since $w(f) = ne - 1$, we have

$$w(f_1) = w(f) - nr^2 = ne - 1 - nr^2$$

$$= n \left( \sum_{i=0}^{t} r_i^2 \right) - 1 - nr^2 = n \left( \sum_{i=1}^{t} r_i^2 \right) - 1$$

$$= ne_1 - 1,$$

where $e_1 = e(J_1)$. Recall that $e(J) = (J \cdot J) = \sum_{i=0}^{t} r_i^2$ ([6, (23.3)], [7, (3.8)]). Therefore $f_1 \notin J_i^n$ and $f_1 \in J_i^n$ since $w(f_1) = ne_1 - 1 \geq w(J^n_1 : m_1) = ne_1 - r_1$, where $r_1 = o(J_1)$. Note that $J_i^n : m_1$ is a $w$-ideal since $J_i^n$ is a $w$-ideal in $R$, [11, Lemma 1, Appendix 3]. By induction hypothesis, $I_1 = I_1$ since $r_k(J_1) < t$ and $J_1$ is the simple complete ideal corresponding to $w$ in $R_1$. Therefore $I = I$ by ZPT and [11, Proposition 5, Appendix 5].

Conversely suppose $I = I$. Use induction on $r_k(J) = t$ (cf. [11, Theorem (F), Appendix 5]) to show that $w(f) = ne - 1$. Suppose $t = 0$, i.e., $J = m$. Then $w(f) = o(f) = n - 1$ since $I \supset m^n$ are adjacent. Assume $t > 0$. By Lemma 2.7, $I_1 = (J^n_1, f_1) \supset J_i^n$ are adjacent. Furthermore, $I_1$ is complete since $I$ is complete by [11, Proposition 5, Appendix 5]. Hence by induction hypothesis, $w(f_1) = ne_1 - 1$ since $r_k(J_1) = t - 1 < t$. Therefore, $w(f) = w(x^n r) + w(f_1) = nr^2 + ne_1 - 1 = n(e_1 + r^2) - 1 = ne - 1$. $\square$

In Lemma 3.6, we showed that there exists a unique complete ideal of order $o(m^n J)$ right above $m^n J$. We can give an equivalent condition for an adjacent ideal $I = (m^n J, f)$ right above $m^n J$ to be complete if $o(I) = o(m^n J)$.

**Proposition 3.13.** Let $J$ be a simple complete ideal of order $r$ with $T(J) = \{w\}$. Let $I$ be an adjacent ideal right above $m^n J$ such that $o(I) = o(m^n J)$. Then $I = I$ if and only if $w(I) = w(m^n J) - 1$. Furthermore, such a complete ideal is unique.

**Proof.** Suppose $I$ is complete. By Lemma 3.6, $I = m^n M$ for a complete ideal $M$ which is the unique complete ideal right above $J$. Hence $w(I) = w(m^n M) = w(m^n J) - 1$.

Conversely, suppose $w(I) = w(m^n J) - 1$. Let $w(J) = e$. Since $o(I) = o(m^n J)$, $\deg(c(J)) \leq \deg(c(m^n J)) = r$. Therefore, $m^n \mid I$. Let $I = m^n L$ for some ideal $L$. Then $w(L) = e - 1$, i.e., $L \subseteq M$ for the predecessor $M$ of $J$ as a $w$-ideal. Therefore, $L = M$ since $\lambda(M:J) = 1$. Hence $I = m^n L = m^n M$ is complete by ZPT. $\square$

Now we consider the adjacent ideals of order $o(m^n J)$ right above $m^n J$. 
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Proposition 3.14. Let $J$ be a simple complete ideal of order $r$ with $T(J) = \{w\}$. Let $I$ be an adjacent ideal right above $mJ^n$ such that $o(I) = o(mJ^n)$. If $I = I$, then $w(I) = w(mJ^n) - 1$.

Proof. Let $w(J) = e$. By Lemma 1.2, $I = mL$ for some complete ideal $L$. Since $\lambda(I/mJ^n) = \lambda(mL/mJ^n) = (m \cdot J) - (m \cdot L) + \lambda(L/J^n) = o(J^n) - o(L) + \lambda(L/J^n) = 1$, we have $o(J^n) = o(L)$ and $\lambda(L/J^n) = 1$. Therefore, $L$ is a complete ideal right above $J^n$. Hence $w(L) = ne - 1$ by Proposition 3.12. Therefore, $w(I) = w(mL) = r + ne - 1 = w(mJ^n) - 1$. \(\square\)

In Proposition 3.13 (3.14, respectively), we showed that $w(I) = w(mJ^n) - 1$ ($w(I) = w(mJ^n) - 1$, respectively) for an adjacent complete ideal $I$ right above $mJ^n$ (respectively) if $o(I) = o(mJ^n)$ ($o(I) = o(mJ^n)$, respectively). These are not true if they have different orders. The following example is given by C. Huneke.

Example 3.15 (Huneke). Let $I = (x^r + y^{r+1}, m^n)$ for $m = (x, y)$, $n > r + 1$, and $r \geq 2$. Then $I$ is a simple complete ideal of order $r$ and $I : m = (x^r + y^{r+1}, m^{n-1})$ is also a simple complete ideal of order $r$. Furthermore, $I \supset m(I : m)$ are adjacent complete ideals such that $o(I) = o(m(I : m)) - 1$. Let $J = I : m$ and $T(J) = \{w\}$. Then $w \not\in T(I)$, $w(I) = r(n - 1)$, and $w(mJ) = r + r(n - 1)$. Therefore, $w(I) = w(mJ) - r < w(mJ) - 1$.

More generally we can show the following for the adjacent ideals of order $o(mJ^n) - 1$ right above $mJ^n$.

Proposition 3.16. Let $J$ be a simple complete ideal of order $r$ with $T(J) = \{w\}$. Let $I$ be an adjacent ideal right above $mJ^n$ of order $o(J^n)$. Then $w(I) = w(J^n)$.

Proof. Let $w(J) = e$ and $I_i$ be the $w$-ideal of $w$-value $e + i$ for $i = 1, \ldots, r$ as in the proof of Lemma 3.9. Since $mJ : J, J/m(J : m)$, and $o(J : m) = r$ (cf. [7, (3.5)]). Therefore, $o(J : m) = r$ for all $J \subseteq J : m \subseteq J : m$. However, $J_i : m \neq J_{i+1} : m$ since $J_i$ and $J_{i+1}$ are different $w$-ideals for all $i$. Therefore, $\lambda(J_i : m/J_{i+1} : m) = 1$ for each $i$. That implies that $m(J_i : m) = J_i$ since $\mu(J_i : m) = \lambda(J_i : m/J_{i+1}) + \lambda(J/J_i) = r + 1$ (cf. [6, Theorem 1.5]). Therefore, $o(J_i) = r + 1$. Since $J_i \supset J_i \supset mJ$, $o(J_i) = r + 1$ for all $i$. Since $I$ is adjacent to $mJ^n$, $w(I) \geq ne$. However, $I \nsubseteq J, J^{n-1}$ for all $i$ since $o(I) = nr$ and $o(J, J^{n-1}) = nr + 1$ for all $i$. Since $J, J^{n-1}$ is a $w$-ideal of $w$-value $ne + i$ [9, Theorem 2.4], $w(I) < ne + i$ for all $i$. Therefore, $w(I) = ne$. \(\square\)

In Theorem 2.1 and Corollary 2.4, for adjacent complete ideals $I \supset J$, we showed that $e(J) - e(I) \leq o(J)$ and the equality holds if and only if $T(I) \cap T(J) = \emptyset$ and $w_j(J) - w_j(I) = o(J)$ for each Rees valuation $w_j$ of $J$ corresponding to the
simple factor $J$. This seems to be hardly true in general (cf. Questions 3.18), but we find a case in which the multiplicity difference is the possible maximum.

**Corollary 3.17.** Let $I,J$ be as in Proposition 3.16. Then $e(mJ^n) - e(I) = o(mJ^n)$.

**Proof.** By Theorem 2.1 and Proposition 3.16,

$$e(mJ^n) - e(I) = \left[ o(mJ^n) - o(I) \right] + n \left[ w(mJ^n) - w(I) \right]$$

$$= 1 + nr = o(mJ^n). \quad \square$$

**Questions 3.18.** (1) Is the converse of Proposition 3.14 true? That is, if $w(I) = w(mJ^n) - 1$, then is $I$ complete?

(2) Corollary 3.17 is not true for $m^nJ$. For example [2, p. 70], if $m = (x, y)$, $I = (y, x^2)(y^2 - x^3, x^4, x^3y)$, and $J = (x^3, x^2y, y')$, then $I \cap m^nJ$ are adjacent complete ideals such that $T(I) \cap T(m^nJ) = \emptyset$. But $w(m^nJ) - w(I) = 1 < o(J) = 2$ for $T(J) = \{w\}$. Hence $e(m^nJ) - e(I) < o(m^nJ)$ in this case.

(3) How many complete ideals of order $o(m^nJ) - 1$ do exist right above $m^nJ$?

(4) How many complete ideals of order $o(mJ^n) - 1$ do exist right above $mJ^n$?

(5) Let $I \supset J$ be adjacent complete ideals of the same order. For $w \in T(J) \setminus T(I)$, is it true that $w(I) - w(J) - 1$? If this is true, then $e(J) - e(I) < \deg(c(J))$ in Theorem 2.5.

(6) Is it possible to have adjacent complete ideals $I \supset J = J_1 \cup J_2^\prime$, where $T(I) \cap T(J) = \emptyset$ and $s_i > 1$ for $i = 1,2$?

(7) In Theorem 3.4, are there exactly $n$ complete ideals right above $J$?

**References**


