Sign Patterns That Require Repeated Eigenvalues

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ABSTRACT

Motivated by the question of which sign patterns allow a diagonalizable matrix, we relate a number of properties to that of requiring repeated eigenvalues. These are then used to make several observations about sign patterns that allow diagonalizability. The only barrier to diagonalizability is a required nontrivial Jordan structure associated with zero. We note that the question of sign patterns that require diagonalizability is also open.

INTRODUCTION

By a sign-pattern matrix we mean an $n$-by-$n$ array $B = (b_{ij})$ each of whose entries $b_{ij}$ is an element of the set $\{+, -, 0\}$. The sign-pattern class $Q(B)$ associated with $B$ consists of those $n$-by-$n$ real matrices $A = (a_{ij})$ such

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that \(a_{ij}\) is positive (respectively, negative, zero) if and only if \(b_{ij}\) is + (respectively, -, 0). Given a property \(P\) that a real matrix may enjoy, we say that the sign-pattern matrix \(B\) requires \(P\) if \(A \in Q(B)\) implies that property \(P\) holds for \(A\). Similarly, \(B\) allows \(P\) if there is a matrix in \(Q(B)\) for which \(P\) holds (i.e., \(B\) does not require not \(P\)).

The motivating issue for this note is the understanding of sign-pattern matrices \(B\) that allow the property of diagonalizability (by similarity). We do not give here a complete description of such sign patterns; however, it is clear that a sign-pattern matrix \(B\) allows diagonalizability if it does not require repeated eigenvalues. [We say that a sign pattern matrix \(B\) requires \(k\) repeated eigenvalues if every \(A \in Q(B)\) has an eigenvalue of algebraic multiplicity at least \(k\), and \(k\) is a minimum with respect to this requirement. Note that there is no predisposition in this definition as to the constancy of such an eigenvalue over \(A\)'s in \(Q(B)\).] Most of our observations in the next section are centered around properties related to repeated eigenvalues. In the final section we use our basic observations and some examples to discuss the issue of allowing diagonalizability and the broader problem of understanding possible Jordan forms in a sign-pattern class.

With an \(n\)-by-\(n\) sign pattern or conventional matrix we associate a directed graph on vertices \{1, \ldots, \(n\)\} in the usual way: an edge from \(i\) to \(j\) occurs in the directed graph if and only if the \(i, j\) entry of the matrix is not 0. A sequence of edges \((i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)\) in a directed graph is called a simple cycle of length \(k\) if the set \(\{i_1, \ldots, i_k\}\) contains no repeated vertices. A composite cycle of length \(k\) is a set of simple cycles whose total length is \(k\) and whose index sets are mutually disjoint. We may also refer to the product of (nonzero) entries from the associated matrix corresponding to the edges as a cycle. A matrix or graph is called \(k\)-cyclic if every cycle length is a positive integer multiple of \(k\). As a cycle corresponds to a term in the determinant of the principal submatrix associated with the indices/vertices of the cycle, it is clear that the characteristic polynomial \(p_A(\lambda)\) of a \(k\)-cyclic matrix \(A\) is of the form \(\lambda^r q(\lambda^k)\), in which \(q\) is a polynomial. A matching of cardinality \(k\) in a directed graph is a set of \(k\) edges among whose collective initial vertices there are no repetitions and among whose collective terminal vertices there are no repetitions. We call a matching a principal matching if the set of initial vertices is the same as the set of terminal vertices. Of course, a principal matching corresponds exactly to a cycle. Again the notion of a matching or a principal matching may be transferred to matrices in an obvious way. We note that some of the concepts we have mentioned also relate to “term rank” and “systems of distinct representatives,” whose definitions may be found in [1].

In order to simplify some of our statements, we define several quantities associated with a sign-pattern matrix \(B\). We denote the minimum rank
deficiency occurring among matrices $A \in Q(B)$ by $d(B)$; thus $d(B) = n - \text{maximum rank occurring in } Q(B)$. Similarly $D(B)$ is the maximum rank deficiency among matrices in $Q(B)$. When we wish to refer to the actual rank deficiency of a conventional matrix $A$, we also use $d(A)$. We denote the minimum algebraic multiplicity of the eigenvalue 0 occurring among matrices $A \in Q(B)$ by $z(B)$, and the maximum by $Z(B)$. If $A$ is a conventional matrix, we also use $z(A)$ for the multiplicity of the eigenvalue 0. Thus $z(B) \leq z(A) \leq Z(B)$ for $A \in Q(B)$, with equality possible at either extreme. Recall that the geometric multiplicity of an eigenvalue $\lambda$ of a square matrix $A$ is the number of Jordan blocks associated with $\lambda$ in the Jordan form of $A$. Let $g(B)$ be the minimum geometric multiplicity of 0 over all $A \in Q(B)$. Finally, we denote the maximum cycle length occurring in $B$ by $c(B)$. Note that $c(A) = c(B)$ for all $A \in Q(B)$.

OBSERVATIONS

Our purpose here is to sort out the relations among several algebraic and combinatorial concepts for sign-pattern matrices. Some of these are already known, via folklore or more explicitly, in other contexts, but we wish to give a self-contained treatment.

Our first observation, involves the necessity of a zero eigenvalue and motivates further discussion. Each of the implications (1.1) $\Rightarrow$ (1.2) $\Rightarrow$ (1.3) $\Rightarrow$ (1.4) $\Rightarrow$ (1.5) $\Rightarrow$ (1.6) $\Rightarrow$ (1.1) is straightforward, and most are generally known.

**Theorem 1.** For an $n$-by-$n$ sign-pattern matrix $B$, the following statements are equivalent:

among $p$-by-$q$ 0 submatrices of $B$, $\max(p + q) \geq n + 1$; (1.1)

$$d(B) \geq 1; \quad (1.2)$$

$$g(B) \geq 1; \quad (1.3)$$

$$z(B) \geq 1; \quad (1.4)$$

$$c(B) \leq n - 1; \quad (1.5)$$
and

\[ A \in Q(B) \text{ implies } \det A = 0. \quad (1.6) \]

Note that (1.1), (1.2), and (1.3) remain equivalent if we replace the inequalities by equalities. Of course \( z(B) \geq g(B) \), with strict inequality possible. Theorems 2 and 3 further clarify the relations between these quantities, but first we need an important lemma.

**Lemma.** If \( B \) is an \( n \)-by-\( n \) sign pattern matrix, then

(i) \( z(B) = n - c(B) \), and

(ii) there is a matrix \( A \in Q(B) \) with \( c(B) \) distinct nonzero eigenvalues

**Proof.** We first demonstrate (ii), so that to demonstrate (i) we need only show that \( z(B) \geq n - c(B) \).

Consider a cycle in \( B \) that attains \( c(B) \), and suppose that it consists of \( s \) simple cycles

\[
(i_{11}, i_{12}), \ldots, (i_{1k_1}, i_{11}) \\
\vdots \\
(i_{s1}, i_{s2}), \ldots, (i_{sk_s}, i_{s1})
\]

in which \( i_{11}, \ldots, i_{1k_1}, i_{s1}, \ldots, i_{sk_s} \in N = \{1, \ldots, n\} \) are distinct and \( k_1 + \cdots + k_s = c(B) \). Define a matrix \( E = (e_{ij}) \) in which

\[ e_{ij} = t \sgn b_{ij} \]

if \((i, j)\) is an edge in the \( t \)th cycle, \( t = 1, \ldots, s \), and \( e_{ij} = 0 \) otherwise. Then, \( t \) times each of the \( k \)th roots of \( \pm 1 \) is an eigenvalue of \( E \); the sign \( \pm \) coincides with the product of the signs associated with the entries of \( B \) in the \( t \)th cycle. Thus \( E \) has \( c(B) \) distinct nonzero eigenvalues. Now, let \( A \in Q(B) \) be a sufficiently small perturbation of \( E \) so that assertion (ii) holds.

For assertion (i), recall that the coefficient of \( \lambda^{n-k} \) in the characteristic polynomial of \( A \in M_n \), is \( \pm \) the sum of the \( k \)-by-\( k \) principal minors of \( A \). Note that, for a \( k \)-by-\( k \) principal minor to be nonzero, it must include a \( k \)-cycle, which must be a \( k \)-cycle of \( A \). Thus, for \( k > c(B) \) every \( k \)-by-\( k \) principal minor of each \( A \in Q(B) \) is 0. We conclude that \( \lambda^{n-c(B)} \) divides the characteristic polynomial of \( A \in Q(B) \), that \( z(B) \geq n - c(B) \), and thus that \( z(B) = n - c(B) \), based upon (ii).
THEOREM 2. For an n-by-n sign-pattern matrix B, the following statement are equivalent for each positive integer k, with the understanding that the statement (2.2) is omitted in the case $k = 1$:

\[ z(B) = k; \]  \hspace{1cm} (2.1)

\[ B \text{ requires } k \text{ repeated eigenvalues}; \]  \hspace{1cm} (2.2)

\[ c(B) = n - k; \]  \hspace{1cm} (2.3)

and

a maximum-cardinality principal matching in $B$ has cardinality $n - k$. (2.4)

Proof. According to the lemma, $z(B) + c(B) = n$; hence, (2.1) and (2.3) are equivalent. Also, due to the lemma, a sign pattern may require only the eigenvalue 0 to be repeated. This is an important principle that we wish to emphasize; thus (2.1) and (2.2) are equivalent. The equivalence of (2.3) and (2.4) is entirely combinatorial. Any cycle yields a principal matching of the same cardinality, simply using the edges of the cycle. Conversely, the edges of a principal matching must be the edges of a cycle of like cardinality; start at any vertex of the principal matching and match the initial vertex of the next edge to the terminal vertex of the last. Restart after each simple cycle until all edges are exhausted.

It is clear that the following statement may also be added to the list in Theorem 2:

$A \in Q(B)$ and $A[\alpha]$ an $m$-by-$m$ principal submatrix of $A$, $m > n - k$,

imply \[ \det A = 0 \]  \hspace{1cm} (2.5)

THEOREM 3. For an n-by-n sign-pattern matrix B, the following statements are equivalent for each positive integer k:

\[ \text{among p-by-q 0 submatrices of } B, \quad \max( p + q ) = n + k; \]  \hspace{1cm} (3.1)

\[ d(B) = k; \]  \hspace{1cm} (3.2)

\[ g(B) = k; \]  \hspace{1cm} (3.2)

a maximum-cardinality matching in $B$ has cardinality $n - k$. (3.4)
Proof. As \( d(A) = g(A) \) for any conventional matrix \( A \) (see e.g. [2]), (3.2) and (3.3) are equivalent. The term rank of a matrix is the minimum number of lines (rows or columns) that cover all its nonzero entries. It is a classical fact [1, pp. 55–56] that the statement

\[
\text{the term rank of } B \text{ is } n - k
\]  

(3.5)

is equivalent to (3.4). The statement (3.1) may be seen to be equivalent to (3.5) by using all the lines \([2n - (n + k) = n - k]\) not in a maximum-cardinality zero submatrix; thus (3.1) and (3.4) are equivalent. Assuming (3.4), every \((n - k + 1)\)-by-\((n - k + 1)\) submatrix of \( A \in Q(B) \) has determinant equal to zero, so that \( d(B) \geq k \). By choosing the entries sufficiently (relatively) large associated with some maximum-cardinality matching, we may obtain \( A \in Q(B) \) with \((n - k)\)-by-\((n - k)\) nonsingular submatrix; hence rank \( A = n - k \), and \( d(B) = k \) [i.e. (3.2)]. On the other hand, assuming (3.2), there is an \( A \in Q(B) \) with rank \( A = n - k \) [and no \( A \in Q(B) \) with rank \( A = n - k + 1 \)]. Such an \( A \) must have an \((n - k)\)-by-\((n - k)\) submatrix with nonzero determinant. As this submatrix must have a nonzero term in its determinant, it (and therefore \( A \) and \( B \)) must enjoy a matching of cardinality \( n - k \). Any matching of cardinality \( n - k + 1 \) is precluded by the fact that rank \( A \) cannot be \( n - k + 1 \) if \( A \in Q(B) \) (otherwise, we would emphasize the entries associated with such a matching, as above). We conclude (3.4), to complete a proof.  

An explicit, self-contained proof of Theorem 3 was replaced with the above in deference to the editors. We note also that the following statements may be added to the list (3.1)-(3.5) of equivalent statements:

\[
\text{a system of distinct (row) representatives of the (columns of) } B \\
\text{has maximum cardinality } n - k z
\]

(3.6)

and

\[
A \in Q(B) \text{ and } A[\alpha, \beta] \text{ an } m\text{-by-}m \text{ submatrix of } A, \ m > n - k, \\
\text{imply } \det A[\alpha, \beta] = 0.
\]

(3.7)
DIAGONALIZABILITY

As noted earlier, if an $n$-by-$n$ sign-pattern matrix $B$ does not require repeated eigenvalues (i.e., does not require $k$ repeated eigenvalues for any $k \geq 2$), then it allows diagonalizability. Thus, by Theorem 2, if $c(B) \geq n - 1$, then $B$ allows diagonalizability. However, if $c(B) < n - 1$, $B$ may or may not allow diagonalizability; both imaginable situations occur. Using Theorem 3 along with Theorem 2, we may make an observation stronger than the sufficiency of $c(B) \geq n - 1$.

**Corollary 1.** If $B$ is an $n$-by-$n$ sign-pattern matrix for which there is a principal matching of maximum cardinality among all matchings, then $B$ allows diagonalizability.

**Proof.** Because of (2.4) and (3.4), the statements of Theorem 2 become equivalent to those of Theorem 3 under the hypothesis of the corollary. Then, (2.1) and (3.3) show that the zero eigenvalues are not a barrier to diagonalizability, while the construction of the lemma ensures a maximum number of distinct nonzero eigenvalues at the same time.

**Corollary 2.** For any $n$-by-$n$ sign pattern matrix $B$, there is a permutation matrix $P$ such that $PB$ allows diagonalizability.

**Proof.** Choose $P$ so that the row indices of a maximum-cardinality matching are permuted to the set of column indices of the matching, and apply Corollary 1.

**Example 1.** It is worth noting that the corresponding statement for conventional matrices is false. For example, both

$$
\begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}
\quad\text{and}\quad
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
-1 & -1
\end{pmatrix}
$$

are rank-1 nilpotent matrices.

Even when a sign pattern $B$ does not have a maximum-cardinality matching that is principal, $B$ may still allow diagonalizability.
**Example 2.** Let

\[
B = \begin{bmatrix}
0 & + & 0 & 0 & + \\
0 & 0 & + & 0 & 0 \\
+ & 0 & 0 & + & 0 \\
0 & + & 0 & 0 & + \\
0 & 0 & + & 0 & 0
\end{bmatrix}.
\]

Since \(c(B) = 3\), \(z(b) = 2\). However, \(B\) has a matching of cardinality 4 (\(((1, 2), (2, 3), (3, 4), (4, 5))\)) and, equivalently, maximal 0 submatrices of sizes 4 by 2 and 3 by 3, so that \(g(B) = 1\). Of course, no \(A \in Q(B)\) for which \(g(A)\) and \(z(A)\) are minimal is diagonalizable. Though the maximum rank in \(Q(B)\) is 4, the minimum rank is 3. In fact, for \(A = (a_{ij}) \in Q(B)\), rank \(A = 3\) if and only if

\[
\det \begin{bmatrix}
a_{12} & a_{15} \\
a_{42} & a_{45}
\end{bmatrix} = 0.
\]

In particular,

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix} \in Q(B)
\]

satisfies \(g(A) = 2\) and \(z(A) = 2\) and, since its nonzero eigenvalues are distinct, is diagonalizable.

**Example 3.** A variation upon the prior example is also of interest. Let

\[
B = \begin{bmatrix}
0 & + & 0 & 0 & - \\
0 & 0 & + & 0 & 0 \\
+ & 0 & 0 & + & 0 \\
0 & + & 0 & 0 & + \\
0 & 0 & + & 0 & 0
\end{bmatrix}.
\]

Again \(z(B) = 2\), but now \(D(B) = 1 = d(B)\) [i.e., \(A \in Q(B)\) implies rank \(A = 4\)]. Thus, \(g(A) = 1\) for all \(A \in Q(B)\), and \(B\) does not allow diagonalizability.
Observation. Let \( B \) be an \( n \times n \) \( k \)-cyclic sign-pattern matrix with \( n = mk + s, 0 \leq s < k \). Then

(i) \( z(B) \geq s \);
(ii) \( s \geq 1 \) implies \( d(B) \geq 1 \); and
(iii) \( s = 0 \) and \( d(B) \geq 1 \) implies \( z(B) \geq k \).

Proof. For \( A \in Q(B) \), the characteristic polynomial \( p_A(\lambda) = \lambda q(\lambda^k) \), for some polynomial \( q \). Each of the conclusions then follows easily.

Of course, for any sign pattern \( B \), \( d(B) \leq z(B) \). For a conventional matrix \( A \), \( d(A) < z(A) \) precludes diagonalizability of \( A \). As we have seen, \( d(B) < z(B) \) does not, by itself, preclude a sign-pattern matrix \( B \) from allowing diagonalizability. However, we can say the following.

Observation. Suppose that \( B \) is an \( n \times n \) sign-pattern matrix that allows diagonalizability. Then \( d(B) \leq z(B) \leq D(B) \).

Remarks. The problem of characterizing sign-pattern matrices that allow diagonalizability remains open. We have observed that it suffices to understand sign patterns for which no principal matching has maximum cardinality among all matchings, or, more precisely patterns \( B \) for which \( d(B) < z(B) \leq D(B) \). It is a reasonable working conjecture that the necessary condition \( z(B) \leq D(B) \) is also sufficient. However, this would mean that effective recognition of sign patterns that allow diagonalizability might be quite difficult. Obtaining \( D(B) \), in general, is equivalent to determining minimum rank in a sign-pattern class, a problem that has been open for some time and appears quite difficult. (Maximum rank is relatively easy, and all intermediate ranks are attained.) There may be some hope, however, in the fact that allowing diagonalizability is clear-cut for many patterns, and \( d(B) < z(B) \) requires some degree of sparsity, which may make the minimum-rank problem easier.

We note, however, that not all algebraic multiplicities (for the eigenvalue zero) lying between \( z(B) \) and \( Z(B) \) may be attained in \( Q(B) \).

Example 4. Let

\[
B = \begin{bmatrix}
0 & + & 0 & 0 \\
+ & 0 & + & 0 \\
0 & 0 & 0 & + \\
+ & 0 & + & 0
\end{bmatrix}.
\]
Then, for any $A \in Q(B)$, the characteristic polynomial $p_A(\lambda)$ has the form

$$p_A(\lambda) = \lambda^4 + c_2 \lambda^2 + c_1$$

with $c_2 < 0$ and $c_1$ ambiguous in sign. Thus $A$ may have 0 or 2 (when $c_1 = 0$) eigenvalues equal to zero, but not just one. We note more generally that if $B$ is an $n$-by-$n$ sign-pattern matrix for which the coefficient of $\lambda^j$ in the characteristic polynomial is combinatorially zero [i.e., 0 for each $A \in Q(B)$], then even if $z(B) < j < Z(B)$, there is no $A \in Q(B)$ such that $z(A) = j$.

Thus far, our remarks have been independent of whether the sign-pattern matrix $B$ is irreducible. If $B$ is reducible, some useful observations may be made. In this event, we may assume that $B$ is in Frobenius (irreducible) normal form:

$$B = \begin{bmatrix}
B_{11} & B_{12} & \cdots & B_{1k} \\
0 & B_{22} & \cdots & B_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & B_{kk}
\end{bmatrix},$$

in which each $B_{ii}$ is either 1-by-1 or irreducible. It is clear that

$$D(B) \leq \sum_{i=1}^{k} D(B_{ii}) \text{ and } d(B) \leq \sum_{i=1}^{k} d(B_{ii}), \text{ while } z(B) = \sum_{i=1}^{k} z(B_{ii}).$$

Furthermore, it is straightforward to see that if $B$ allows diagonalizability, then each of the $B_{ii}$ must allow diagonalizability, $i = 1, \ldots, k$. Unfortunately, the converse is false, and if each $B_{ii}$ allows diagonalizability and $\sum_{i=1}^{k} z(B_{ii}) \geq 2$, whether $B$ allows diagonalizability depends very much upon the $B_{ij}$, $i < j$. In some sense, it is more difficult for $B$ to allow diagonalizability, and it is an interesting question what the exact conditions on the $B_{ij}$ are. It must happen that for some $A \in Q(B)$ for which each $A_{ii}$ is diagonalizable, rank $A = \sum_{i=1}^{k} \text{rank } A_{ii}$. 
We also note that the problem of characterizing sign patterns that require diagonalizability is an interesting open one too. It is possible to identify a variety of sign patterns that do, and simple cycle lengths and overlap are clearly important.

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