Stability of Inequalities in the Dual Brunn-Minkowski Theory*

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Stability versions are given of several inequalities from E. Lutwak’s dual Brunn-Minkowski theory. These include the dual Aleksandrov-Fenchel inequality, the dual Brunn-Minkowski inequality, and the dual isoperimetric inequality. Two methods are used. One involves the application of strong forms of Clarkson’s inequality for $L^p$ norms that hold for nonnegative functions, and the other utilizes a refinement of the arithmetic-geometric mean inequality. A new and more informative proof of the equivalence of the dual Brunn-Minkowski inequality and the dual Minkowski inequality is given. The main results are shown to be the best possible up to constant factors.

Key Words: geometric tomography; star body; convex body; isoperimetric inequality; Aleksandrov-Fenchel inequality; Brunn-Minkowski inequality; arithmetic-geometric mean inequality; Clarkson’s inequality; dual mixed volume; stability.

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1. INTRODUCTION

The classical Brunn-Minkowski theory has proved extraordinarily useful in many ways. Its success in dealing with projections of convex bodies has made it a fundamental ingredient in geometric tomography, in which information about a geometric object is deduced from data concerning its projections or sections. The more recent dual Brunn-Minkowski theory, initiated by E. Lutwak [14] and developed by him and others, earns its place alongside the Brunn-Minkowski theory as an essential tool in geometric tomography by virtue of its success in dealing with sections of star bodies. Lutwak’s theory also generated the tools for the recent solution of the Busemann-Petty problem (see [1], [2], [22], [21]).

The term “dual Brunn-Minkowski theory” is appropriate since there is indeed a well-established, but not fully understood, duality between projections of convex bodies and sections of star bodies. See [3] and [4] for more information and surveys of geometric tomography and its applications and connections to stereology, geometric probing in robotics, crystallography, and other fields.

Perhaps the most widely known of the many inequalities from the Brunn-Minkowski theory is the celebrated Brunn-Minkowski inequality ([19, Section 6.1])

\[
V((1-t)K + tL)^{1/n} \geq (1-t)V(K)^{1/n} + tV(L)^{1/n},
\]

which holds for convex bodies \(K\) and \(L\) in \(\mathbb{E}^n\), where \(V(K)\) denotes the volume of \(K\), \(+\) stands for Minkowski or vector addition, and \(0 \leq t \leq 1\). (Section 2 explains other terms and notation.) Using the homogeneity property of volume, this can be written in the simpler but equivalent form

\[
V(K + L)^{1/n} - 2 \geq 0,
\]

where \(K\) and \(L\) are convex bodies in \(\mathbb{E}^n\) with \(V(K) = V(L) = 1\). The isoperimetric inequality is a very special case of (2); see [3, pp. 368–372]. Stronger forms of such inequalities, sometimes called stability versions, were found by Minkowski himself in some instances and have been of interest ever since. Groemer’s survey [10] provides a lucid account. For example, Groemer [8] proved that if \(K\) and \(L\) have volume 1 and the same centroid, then

\[
V(K + L)^{1/n} - 2 \geq \alpha_n \frac{1}{D^{n+1}} \delta(K, L)^{n+1},
\]

where \(\delta\) is the Hausdorff metric, \(D\) is the maximum of the diameters of \(K\) and \(L\), and \(\alpha_n\) is an explicit constant depending only on \(n\).
Corresponding to (2) in the dual Brunn-Minkowski theory is the *dual Brunn-Minkowski inequality* ([14] or [3, p. 374])

\[
2 - V(M \uplus N)^{1/n} \geq 0,
\]  

which holds for star bodies \( M \) and \( N \) in \( \mathbb{R}^n \) with \( V(M) = V(N) = 1 \), where \( \uplus \) denotes radial addition. There is a dual isoperimetric inequality, stability versions of which were presented in [5], that is a special case of (4) (see [3, p. 373]). Here we obtain the following stability version for convex bodies \( K \) and \( L \) in \( \mathbb{R}^n \) of volume 1:

\[
2 - V(K \uplus L)^{1/n} \geq \tilde{\alpha}_n \frac{r^{2n+1}}{R^{2n+2}} \tilde{d}(K, L)^{n+1},
\]

where \( \tilde{d} \) is the radial metric, \( K \) and \( L \) contain a ball of radius \( r \) centered at the origin and are contained in a ball of radius \( R \) centered at the origin, and \( \tilde{\alpha}_n \) is an explicit constant depending only on \( n \).

We stress that the assumption concerning the volume of the convex bodies \( K \) and \( L \) in (3) and (5) occasions no real loss in generality. General versions of these inequalities can easily be obtained from them using the homogeneity property of volume, and the normalization used here is simply a matter of convenience.

The (necessary) restriction to convex bodies in inequality (5) is an artifact of the metric used on the right-hand side. In Theorem 3.6 below we offer a stability version of (4) that holds for all star bodies in \( \mathbb{R}^n \). We obtain this by means of an inequality for nonnegative functions in \( L^p, p \geq 2 \); see Lemma 3.3 below. The latter inequality is closely related to Clarkson's inequality (see, for example, [11, p. 225])

\[
\|f - g\|_p^p + \|f + g\|_p^p \leq 2^{p-1}(\|f\|_p^p + \|g\|_p^p)
\]

for \( f, g \in L^p, p \geq 2 \). Indeed, a weaker form of Theorem 3.6 could be derived from Clarkson's inequality. These matters are discussed at length in Section 3.

The Brunn-Minkowski inequality is a consequence of the Aleksandrov-Fenchel inequality (see, for example, [19, Section 6.3]). A nother is Minkowski's (first) inequality

\[
V_1(K, L) - 1 \geq 0,
\]

where \( K \) and \( L \) are convex bodies in \( \mathbb{R}^n \) with \( V(K) = V(L) = 1 \); see [19, Section 6.2]. Here \( V_1(K, L) \) denotes the mixed volume of \( K \) and \( L \) in which \( K \) appears \( (n - 1) \) times. A stability version of (7) was found by Groemer [9]. In Section 4 we employ a refinement of the arithmetic-geometric mean inequality to produce a stability version of the dual Aleksandrov-Fenchel inequality (see Theorem 4.3). (A stability version of a restricted form of the
Aleksandrov-Fenchel inequality was proved by Schneider [18]. From this stability versions of other inequalities flow, including one for the following dual of (7):

$$1 - \tilde{V}_1(M, N) \geq 0,$$

where $M$ and $N$ are star bodies in $\mathbb{E}^n$ with $V(M) = V(N) = 1$. See Corollary 4.5 and also Corollary 5.5, from which a version for convex bodies, relating to Groemer's as (5) relates to (3), can be derived. The dual Minkowski inequality (8), incidentally, plays a part in the solution of the Busemann-Petty problem mentioned above (see, for example, [3, p. 295]).

In the final Section 5, we discuss the relationship between our two methods and in particular prove in Theorem 5.1 that

$$2 - V(M ^\ast N)^{1/n} \leq 1 - \tilde{V}_1(M, N) \leq 2(n - 1)(2 - V(M ^\ast N)^{1/n}),$$

where $M$ and $N$ are star bodies in $\mathbb{E}^n$ with $V'(M) = V'(N) = 1$. This implies the known fact that (4) and (8) are equivalent, just as their classical counterparts (2) and (7) are. However, (9) provides a much more informative proof than the usual one (see, for example, [3, Theorem B.2.1]). Moreover, in Example 5.2 we show that (9) is the best possible up to constant factors. The same example is also used to show that our main stability results are also the best possible up to constant factors.

Our proof of (9) does not seem to carry over directly to the classical case, but we have found a different proof of a version of (9) that does apply equally to the classical case, giving the following new relationship between the left-hand sides of (2) and (7). Let $K$ and $L$ be convex bodies in $\mathbb{E}^n$ such that $V(K) = V(L) = 1$. There is an $\epsilon > 0$ such that if $V_1(K, L) < 1 + \epsilon$, then

$$V_{1}(K + L)^{1/n} - 2 \leq V_1(K, L) - 1 \leq \alpha (n - 1)(V(K + L)^{1/n} - 2),$$

for any $\alpha > 2^{n-1}/(2^{n-1} - 1)$. (We intend to examine whether this can be improved in a further investigation.) As far as we know (9) and (10) represent a new type of result. This supports our belief that quite apart from its intrinsic merit, the dual Brunn-Minkowski theory is worth developing because of the light it sheds on the classical Brunn-Minkowski theory.

2. DEFINITIONS AND PRELIMINARIES

We denote the origin, unit sphere, and closed unit ball in $n$-dimensional Euclidean space $\mathbb{E}^n$ by $o$, $S^{n-1}$, and $B$, respectively.

Lebesgue $k$-dimensional measure $\lambda_k$ in $\mathbb{E}^n$, $k = 1, \ldots, n$, can be identified with $k$-dimensional Hausdorff measure in $\mathbb{E}^n$. Then spherical Lebesgue
measure in $S^{n-1}$ can be identified with $\lambda_{n-1}$ in $S^{n-1}$. In this paper integration over $S^{n-1}$ with respect to $\lambda_{n-1}$ will be denoted by $du$. We write $V = \lambda_n$, and call this volume in $E^n$. We also write $\kappa_n = V(B)$.

We say that a set is centered if it is centrally symmetric, with center at the origin.

A convex body is a compact convex set with nonempty interior. If $0 \leq r \leq R \leq \infty$, we denote the class of convex bodies in $E^n$ that contain $rB$ and are contained in $RB$ by $\mathcal{K}^n(r, R)$. The reader is referred to [19] for a comprehensive account of the Brunn-Minkowski theory.

A set $M$ is star shaped at the origin if every line through the origin that meets $M$ does so in a (possibly degenerate) closed line segment. If $M$ is a set that is star shaped at the origin, its radial function $\rho_M$ is defined, for all $u \in S^{n-1}$ such that the line through the origin parallel to $u$ intersects $M$, by

$$\rho_M(u) = \max \{ c : cu \in M \}.$$ 

In this paper, a star body is a set that is star shaped at the origin and whose radial function is positive and continuous on $S^{n-1}$. There are other definitions of this term in the literature; see, for example [6]. If $0 \leq r \leq R \leq \infty$, we denote the class of star bodies in $E^n$ that contain $rB$ and are contained in $RB$ by $\mathcal{S}^n(r, R)$.

If $x, y \in E^n$, then the radial sum $x \oplus y$ of $x$ and $y$ is defined to be the usual vector sum $x + y$ if $x$ and $y$ are contained in a line through $o$, and $o$ otherwise. If $M$ and $N$ are star bodies in $E^n$ and $s, t \in \mathbb{R}$, then

$$sM \oplus tN = \{ sx \oplus ty : x \in M, y \in N \},$$

and

$$\rho_{sM \oplus tN} = s \rho_M + t \rho_N.$$ 

In [14], Lutwak defined the dual mixed volume $\tilde{V}(M_1, \ldots, M_n)$ of star bodies $M_1, \ldots, M_n$ in $E^n$ by

$$\tilde{V}(M_1, \ldots, M_n) = \frac{1}{n} \int_{S^{n-1}} \rho_{M_1}(u) \rho_{M_2}(u) \cdots \rho_{M_n}(u) \, du.$$

If $M$ is a star body, the dual volume $\tilde{V}_i(M)$, where $i$ is an integer with $0 \leq i \leq n$, is defined by

$$\tilde{V}_i(M) = \tilde{V}(M, i; B, n - i) = \frac{1}{n} \int_{S^{n-1}} \rho_M(u)^i \, du,$$

where the notation $\tilde{V}(M, i; B, n - i)$ signifies that $M$ appears $i$ times and $B$ appears $(n - i)$ times. The special cases $i = 0$ and $i = n$ yield $\tilde{V}_0(M) = \kappa_n$ and $\tilde{V}_n(M) = V(M)$. 

One can relax the restriction on \(i\) and define \(\tilde{V}_i(M)\) for any \(i \in \mathbb{R}\) by the integral above. Note that \(\tilde{V}_i(B) = \kappa_n\) for all \(i\).

We shall also use the notation
\[
\tilde{V}_1(M, N) = \tilde{V}(M, n-1; N)
\]
and
\[
\tilde{V}_1(M, N) = \tilde{V}(M, N; B, n-2).
\]

Dual volumes can also be defined as averages of volumes of sections by subspaces, just as the classical intrinsic volumes, special cases of general mixed volumes, can be defined as averages of volumes of projections on subspaces.

Suppose that \(M\) and \(N\) are star bodies in \(\mathbb{E}^n\) and \(p \geq 1\). The \(L^p\) distance and the \(L^\infty\) distance between \(M\) and \(N\) are defined by
\[
\tilde{d}_p(M, N) = \left( \int_{S^{n-1}} |\rho_M(u) - \rho_N(u)|^p du \right)^{1/p}
\]
and
\[
\tilde{d}(M, N) = \tilde{d}_\infty(M, N) = \max_{u \in S^{n-1}} |\rho_M(u) - \rho_N(u)|,
\]
respectively. The metric \(\tilde{d}\) is often called the radial metric.

3. The Dual Brunn-Minkowski Inequality

In this section we obtain stability versions of the dual Brunn-Minkowski inequality. We require several technical lemmas.

**Lemma 3.1.** Let \(X\) be a measurable set of finite measure, let \(p \geq 2\), and let \(f, g \in L^p(X)\). Let \(a = \min\{\|f\|_p, \|g\|_p\}\), \(A = \max\{\|f\|_p, \|g\|_p\}\), \(0 \leq \varepsilon \leq a\), and \(0 \leq t \leq 1\), and suppose that
\[(1 - t)\|f\|_p + t\|g\|_p - \|(1 - t)f + tg\|_p \leq \varepsilon.\]

If \(\tilde{f} = f/\|f\|_p\), \(\tilde{g} = g/\|g\|_p\), and
\[
\tilde{t} = \frac{t\|g\|_p}{(1 - t)\|f\|_p + t\|g\|_p},
\]
then
\[
1 - \|(1 - \tilde{t})\tilde{f} + \tilde{t}\tilde{g}\|_p^p \leq 1 - \left(1 - \frac{\varepsilon}{a}\right)^p \leq \frac{p\varepsilon}{a}.
\]
Proof. Suppose that \( z = (1 - t)\|f\|_p + t\|g\|_p - (1 - t)f + tg \leq \varepsilon \). We have

\[
1 - \| (1 - i)\tilde{f} + i\tilde{g} \|_p^p = 1 - \left( \frac{(1 - t)f + tg}{(1 - t)f + tg} \right)^p \\
= 1 - \left( 1 - \frac{z}{(1 - t)f + tg} \right)^p \\
\leq 1 - \left( 1 - \frac{\varepsilon}{a} \right)^p \leq \frac{pe}{a}.
\]

Lemma 3.2. Let \( X \) be a measurable set of finite measure, let \( p \geq 2 \), and suppose that

\[
t(1 - t)F(f, g) + \| (1 - t)f + tg \|_p^p \leq (1 - t)f + tg \|_p^p \quad (11)
\]

holds for nonnegative \( f, g \in L^p(X) \). Suppose that \( f, g \in L^p(X) \) are nonnegative, with

\[
a = \min\{\|f\|_p, \|g\|_p\} \text{ and } A = \max\{\|f\|_p, \|g\|_p\}.
\]

If \( \tilde{f} = f/\|f\|_p, \tilde{g} = g/\|g\|_p \), and \( 0 \leq t \leq 1 \), then

\[
(1 - t)\|f\|_p + t\|g\|_p - \| (1 - t)f + tg \|_p \geq t(1 - t) \frac{a^2}{pa}F(\tilde{f}, \tilde{g}). \quad (12)
\]

Proof. Let \( \varepsilon = (1 - t)\|f\|_p + t\|g\|_p - (1 - t)f + tg \|_p \). By (11) (applied to \( \tilde{f} \) and \( \tilde{g} \)) and Lemma 3.1,

\[
\tilde{t}(1 - t)F(\tilde{f}, \tilde{g}) \leq 1 - \| (1 - i)\tilde{f} + i\tilde{g} \|_p^p \leq \frac{pe}{a}.
\]

Furthermore,

\[
\tilde{t}(1 - t) = \frac{(1 - t)\|f\|_p\|g\|_p}{(1 - t)\|f\|_p + t\|g\|_p} \geq \frac{at(1 - t)}{A},
\]

as can be seen by minimizing the function

\[
\frac{\|f\|_p\|g\|_p}{(1 - t)\|f\|_p + t\|g\|_p}^2
\]

with respect to \( t \). The lemma follows directly.

Lemma 3.3. Let \( X \) be a measurable set of finite measure, let \( p \geq 2 \), and let \( f, g \in L^p(X) \) be nonnegative. If \( 0 \leq t \leq 1 \), then

\[
t(1 - t)\|f^{p/2} - g^{p/2}\|_p^2 + \| (1 - t)f + tg \|_p^p \leq (1 - t)\|f\|_p^p + t\|g\|_p^p.
\]
Proof. Let \( p \geq 2 \). For \( a \geq b \geq 0 \), let

\[
G(a, b) = (1 - t)a^p + tb^p - ((1 - t)a + tb)^p - t(1 - t)(a^{p/2} - b^{p/2})^2.
\]

The inequality

\[
(1 - t)a + tb)^p - 1 \leq (1 - t)a^{p-1} + tb^{p-1}
\]

holds by Jensen's inequality for means; see, for example, [3, p. 367]. Using this, we obtain

\[
G(a, b) = p(1 - t)(a^{p-1} - ((1 - t)a + tb)^p - t(a^{p/2} - b^{p/2})a^{(p-2)/2})
\]

\[
\geq p(1 - t)(a^{p-1} - (1 - t)a^{p-1} - tb^{p-1} - t(a^{p/2} - b^{p/2})a^{(p-2)/2})
\]

\[
= pt(1 - t)b^{p/2}(a^{(p-2)/2} - b^{(p-2)/2}) \geq 0.
\]

Since \( G(a, a) = 0 \), it follows that \( G(a, b) \geq 0 \). Let \( x \in X \), \( a = f(x) \), and \( b = g(x) \). Then we have

\[
t(1 - t)(f(x)^{p/2} - g(x)^{p/2})^2 + ((1 - t)f(x) + tg(x))^p
\]

\[
\leq (1 - t)f(x)^p + tg(x)^p.
\]

Integration over \( X \) now yields the required inequality. \( \square \)

Lemma 3.4. Let \( X \) be a measurable set of finite measure, let \( p \geq 2 \), and let \( f, g \in L^p(X) \) be nonnegative. Then

\[
\|f^{p/2} - g^{p/2}\|_2^2 \geq \|f - g\|_p^2.
\]

Proof. If \( a, b \geq 0 \) and \( r \geq 1 \), then

\[
|a^r - b^r|^{1/r} \geq |a - b|,
\]

as can be seen by differentiating with respect to \( r \). If \( r = p/2 \), we obtain

\[
(a^{p/2} - b^{p/2})^2 \geq |a - b|^p.
\]

Let \( x \in X \), \( a = f(x) \), and \( b = g(x) \), and integrate over \( X \). \( \square \)

Remark 3.5. For arbitrary \( f, g \in L^p(X) \) and \( p \geq 2 \), the following refinement of Clarkson's inequality (see [11, p. 225] or [16, p. 534]) was proved in [12] (see also [16, p. 562]). If \( 0 \leq t \leq 1 \), then

\[
t^{p/2}(1 - t)^{p/2}\|f - g\|_p^2 + \|(1 - t)f + tg\|_p^2 \leq (1 - t)\|f\|_p^2 + t\|g\|_p^2.
\]

By Lemma 3.4, Lemma 3.3 implies that for nonnegative \( f, g \in L^p(X) \), the stronger inequality

\[
t(1 - t)\|f - g\|_p^2 + \|(1 - t)f + tg\|_p^2 \leq (1 - t)\|f\|_p^2 + t\|g\|_p^2
\]

holds. Therefore Lemma 3.3 represents a considerable improvement of Clarkson's inequality for nonnegative functions; however, it is not generally true for arbitrary functions.
Theorem 3.6. Let $M$ and $N$ be star bodies in $\mathbb{R}^n$ and let $p \geq 2$. Suppose that $M', N'$ are the dilatates of $M$, $N$, respectively, such that $\tilde{V}_p(M') = \tilde{V}_p(N') = 1$. Let

$$c = \min\{\tilde{V}_p(M)^{1/p}, \tilde{V}_p(N)^{1/p}\} \text{ and } C = \max\{\tilde{V}_p(M)^{1/p}, \tilde{V}_p(N)^{1/p}\}.$$ 

If $0 \leq t \leq 1$, then

$$(1 - t)\tilde{V}_p(M)^{1/p} + t\tilde{V}_p(N)^{1/p} - \tilde{V}_p((1 - t)M \tilde{+} tN)^{1/p} \geq t(1 - t)\frac{c^2}{npC}(p_M^{p/2} - p_N^{p/2})^2/2 \geq t(1 - t)\frac{c^2}{npC}\delta(M', N')p.$$

Proof. Lemma 3.3 shows that in Lemma 3.2, (12) holds when 

$$F(\tilde{f}, \tilde{g}) = \|\tilde{f}^{p/2} - \tilde{g}^{p/2}\|_2^2.$$ 

We apply this case of Lemma 3.2 when $X = S^{n-1}$ with spherical Lebesgue measure, $f = \rho_M$, and $g = \rho_N$. Then

$$\|f\|_p = \|\rho_M\|_p = (n\tilde{V}_p(M))^{1/p},$$

and similarly $\|g\|_p = (n\tilde{V}_p(N))^{1/p}$. Also, $a = n^{1/p}c$ and $A = n^{1/p}C$. Furthermore, we have

$$M' = \frac{1}{\tilde{V}_p(M)^{1/p}}M, \ N' = \frac{1}{\tilde{V}_p(N)^{1/p}}N,$$

and

$$\|\tilde{f}^{p/2} - \tilde{g}^{p/2}\|_2^2 = \frac{1}{n}\|p_M^{p/2} - p_N^{p/2}\|_2^2.$$ 

Substitution of these quantities in (12) immediately provides the first inequality in the statement of the theorem, and the second follows by Lemma 3.4.

The following corollary can be regarded as a dual form of the stability version of the Brunn-Minkowski theorem obtained by Groemer [8] (see also [10, p. 135]).

Corollary 3.7. Let $M$ and $N$ be star bodies in $\mathbb{R}^n$ with $V(M) = V(N) = 1$. Then

$$2 - V(M \tilde{+} N)^{1/n} \geq \frac{1}{2n^2}\|\rho_M^{n/2} - \rho_N^{n/2}\|_2^2.$$
Lema 3.8. Let $X$ be a measurable set of finite measure, let $p \geq 2$, and let $f, g \in L^p(X)$ be nonnegative. If $0 \leq t \leq 1$, then
\[
t(1-t)\frac{p(p-1)}{2} \min \{f, g\}^{p-2} \|f - g\|^2 \leq (1-t)\|f\|^p_p + t\|g\|^p_p.
\]
Proof. Let $p \geq 2$. For $a \geq b \geq 0$, let
\[
H(a, b) = (1-t)a^p + tb^p - ((1-t)a + tb)^p - t(1-t)\frac{p(p-1)}{2}b^{p-2}(a-b)^2.
\]
Using Jensen’s inequality for means, we obtain
\[
H_a(a, b) = p(1-t)(a^{p-1} - ((1-t)a + tb)^{p-1}) - t(p-1)b^{p-2}(a-b))
\geq p(1-t)(a^{p-1} - (1-t)a^{p-1} - tb^{p-1} - t(p-1)b^{p-2}(a-b))
= pt(1-t)(a^{p-1} - b^{p-1} - (p-1)b^{p-2}(a-b)).
\]
Now for $a > b$ we have
\[
\frac{a^{p-1} - b^{p-1}}{a-b} > (p-1)b^{p-2},
\]
since the right-hand side is the derivative of the convex function $s^{p-1}$ at $s = b$. Therefore $H_a(a, b) \geq 0$, and since $H(a, a) = 0$, it follows that $H(a, b) \geq 0$. Letting $x \in X$, $a = f(x)$, and $b = g(x)$, and integrating over $X$, we obtain the required inequality. □

Rem. 3.9. Lemmas 3.3 and 3.8 provide inequalities of the same type, neither of which is stronger than the other. It is not hard to see that the inequality from Lemma 3.8 is stronger when $f \geq g$ and $f$ is close to $g$, while that from Lemma 3.3 is stronger when $f$ is much larger than $g$. The same considerations extend to Theorem 3.6 and the next theorem.

Theorem 3.10. Let $M$ and $N$ be star bodies in $\mathbb{R}^n(0, \infty)$ and let $p \geq 2$. Suppose that $M', N'$ are the dilatates of $M$, $N$, respectively, such that $\tilde{V}_p(M') = \tilde{V}_p(N') = 1$. Let
\[
c = \min \{\tilde{V}_p(M)^{1/p}, \tilde{V}_p(N)^{1/p}\} \quad \text{and} \quad C = \max \{\tilde{V}_p(M)^{1/p}, \tilde{V}_p(N)^{1/p}\}.
\]
Then for $0 \leq t \leq 1$,
\[
(1-t)\tilde{V}_p(M)^{1/p} + t\tilde{V}_p(N)^{1/p} - \tilde{V}_p((1-t)M \ast tN)^{1/p}
\geq t(1-t)\frac{(p-1)c^2r^2-p^2}{2n(2p-2)|pCp^{-1}D_2(M', N')^2}.
\]
Proof. Lemma 3.8 shows that in Lemma 3.2, (12) holds when
\[ F(\tilde{f}, \tilde{g}) = \frac{p(p-1)}{2} \min\{\tilde{f}, \tilde{g}\}^{p-2} \|\tilde{f} - \tilde{g}\|^2. \]

Since
\[ \min\{\tilde{f}, \tilde{g}\} = \min \left\{ \frac{f}{\|f\|_p}, \frac{g}{\|g\|_p} \right\} \geq \frac{\min\{f, g\}}{A}, \]
(12) becomes
\[ (1-t)\|f\|_p + t\|g\|_p - \|1-t)f + tg\|_p \]
\[ \geq t(1-t) \frac{(p-1)a^2 \min\{f, g\}^{p-2}}{2A^{p-1}} \|\tilde{f} - \tilde{g}\|^2. \]

The theorem follows immediately when \( X = S^{n-1} \) with spherical Lebesgue measure, \( \tilde{f} = \rho_M \), and \( \tilde{g} = \rho_N \). \( \blacksquare \)

The results above can be combined with known relations between the \( \tilde{\delta}_p \) and \( \delta \) metrics for convex bodies to give a stability version of the dual Brunn-Minkowski theorem for convex bodies in terms of the \( \tilde{\delta} \) metric. We state just one of many possible such versions next.

**Theorem 3.11.** Let \( K \) and \( L \) be convex bodies in \( \mathbb{R}^n(r, R) \) such that \( V(K) = V(L) = 1 \). If \( 0 \leq t \leq 1 \), then
\[ 1 - V((1-t)K \overset{\approx}{+} tL)^{1/n} \geq t(1-t) \frac{(n-1)c_n}{2^n(2n-2)!/2^n} \tilde{\delta}(K, L)^{n+1}, \]
where
\[ c_n = \frac{\kappa_{n-1}}{2^{n-2}n(n+1)}. \]

**Proof.** Groemer [9, Lemma 3], using a result of Vitale [20], proved that if \( K \) and \( L \) are convex bodies in \( \mathbb{R}^n(r, R) \), then
\[ \tilde{\delta}_2(K, L)^2 \geq c_n \frac{r^{n+3}}{R^{n+2}} \tilde{\delta}(K, L)^{n+1}. \]

The corollary follows from this and Theorem 3.10 with \( p = n \). \( \blacksquare \)

**Remark 3.12.** When \( M = B \) and \( N \) is the star body obtained by adding to the unit ball a long thin “spike,” \( \tilde{\delta}(M, N) \) is large but the left-hand side of the inequality in Theorem 3.11 is small. Examples of this type show that there is no version of Theorem 3.11 that applies to star bodies in \( \mathbb{R}^n(r, R) \). Compare [5, Example 3.3].
Remark 3.13. The methods used above also provide upper bounds for the quantity

\[(1 - t)\tilde{V}_p(M)^{1/p} + t\tilde{V}_p(N)^{1/p} - \tilde{V}_p((1 - t)M + tN)^{1/p}.\]

To see this, note that the assumption that \((1 - t)|f|_p + tg|_p \geq (1 - t)f + tg|_p \geq \varepsilon\) in Lemma 3.1 leads similarly to

\[1 - \| (1 - t)f + tg \|_p \geq 1 - \left(1 - \frac{\varepsilon}{A}\right)^p \geq \frac{\varepsilon}{A}.\]

If \(f, g \leq R\), we also have

\[\tilde{f}, \tilde{g} \leq \max \left\{ \frac{f}{\|f\|_p}, \frac{g}{\|g\|_p} \right\} \leq \frac{R}{a}\]

and \(\tilde{t}(1 - \tilde{t}) \leq \tilde{t}(1 - t)/a\). The proof of Lemma 3.8 shows that the inequality in that lemma is reversed when \(\min \{f, g\}\) is replaced by \(\max \{f, g\}\). Using these facts applied to \(f\) and \(g\), one concludes that if \(f, g \leq R\) then

\[(1 - t)|f|_p + tg|_p \geq (1 - t)f + tg|_p \leq t(1 - t)\frac{p(p - 1)A^2R^{p-2}}{2a^{p-1}} \|\tilde{f} - \tilde{g}\|_2^2.\]

Now with other assumptions as in Theorem 3.10, let \(M\) and \(N\) be star bodies in \(\mathcal{P}^n(0, R)\). Then

\[(1 - t)\tilde{V}_p(M)^{1/p} + t\tilde{V}_p(N)^{1/p} - \tilde{V}_p((1 - t)M + tN)^{1/p} \leq t(1 - t)\frac{p(p - 1)C^2R^{p-2}}{2n^{(p-2)(p-1)} \|g\|_p} - \delta_2(M', N'^2).\]

Remark 3.14. Clarkson’s inequality and its refinement mentioned in Remark 3.5 reverse for \(1 < p \leq 2\); see [11, p. 227] and [12]. We stated the results above for \(p \geq 2\) for convenience and since our main interest is in the case \(p = \infty\). It would be a routine matter to obtain similar results for other, even negative, values of \(p\), by the methods employed above. (Negative values of \(p\) are occasionally of use, even, surprisingly, in the classical Brunn-Minkowski theory; see, for example, [3, p. 320].)

4. THE DUAL ALEKSANDROV-FENCHEL INEQUALITY

In this section we obtain stability versions of the dual Aleksandrov-Fenchel inequality and some of the standard inequalities it implies. The inequalities themselves are of course just those given below when the right-hand side of the inequality is replaced by zero.

The following refinement of the arithmetic-geometric mean inequality is due to Kober [13]. A proof is also given in [15, pp. 81-83].
Proposition 4.1. Let $a_i \geq 0$ and $w_i > 0$, $i = 1, \ldots, m$, where $\sum w_i = 1$. Define

$$w = \min_{1 \leq i \leq m} w_i \text{ and } W = \max_{1 \leq i \leq m} w_i.$$ 

Then

$$\frac{w}{m-1} \sum_{1 \leq i \leq j \leq m} (a_i^{1/2} - a_j^{1/2})^2 \leq \sum_{i=1}^m w_i a_i - \prod_{i=1}^m a_i^{w_i} \leq W \sum_{1 \leq i \leq j \leq m} (a_i^{1/2} - a_j^{1/2})^2.$$ 

Lemma 4.2. Let $f_0, f_1, \ldots, f_m$ be positive Borel functions on $S^{n-1}$, and let $w_i > 0$, $i = 1, \ldots, m$, where $\sum_i w_i = 1$ and $w = \min \{w_i; 1 \leq i \leq m\}$. Then

$$1 - \frac{\int_{S^{n-1}} f_0(u) \cdots f_m(u) \, du}{\prod_{i=1}^m \left( \int_{S^{n-1}} f_0(u) f_i(u)^{1/w_i} \, du \right)^{w_i}} \geq \frac{w}{m-1} \sum_{1 \leq i \leq j \leq m} \int_{S^{n-1}} \frac{f_i(u)^{1/(2w_i)}}{f_0(u)^{1/(2w)}} \cdot \left( \frac{f_i(u)^{1/(2w)}}{f_0(u)^{1/(2w)}} \right)^{1/2} \, du.$$ 

Proof. In Proposition 4.1, we let

$$a_i = \frac{f_0(u) f_i(u)^{1/w_i}}{\int_{S^{n-1}} f_0(u) f_i(u)^{1/w_i} \, du},$$

for $i = 1, \ldots, m$ and integrate over $S^{n-1}$. 

Theorem 4.3. (Stability version of the dual Aleksandrov-Fenchel inequality.) Let $M_1, \ldots, M_n$ be star bodies in $\mathbb{E}^n$, and let $m$ be an integer with $1 \leq m \leq n$. Then

$$1 - \frac{\widetilde{V}(M_1, \ldots, M_n)}{\prod_{i=1}^m \widetilde{V}(M_i, m; M_{m+1}, \ldots, M_n)^{1/m}} \geq \frac{1}{nm(m-1)} \sum_{1 \leq i \leq j \leq m} \int_{S^{n-1}} \rho_{M_{m+1}}(u) \cdots \rho_{M_n}(u)$$

$$\times \left( \frac{\rho_{M_i}(u)^{m/2}}{\widetilde{V}(M_i, m; M_{m+1}, \ldots, M_n)^{1/2}} - \frac{\rho_{M_j}(u)^{m/2}}{\widetilde{V}(M_j, m; M_{m+1}, \ldots, M_n)^{1/2}} \right)^2 \, du.$$
Proof. In Lemma 4.2, we let $w_i = 1/m$,

$$f_0 = \frac{1}{n} \rho_{M_1} \cdots \rho_{M_n},$$

if $m < n$ and $f_0 = 1/n$ if $m = n$, and $f_i = \rho_{M_i}$ for $i = 1, \ldots, m$. □

**Corollary 4.4.** Let $M_1, \ldots, M_n$ be star bodies in $\mathbb{R}^n$, and let $M_i'$ be the dilatate of $M_i$ such that $V(M_i') = 1$, $i = 1, \ldots, n$. Then

$$1 - \frac{\tilde{V}(M_1, \ldots, M_n)}{\prod_{i=1}^n V(M_i)^{1/n}} \geq \frac{1}{n^2 (n-1)} \sum_{1 \leq i \leq j \leq n} \|\rho_{M_i}^{n/2} - \rho_{M_i}^{n/2}\|^2.$$

Proof. Let $m = n$ in Theorem 4.3. □

**Corollary 4.5.** (Stability version of the dual Minkowski inequality.) Let $M$ and $N$ be star bodies in $\mathbb{R}^n$, and let $M'$, $N'$ be the dilatates of $M$, $N$, respectively, such that $V(M') = V(N') = 1$. Then

$$1 - \frac{\tilde{V}(M, N)}{V(M)^{(n-1)/n} V(N)^{(n-1)/n}} \geq \frac{1}{n^2} \|\rho_{M'}^{n/2} - \rho_{N'}^{n/2}\|^2.$$

Proof. For the first inequality, let $M_1 = M_2 = \cdots = M_{n-1} = M$ and $M_n = N$ in the previous corollary. □

**Corollary 4.6.** (Stability version of the dual isoperimetric inequality.) Let $M$ be a star body in $\mathbb{R}^n$. Then

$$\left(\frac{V(M)}{V(B)}\right)^{(n-1)/n} - \frac{\tilde{V}_{n-1}(M)}{\tilde{V}_{n-1}(B)} \geq \frac{1}{n^2} \left(\frac{V(M)}{V(B)}\right)^{(n-1)/n} \left\|\rho_{M}^{n/2} - \frac{1}{V(M)^{1/2}}\right\|^2.$$

Proof. Let $N = B$ in the previous corollary, and recall that $\tilde{V}_{n-1}(B) = V(B)$. □

Other inequalities implied by the dual Aleksandrov-Fenchel inequality (see, for example, [3, p. 373]) can be treated in the same way as those above. Note that by using Lemma 3.4 with $p = n$ and $f$ and $g$ replaced by the radial functions of the bodies concerned, we can easily replace the right-hand sides in the previous three corollaries by expressions involving the $\delta_n$ distance between these bodies.
5. FURTHER RESULTS AND COMMENTS

The dual Brunn-Minkowski inequality and the dual Minkowski inequality are equivalent. The proof in [3, Theorem B.2.1] that the Brunn-Minkowski inequality and Minkowski’s first inequality are equivalent is easily adapted to the dual situation; however, the equivalence is also a consequence of the following theorem, which provides much more information.

**Theorem 5.1.** Let $M$ and $N$ be star bodies in $\mathbb{E}^n$ such that $V(M) = V(N) = 1$. Then

$$2 - V(M \bar{+} N)^{1/n} \leq 1 - \tilde{V}_1(M, N) \leq 2(n - 1)(2 - V(M \bar{+} N)^{1/n}).$$

**Proof.** To establish the left-hand inequality, let

$$f(t) = V((1 - t)M \bar{+} tN)^{1/n},$$

for $0 \leq t \leq 1$. Then $f$ is differentiable on $(0, 1)$, and a straightforward computation shows that

$$f'(0) = \tilde{V}_1(M, N) - 1.$$

The dual Brunn-Minkowski inequality implies that $f$ is convex on $[0, 1]$, and this in turn implies that for $0 \leq t \leq 1$, we have

$$f'(0) \leq \frac{f(t) - f(0)}{t}.$$

Using $f(0) = 1$, we obtain with $t = 1/2$,

$$2 - V(M \bar{+} N)^{1/n} = 2(1 - f(\frac{1}{2})) \leq -f'(0) = 1 - \tilde{V}_1(M, N),$$

as required.

For the right-hand inequality we first note that

$$1 - \tilde{V}_1(M, N) \leq \frac{n - 1}{n^2} \|\rho_M^{n/2} - \rho_N^{n/2}\|^2_2.$$

This can be obtained from the right-hand inequality in Proposition 4.1, just as Corollary 4.5 was obtained from the left-hand inequality in Proposition 4.1. The required inequality now follows from Corollary 3.7. \(\blacksquare\)

The argument for the left-hand inequality in Theorem 5.1 is essentially that employed by Groemer [9, p. 121] (see also [19, p. 319]) in obtaining a stability version of the Minkowski inequality from a stability version of the Brunn-Minkowski inequality.
Example 5.2. We now show that some of the results above are the best possible up to a constant factor.

Let $M$ be the centered ball with $V(M) = 1$; that is, $M = \kappa_n^{-1/n} B$. For $0 \leq s \leq 1/2$, let $N = N(s)$ be the union of two closed half balls with center at the origin and disjoint interiors, one with volume $(1/2) + s$, and the other with volume $(1/2) - s$, so that $V(N) = 1$. Then

$$\rho_N(u) = \left( \frac{1+2s}{\kappa_n} \right)^{1/n}$$

when $u$ is in the upper closed half of $S^{n-1}$, and the same but with $s$ replaced by $-s$ when $u$ is in the lower open half of $S^{n-1}$. Of course $N$ is not a star body as defined above because $\rho_N$ is not continuous, but we can approximate $N$ arbitrarily closely by star bodies in such a way that our main claims below still hold.

By direct computation we obtain

$$1 - \tilde{V}_1(M, N) = 1 - \frac{1}{2}((1+2s)^{1/n} + (1-2s)^{1/n}) = f(n, s)$$

and

$$2 - V(M) = 2 - \left( \frac{(1+(1+2s)^{1/n})^n + (1+(1-2s)^{1/n})^n}{2} \right)^{1/n}$$

$$= g(n, s),$$

say. With a little calculation we see that

$$f(n, s) = \frac{2(n-1)}{n^2}s^2 + o(s^2)$$

and

$$g(n, s) = \frac{(n-1)}{n^2}s^2 + o(s^2).$$

Therefore

$$\lim_{s \to 0^+} \frac{f(n, s)}{g(n, s)} = 2.$$  

It is also not difficult to show that $f(n, 1/2)/(2(n-1)g(n, 1/2))$ decreases to $1/(4\ln 2)$ as $n \to \infty$. This shows that Theorem 5.1 is the best possible, up to constant factors 2 and $1/(4\ln 2)$.

A further computation gives

$$\frac{1}{n^2} \| \rho_M^{n/2} - \rho_N^{n/2} \|_2^2 = \frac{1}{n}(2 - \sqrt{1+2s} - \sqrt{1-2s}) = h(n, s),$$
say. Now
\[
\lim_{n \to \infty} \frac{f(n, s)}{h(n, s)} = \frac{\log(1 + 2s) + \log(1 - 2s)}{4 - 2\sqrt{1 + 2s} - 2\sqrt{1 - 2s}} = j(s),
\]
say. The function \( j(s) \) satisfies \( j(s) \to 2 \) as \( s \to 0^+ \) and \( j(s) \) increases to infinity as \( s \to 1/2^- \). Moreover, \( f(n, s)/h(n, s) \) increases with \( n \) for fixed \( s \). By taking \( s \) close to 0 we see that
\[
1 - \tilde{V}_1(M, N) \leq c \frac{n^2}{n^2} \|\rho_M^{n/2} - \rho_N^{n/2}\|_2^2
\]
where \( c \) is a constant arbitrarily close to 2. Therefore the first inequality in Corollary 4.5 is the best possible up to a constant factor 2. It now follows from Theorem 5.1 that Corollary 3.7 is also the best possible up to a constant factor 4.

In view of Theorem 5.1, it is appropriate to compare the results in the previous two sections.

The method in the previous section yields stability versions of the dual Aleksandrov-Fenchel inequality and all the other inequalities it implies. Moreover, as we now show, Corollary 4.5 implies Corollary 3.7.

Let \( M \) and \( N \) be star bodies in \( \mathbb{R}^n \) with \( V(M) = V(N) = 1 \), and define \( Q = r^{-1}(M \overset{\infty}{\oslash} N) \), where \( r = V(M \overset{\infty}{\oslash} N)^{1/n} \), so that \( V(Q) = 1 \). By Corollary 4.5, we have
\[
V(M \overset{\infty}{\oslash} N)^{1/n} = \tilde{V}(Q, n - 1; M \overset{\infty}{\oslash} N) = \tilde{V}_1(Q, M \overset{\infty}{\oslash} N)
\]
\[
= \tilde{V}_1(Q, M) + \tilde{V}_1(Q, N)
\]
\[
\leq 2 - \frac{1}{n^2} \int_{S^{n-1}} ((\rho_Q(u)^{n/2} - \rho_M(u)^{n/2})^2 + (\rho_Q(u)^{n/2} - \rho_N(u)^{n/2})^2) \, du.
\]
The function \( F(x) = (x - a)^2 + (x - b)^2 \) takes its minimum value for \( x = (a + b)/2 \), and
\[
F\left(\frac{a + b}{2}\right) = \frac{(a - b)^2}{2}.
\]
Applying this to the previous integrand, with \( x = \rho_Q(u)^{n/2} \), \( a = \rho_M(u)^{n/2} \), and \( b = \rho_N(u)^{n/2} \), we obtain
\[
2 - V(M \overset{\infty}{\oslash} N)^{1/n} \geq \frac{1}{2n^2} \|\rho_M^{n/2} - \rho_N^{n/2}\|_2^2,
\]
which is the inequality in Corollary 3.7.

We now show that the methods used in Section 3 can provide inequalities related to those in Section 4.
Corollary 3.7 and the left-hand inequality of Theorem 5.1 give
\[ 1 - \tilde{V}_1(M, N) \geq \frac{1}{2n^2} \| \rho_M^{n/2} - \rho_N^{n/2} \|^2. \]

Though Corollary 4.5 provides the better inequality
\[ 1 - \tilde{V}_1(M, N) \geq \frac{1}{n^2} \| \rho_M^{n/2} - \rho_N^{n/2} \|^2, \]
the results of Section 3 can be profitably applied as follows.

**Lemma 5.3.** Let \( X \) be a measurable set of finite measure, let \( p \geq 2 \), and let \( f, g \in L^p(X) \) satisfy \( f, g \geq 0 \) and \( \| f \|_p = \| g \|_p = b \). Then
\[ b^p - \int_X f^{p-1}g \geq \frac{(p-1)r^{p-2}}{2} \| f - g \|_2^2. \]

**Proof.** By Lemma 3.8, we have, setting \( a = A = b \) in (12),
\[ h(t) = b - \| (1-t)f + tg \|_p - t(1-t)\frac{(p-1)r^{p-2}}{2b^{p-1}} \| f - g \|_2^2 \geq 0, \]
for \( 0 \leq t \leq 1 \). It follows that \( h'(0) \geq 0 \), and an easy computation shows that this is precisely the required inequality. \( \blacksquare \)

**Theorem 5.4.** Let \( M \) and \( N \) be star bodies in \( S^n \) with \( \tilde{V}_p(M) = \tilde{V}_p(N) = 1 \), and let \( p \) be an integer with \( 2 \leq p \leq n \). Then
\[ 1 - \tilde{V}(M, p-1; N; B, n-p) \geq \frac{(p-1)r^{p-2}}{2n} \tilde{\delta}_2(M, N)^2. \]

**Proof.** In Lemma 5.3, we let \( X = S^{n-1} \) with spherical Lebesgue measure, \( f = \rho_M \), and \( g = \rho_N \). Then
\[ \| f \|_p = \| g \|_p = n^{1/p} = b, \]
and the theorem follows immediately. \( \blacksquare \)

Setting \( p = n \) in the previous theorem, we obtain the following stability version of the dual Minkowski inequality.

**Corollary 5.5.** Let \( M \) and \( N \) be star bodies in \( S^n \) such that \( V(M) = V(N) = 1 \). Then
\[ 1 - \tilde{V}_1(M, N) \geq \frac{(n-1)r^{n-2}}{2n} \tilde{\delta}_2(M, N)^2. \]

A similar but somewhat weaker result can be obtained by combining Theorem 5.1 with the case \( p = n \) of Theorem 3.10. The argument of Theorem 3.11 can of course be applied to Corollary 5.5 to give a stability version of the dual Minkowski inequality for convex bodies.
Theorem 5.6. Let $M$ and $N$ be star bodies in $\mathbb{R}^n$ with $\tilde{V}_{11}(M, M) = \tilde{V}_{11}(N, N) = 1$. Then

$$1 - \tilde{V}_{11}(M, N) = \frac{1}{2n} \delta_2(M, N)^2.$$ 

The proof of this theorem is trivial and we omit it, but it is worth stating as a curiosity. In the form of an inequality, it can be obtained either from Corollary 5.5 with $p = 2$ or from Theorem 4.3 with $m = 2$, $M_1 = M$, $M_2 = N$, and $M_3 = \cdots = M_n = B$. The corresponding inequality

$$\tilde{V}_{11}(M, M)\tilde{V}_{11}(N, N) - \tilde{V}_{11}(M, N)^2 \geq 0$$

is in fact the dual of the inequality

$$V_{11}(K, L)^2 - V_{11}(K, K)V_{11}(L, L) \geq 0$$

of Minkowski that holds for convex bodies $K$ and $L$ in $\mathbb{R}^n$. The stability of the latter inequality was investigated by Schneider [17] and Goodey and Groemer [7] (see also [10, Section 4.2] and [19, Section 6.6]).

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